Another stab at Section G.7 on ω computation PhL 1.11.15

This has been installed into Appendix G, do not edit here.

G.7 Computation of ω (Method 2)

In this section our approach to computing **ω** for the Euler Angle rotation is to find an equation which involves **ω** and solve it for **ω**! More or less at random, we choose (1.7.4),

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \,. \qquad \textit{(1.7.4)} \qquad \text{(G.7.1)}$$

Unlike our Method 1 computation of **ω** in the previous section, here we shall have no need for Goldstein's geometric Figure (G.5.5) or intermediate angles like ξ and ζ. We will, however, need various algebraic results developed in Section G.3.

This calculation is a bit slippery and requires care and precision in the use of notation -- it is easy to go astray. We shall use both matrix and Section 1.1 Dirac notations as seem convenient. A silver lining is that we shall be able to apply many of the results derived earlier in this document.

Recall from (G.5.17a) that the Goldstein Euler angle rotation is given by

$$R = R_z(-\psi)R_x(-\theta)R_z(-\varphi) \,. \qquad \textit{(G.5.17a)} \qquad \text{(G.7.2)}$$

In order to avoid a hundred minus signs, we shall *temporarily negate* all three angles. Then when we are done, we will undo this negation. We therefore temporarily take R to be,

$$R(\psi,\theta,\varphi) = R_z(\psi)R_x(\theta)R_z(\varphi) \equiv R(\boldsymbol{\Phi}) = \exp(-i\,\boldsymbol{\Phi}\bullet\mathbf{J}) \,. \qquad \text{// temp} \qquad \text{(G.7.3)}$$

Comment: To find **Φ** we could write out the matrix $R_z(\psi)R_x(\theta)R_z(\varphi)$,

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R := evalm(Rz(psi) &* Rx(theta) &* Rz(phi));
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$$R := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi) & \sin(\psi)\sin(\theta) \\ \sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & \sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

and decompose it into its symmetric and antisymmetric components S and A. *In theory* we could then compute Φ (=θ) and $\hat{\boldsymbol{\Phi}}$ (=$\hat{\mathbf{n}}$) from (G.2.14) and come up with an explicit expression for $\boldsymbol{\Phi} = \Phi\hat{\boldsymbol{\Phi}}$. The reader is just reminded that this is mechanically possible, but luckily we have no need for the result (which is quite complicated). We could also compute d**Φ** from the following

$$R(\psi+d\psi,\theta+d\theta,\varphi+d\varphi) = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) = \exp(-i\,[\boldsymbol{\Phi}+d\boldsymbol{\Phi}]\bullet\mathbf{J})$$

but again there is no need to do this. Note that **Φ** and d**Φ** will generally not be in the same direction.

We now set about constructing the left side of (G.7.1) starting with our fundamental equation from (1.1.29) which relates the Frame S and Frame S' basis vectors,

$$\mathbf{e}_n(\mathbf{\Phi}) = R(\mathbf{\Phi})\mathbf{e}'_n \qquad \text{or} \qquad |\mathbf{e}_n(\mathbf{\Phi})\rangle = |R(\mathbf{\Phi})\mathbf{e}'_n\rangle = \mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}'_n\rangle \ . \tag{G.7.4}$$

Since we have in mind that $\mathbf{\Phi} = \mathbf{\Phi}(t)$, one sees that the Frame S basis vectors change in time as viewed from Frame S'. In contrast, the basis vectors $\mathbf{e}'_n$ are static. The basis vectors $|\mathbf{e}_n(\mathbf{\Phi})\rangle$ are complete at time t so we can write, in analogy with (1.1.20),

$$\begin{aligned}
&\mathbf{1} = |\mathbf{e}_n(\mathbf{\Phi})\rangle\langle\mathbf{e}_n(\mathbf{\Phi})| && \text{completeness of the } \mathbf{e}_n \text{ at time t} && \text{(a)}\\
&\mathbf{1} = |\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle\langle\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})| && \text{completeness of the } \mathbf{e}_n \text{ at time t+dt} && \text{(b)}\\
&\mathbf{1} = |\mathbf{e}'_n\rangle\langle\mathbf{e}'_n| && \text{completeness of the } \mathbf{e}'_n \text{ at any time} && \text{(c)}
\end{aligned} \tag{G.7.5}$$

In (b) the rotation vector has changed from $\mathbf{\Phi}$ to some $\mathbf{\Phi}+d\mathbf{\Phi}$ as time moved from t to d+dt. A key point is that the $\mathbf{e}_n$ basis vectors are complete at any point in time. The rotation operator $\mathcal{R}(\mathbf{\Phi})$ similarly is real orthogonal at any time, analogous to (1.1.37),

$$\mathcal{R}^{\mathrm{T}}(\mathbf{\Phi}) = \mathcal{R}^{-1}(\mathbf{\Phi}) \qquad \text{and similarly for matrices} \qquad R^{\mathrm{T}}(\mathbf{\Phi}) = R^{-1}(\mathbf{\Phi}) \ . \tag{G.7.6}$$

Now we close the Dirac equation in (G.7.4) on the left with $\langle\mathbf{e}'_m|$ to get

$$\langle\mathbf{e}'_m|\,\mathbf{e}_n(\mathbf{\Phi})\rangle = \langle\mathbf{e}'_m|\mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}'_n\rangle = [R(\mathbf{\Phi})]'_{mn} \ . \tag{G.7.7a}$$

Since this is true for any $\mathbf{\Phi}$, one also has

$$\langle\mathbf{e}'_m|\,\mathbf{e}_n(\mathbf{\Phi}+d\mathbf{\Phi})\rangle = \langle\mathbf{e}'_m|\mathcal{R}(\mathbf{\Phi}+d\mathbf{\Phi})\,|\mathbf{e}'_n\rangle = [R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{mn} \ . \tag{G.7.7.b}$$

Recall from (1.1.35) that $(R)'_{ij} = R_{ij}$. Here we confirm that fact in the current fancier notation,

$$\begin{aligned}
[R(\mathbf{\Phi})]_{mn} &= \langle\mathbf{e}_m(\mathbf{\Phi})|\,\mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}_n(\mathbf{\Phi})\rangle\\
&= \langle\mathbf{e}_m(\mathbf{\Phi})\,|\mathbf{e}'_i\rangle\langle\mathbf{e}'_i|\,\mathcal{R}(\mathbf{\Phi})\,|\mathbf{e}'_j\rangle\langle\mathbf{e}'_j|\mathbf{e}_n(\mathbf{\Phi})\rangle = [R(\mathbf{\Phi})]'_{im}\,[R(\mathbf{\Phi})]'_{ij}[R(\mathbf{\Phi})]'_{jn}\\
&= [\,R^{\mathrm{T}}(\mathbf{\Phi})R(\mathbf{\Phi})R(\mathbf{\Phi})]'_{mn} = [\,R^{-1}(\mathbf{\Phi})R(\mathbf{\Phi})R(\mathbf{\Phi})]'_{mn} = [R(\mathbf{\Phi})]'_{mn} \ .
\end{aligned} \tag{G.7.8a}$$

Since this is true for any $\mathbf{\Phi}$, one also has

$$[R(\mathbf{\Phi}+d\mathbf{\Phi})]_{mn} = [R(\mathbf{\Phi}+d\mathbf{\Phi})]'_{mn} \ . \tag{G.7.8b}$$

DETERMINATION OF THE FRAME S' COMPONENTS OF $\boldsymbol{\omega}$

First expression for: $[(d\mathbf{e}_n/dt)_{S'}]'_i$

We now examine (from Frame S') a small change in the basis vector $\mathbf{e_n(\Phi)}$,

$$|(d\mathbf{e_n(\Phi)})_{S'}\rangle \equiv |\mathbf{e_n(\Phi+d\Phi)}\rangle - |\mathbf{e_n(\Phi)}\rangle$$

$$= |\mathbf{e'_i}\rangle\langle\mathbf{e'_i}|\mathbf{e_n(\Phi+d\Phi)}\rangle - |\mathbf{e'_i}\rangle\langle\mathbf{e'_i}|\mathbf{e_n(\Phi)}\rangle \quad \text{// completeness twice}$$

$$= |\mathbf{e'_i}\rangle\, [R(\mathbf{\Phi+d\Phi})]'_{\mathbf{in}} - |\mathbf{e'_i}\rangle [R(\mathbf{\Phi})]'_{\mathbf{in}} \quad \text{// (G.7.7b,a)}$$

$$= ([R(\mathbf{\Phi+d\Phi})]'_{\mathbf{in}} - [R(\mathbf{\Phi})]'_{\mathbf{in}})\ |\mathbf{e'_i}\rangle \quad \text{// reorder}$$

$$= ([R(\mathbf{\Phi+d\Phi})]_{\mathbf{in}} - [R(\mathbf{\Phi})]_{\mathbf{in}})\ |\mathbf{e'_i}\rangle \quad \text{// (G.7.8b,a) to remove primes}$$

$$= (R(\mathbf{\Phi+d\Phi}) - R(\mathbf{\Phi}))_{\mathbf{in}}\ |\mathbf{e'_i}\rangle$$

$$= (R^{\mathbf{T}}(\mathbf{\Phi+d\Phi}) - R^{\mathbf{T}}(\mathbf{\Phi}))_{\mathbf{ni}}\ |\mathbf{e'_i}\rangle \,. \qquad \text{(G.7.9)}$$

The next step is to replace $|\mathbf{e'_i}\rangle$ as follows

$$|\mathbf{e'_i}\rangle = |\mathbf{e_j(\Phi)}\rangle\langle\mathbf{e_j(\Phi)}|\,\mathbf{e'_i}\rangle = |\mathbf{e_j(\Phi)}\rangle [R(\mathbf{\Phi})]'_{\mathbf{ij}} = [R(\mathbf{\Phi})]_{\mathbf{ij}}\ |\mathbf{e_j(\Phi)}\rangle \,. \qquad \text{(G.7.10)}$$

Then

$$|(d\mathbf{e_n(\Phi)})_{S'}\rangle = (R^{\mathbf{T}}(\mathbf{\Phi+d\Phi}) - R^{\mathbf{T}}(\mathbf{\Phi}))_{\mathbf{ni}} [R(\mathbf{\Phi})]_{\mathbf{ij}}\, |\mathbf{e_j(\Phi)}\rangle$$

$$= (R^{\mathbf{T}}(\mathbf{\Phi+d\Phi})R(\mathbf{\Phi}) - R^{\mathbf{T}}(\mathbf{\Phi})R(\mathbf{\Phi}))_{\mathbf{nj}}\ |\mathbf{e_j(\Phi)}\rangle$$

$$= (R^{\mathbf{T}}(\mathbf{\Phi+d\Phi})R(\mathbf{\Phi}) - 1)_{\mathbf{nj}}\ |\mathbf{e_j(\Phi)}\rangle \,. \qquad \text{(G.7.11)}$$

We then add dt/dt to the left side to get

$$dt\ |(d\mathbf{e_n(\Phi)}/dt\ )_{S'}\rangle = (R^{\mathbf{T}}(\mathbf{\Phi+d\Phi})R(\mathbf{\Phi}) - 1)_{\mathbf{nj}}\ |\mathbf{e_j(\Phi)}\rangle \,. \qquad \text{(G.7.12)}$$

We wish to evaluate the above vector equation in Frame S' components. To do this, we close both sides with $\langle\mathbf{e'_i}|$, obtaining

$$dt\ \langle\mathbf{e'_i}\ |(d\mathbf{e_n(\Phi)}/dt\ )_{S'}\rangle = (R^{\mathbf{T}}(\mathbf{\Phi+d\Phi})R(\mathbf{\Phi}) - 1)_{\mathbf{nj}}\ \ \langle\mathbf{e'_i}\,|\mathbf{e_j(\Phi)}\rangle$$

or

$$dt\ [(d\mathbf{e_n}/dt)_{S'}]'_{\mathbf{i}} = (R^{\mathbf{T}}(\mathbf{\Phi+d\Phi})R(\mathbf{\Phi}) - 1)_{\mathbf{nj}}\ R(\mathbf{\Phi})_{\mathbf{ij}}$$

$$= (R^{\mathbf{T}}(\mathbf{\Phi+d\Phi})R(\mathbf{\Phi}) - 1)_{\mathbf{nj}} R^{\mathbf{T}}(\mathbf{\Phi})_{\mathbf{ji}} = [(R^{\mathbf{T}}(\mathbf{\Phi+d\Phi})R(\mathbf{\Phi}) - 1)R^{\mathbf{T}}(\mathbf{\Phi})]_{\mathbf{ni}}$$

$$= [\ R^{\mathbf{T}}(\mathbf{\Phi+d\Phi}) - R^{\mathbf{T}}(\mathbf{\Phi})\ ]_{\mathbf{ni}} = [\ R(\mathbf{\Phi+d\Phi}) - R(\mathbf{\Phi})\ ]_{\mathbf{in}}$$

$$\equiv (dR)_{\mathbf{in}} \quad \text{where} \quad dR \equiv R(\mathbf{\Phi+d\Phi}) - R(\mathbf{\Phi}) \,. \qquad \text{(G.7.13)}$$

Divide both sides by dt to obtain

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = (dR/dt)_{in} \qquad dR \equiv R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}) \quad . \qquad \text{(G.7.14)}$$

Second expression for: $[(d\mathbf{e}_n/dt)_{S'}]'_i$

Recall (G.7.1),

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \, . \qquad \textit{(1.7.4)} \qquad \textit{(G.7.1)}$$

We evaluate the above vector equation in Frame S' components,

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = - \varepsilon_{ikc}(\omega)'_k (\mathbf{e}_n)'_c = - \varepsilon_{ikc}(\omega)'_k [R(\mathbf{\Phi})]'_{cn} = -\varepsilon_{ikc}(\omega)'_k [R(\mathbf{\Phi})]_{cn} \, . \qquad \text{(G.7.15)}$$

Equate the two expressions for: $[(d\mathbf{e}_n/dt)_{S'}]'_i$

At this point we have shown that doing component evaluations of $(d\mathbf{e}_n/dt)_{S'}$ in Frame S' gives,

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = (dR/dt)_{in} \qquad dR \equiv R(\mathbf{\Phi}+d\mathbf{\Phi}) - R(\mathbf{\Phi}) \qquad \textit{(G.7.14)}$$

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = - \varepsilon_{ikc}(\omega)'_k [R(\mathbf{\Phi})]_{cn} \, . \qquad \textit{(G.7.15)}$$

Setting the right sides equal, one obtains,

$$- \varepsilon_{ikc}(\omega)'_k [R(\mathbf{\Phi})]_{cn} = (dR/dt)_{in} \, .$$

Multiply both sides on the right by $[R^{-1}(\mathbf{\Phi})]_{nj}$ to get

$$- \varepsilon_{ikc}(\omega)'_k [R(\mathbf{\Phi})]_{cn} [R^{-1}(\mathbf{\Phi})]_{nj} = (dR/dt)_{in} [R^{-1}(\mathbf{\Phi})]_{nj}$$

or

$$- \varepsilon_{ikc}(\omega)'_k \delta_{cj} = [\, (dR/dt)R^{-1}(\mathbf{\Phi}) \,]_{ij}$$

or

$$- \varepsilon_{ikj}(\omega)'_k = [\, (dR/dt)R^{-1}(\mathbf{\Phi}) \,]_{ij}$$

or

$$\varepsilon_{ijk}(\omega)'_k = A_{ij} \qquad \text{where} \qquad A \equiv (dR/dt)R^{-1}(\mathbf{\Phi}) \, . \qquad \text{(G.7.16)}$$

It seems that the object A_{ij} must be antisymmetric in its two indices, so the matrix A_{ij} has only three significant elements. Setting ijk = 231 gives the first line below, then the next two lines follow from cyclic permutation :

$$(\omega)'_1 = A_{23}$$
$$(\omega)'_2 = A_{31}$$
$$(\omega)'_3 = A_{12} \, . \qquad \text{(G.7.17)}$$

Thus we have succeeded in solving for the components of $\boldsymbol{\omega}$ in Frame S' . Recall that Frame S' is rotating at rate $\boldsymbol{\omega}$ relative to Frame S as in Fig 1. It remains to compute the three A_{ij} matrix elements so we can learn the specific expressions for the $(\omega)'_i$ in terms of Euler Angles.

Computation of A and Statement of Final Result

Our task is to compute $A \equiv (dR/dt)R^{-1}(\boldsymbol{\Phi})$ where our "temporary" $R(\boldsymbol{\Phi})$ is given by

$$R(\boldsymbol{\Phi}) = R_z(\psi)R_x(\theta)R_z(\varphi) \qquad (G.7.3)$$

and where

$$dR = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R(\boldsymbol{\Phi}) \ . \qquad (G.7.13)$$

The first step is to compute dR in terms of the Euler angles. We make use of the obvious fact that $R_i(\alpha +d\alpha) = R_i(\alpha)R_i(d\alpha)$ and then the fact (1.5.6) that $R_i(d\alpha) \approx 1 - id\alpha\, J_i$ for small $d\alpha$. Keeping only terms of first order in the differential angles, one finds

$$dR = R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R(\boldsymbol{\Phi})$$

$$= R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$$

$$= R_z(\psi)\, R_z(d\psi)\, R_x(\theta)\, R_x(d\theta)\, R_z(\varphi)\, R_z(d\varphi) - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$$

$$= R_z(\psi)\{1-id\psi J_3\}R_x(\theta)\{1-id\theta J_1\}R_z(\varphi)\{1-id\varphi J_3\} - R_z(\psi)\, R_x(\theta)\, R_z(\varphi)$$

$$= -id\psi\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - id\theta\, R_z(\psi)\, R_x(\theta)J_1R_z(\varphi) - id\varphi\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3 \qquad (G.7.18)$$

where the leading terms exactly cancel. Dividing by dt one then has

$$(dR/dt) = -i\dot{\psi}\, R_z(\psi)J_3R_x(\theta)R_z(\varphi) - i\dot{\theta}\, R_z(\psi)R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\, R_z(\psi)R_x(\theta)R_z(\varphi)J_3 \ . \qquad (G.7.19)$$

The next step is to compute $A \equiv (dR/dt)R^{-1}(\boldsymbol{\Phi})$ using $R^{-1}(\boldsymbol{\Phi}) = R_z(-\varphi)R_x(-\theta)R_z(-\psi)$. In doing so, we shall three times in blue use the fact that J_i commutes with R_i as formally stated in (G.3.6) :

$$A \equiv (dR/dt)\ R^{-1}(\boldsymbol{\Phi})$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3R_x(\theta)R_z(\varphi) - i\dot{\theta}\, R_z(\psi)R_x(\theta)J_1R_z(\varphi) - i\dot{\varphi}\, R_z(\psi)R_x(\theta)R_z(\varphi)J_3] \, R_z(-\varphi)R_x(-\theta)R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta) - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)R_z(\varphi)J_3R_z(-\varphi)\,]\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\, -i\dot{\psi}\ R_z(\psi)J_3\, R_x(\theta) - i\dot{\theta}\ R_z(\psi)\, R_x(\theta)J_1 - i\dot{\varphi}\, R_z(\psi)\, R_x(\theta)J_3\,]\, R_x(-\theta)\, R_z(-\psi)$$

$$= [\ -i\dot{\psi}\ R_z(\psi)J_3\ -\ i\dot{\theta}\ R_z(\psi)\ R_x(\theta)J_1R_x(-\theta)\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ ]\ R_z(-\psi)$$

$$= [\ -i\dot{\psi}\ R_z(\psi)J_3\ -\ i\dot{\theta}\ R_z(\psi)\ J_1\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ ]\ R_z(-\psi)$$

$$= [\ -i\dot{\psi}\ R_z(\psi)J_3\ R_z(-\psi)\ -\ i\dot{\theta}\ R_z(\psi)\ J_1\ R_z(-\psi)\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ R_z(-\psi)\ ]$$

$$= [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ R_z(\psi)\ J_1\ R_z(-\psi)\ -\ i\dot{\varphi}\ R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)\ R_z(-\psi)\ ]$$

$$= [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ M_1\ -\ i\dot{\varphi}\ M_2\ ]$$

where (G.7.20)

$$M_1 \equiv R_z(\psi)\ J_1R_z(-\psi) \qquad = [\ R_3(\psi)\ J_1R_3(-\psi)]$$

$$M_2 \equiv R_z(\psi)\ R_x(\theta)J_3R_x(-\theta)R_z(-\psi) \qquad = R_3(\psi)\ [\ R_1(\theta)J_3R_1(-\theta)\ ]\ R_3(-\psi)\ .$$

We now call upon our non-trivial sandwich formulas in (G.3.8) to simplify thing further :

$$M_1 = R_3(\psi)\ J_1R_3(-\psi) = \cos\psi J_1 + \sin\psi\ J_2 \qquad \text{// (G.3.8) line 5}$$

$$M_2 = R_3(\psi)\ [\ R_1(\theta)J_3R_1(-\theta)\ ]\ R_3(-\psi)$$

$$= R_3(\psi)\ [\ \cos\theta\ J_3 - \sin\theta\ J_2\ ]\ R_3(-\psi) \qquad \text{// (G.3.8) line 2}$$

$$= \cos\theta\ [\ R_3(\psi)\ J_3\ R_3(-\psi)]\ - \sin\theta\ [R_3(\psi)J_2R_3(-\psi)]$$

$$= \cos\theta\ J_3\ - \sin\theta\ (\cos\psi J_2 - \sin\psi\ J_1) \qquad \text{// (G.3.6) and (G.3.8) line 6}$$

$$= \sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\ J_3\ . \qquad \text{(G.7.21)}$$

Then,

$$A = [\ -i\dot{\psi}\ J_3\ -\ i\dot{\theta}\ M_1\ -\ i\dot{\varphi}\ M_2\ ]$$

$$= [\ -i\ \dot{\psi}\ J_3\ -\ i\ \dot{\theta}\ (\cos\psi J_1 + \sin\psi\ J_2)\ -\ i\ \dot{\varphi}\ (\sin\theta\sin\psi J_1 - \sin\theta\cos\psi J_2 + \cos\theta\ J_3)\ ]$$

$$= [\ -i\ \dot{\psi}\ J_3\ -\ i\ \dot{\theta}\ \cos\psi J_1\ -\ i\dot{\theta}\ \sin\psi\ J_2\ -\ i\ \dot{\varphi}\ \sin\theta\sin\psi J_1 + i\ \dot{\varphi}\sin\theta\cos\psi J_2 - i\ \dot{\varphi}\ \cos\theta\ J_3\ ]$$

$$= (-i)[\ \dot{\psi}\ J_3\ + \dot{\theta}\ \cos\psi J_1\ + \dot{\theta}\ \sin\psi\ J_2\ + \dot{\varphi}\ \sin\theta\sin\psi J_1 - \dot{\varphi}\sin\theta\cos\psi J_2\ + \dot{\varphi}\ \cos\theta\ J_3\ ]$$

$$= (-i)[\ (\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)J_1 + (\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)J_2 + (\dot\psi + \dot\varphi\cos\theta)J_3\ ]$$

$$= -[\ (\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)(iJ_1) + (\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)(iJ_2) + (\dot\psi + \dot\varphi\cos\theta)(iJ_3)\ ]. \qquad (G.7.22)$$

We now take the ij element of this matrix using the fact (G.1.3) that $(iJ_k)_{ij} = \varepsilon_{kij}$,

$$A_{ij} = -[\ (\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)\varepsilon_{1ij} + (\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)\varepsilon_{2ij} + (\dot\psi + \dot\varphi\cos\theta)\varepsilon_{3ij}\ ]\ . \qquad (G.7.23)$$

The tensor ε_{kij} is antisymmetric in i↔j and therefore the entire matrix A is antisymmetric (as conjectured earlier) and thus has only three distinct matrix elements. They are:

$$A_{23} = -(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)\varepsilon_{123} = -(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)$$

$$A_{31} = -(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)\varepsilon_{231} = -(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)$$

$$A_{12} = -(\dot\psi + \dot\varphi\cos\theta)\,\varepsilon_{312} = -(\dot\psi + \dot\varphi\cos\theta)\ . \qquad (G.7.24)$$

From (G.7.17) we then conclude that

$$(\omega)'_1 = A_{23} = -(\dot\theta\cos\psi + \dot\varphi\sin\theta\sin\psi)$$
$$(\omega)'_2 = A_{31} = -(\dot\theta\sin\psi - \dot\varphi\sin\theta\cos\psi)$$
$$(\omega)'_3 = A_{12} = -(\dot\psi + \dot\varphi\cos\theta)\ . \qquad \text{// angles still negated}$$

We now undo the temporary negation of the angles ψ,θ,φ enacted below (G.7.2). The velocities and sines then negate. Our final result for the Frame S' components of **ω** is then,

$$(\omega)'_1 = \dot\varphi\sin\theta\sin\psi + \dot\theta\cos\psi \equiv \omega_{x'}$$
$$(\omega)'_2 = \dot\varphi\sin\theta\cos\psi - \dot\theta\sin\psi \equiv \omega_{y'}$$
$$(\omega)'_3 = \dot\varphi\cos\theta + \dot\psi \equiv \omega_{z'}\ . \qquad (G.7.25)$$

This is in agreement with our Method 1 calculation (G.6.11) and with Goldstein's result which we again quote from Goldstein page 134 (GPS page 174),

$$\omega_{x'} = \dot\phi\sin\theta\sin\psi + \dot\theta\cos\psi$$
$$\omega_{y'} = \dot\phi\sin\theta\cos\psi - \dot\theta\sin\psi$$
$$\omega_{z'} = \dot\phi\cos\theta + \dot\psi. \qquad (4.87)$$

DETERMINATION OF THE FRAME S COMPONENTS OF **ω**

Here we shall review three different Plans for computing the Frame S components of $\boldsymbol{\omega}$.

Plan A: Since we just computed the $(\omega)'_i$ in Frame S', we just use $\boldsymbol{\omega} = R^{-1}(\boldsymbol{\omega})'$ to get the Frame S components. This was done at the end of Section G.6 with the result stated in (G.6.16).

Plan B: Start with (1.7.1) that $(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n$ in place of (G.7.1) that $(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$, which adds an overall minus sign to the result. The new (G.7.4) becomes $\mathbf{e}'_n(\boldsymbol{\Phi}) = R^{-1}(\boldsymbol{\Phi})\mathbf{e}_n$ with $\mathbf{e}_n$ = constant in Frame S. Things go through as presented above, but since $R(\boldsymbol{\Phi}) \rightarrow R^{-1}(\boldsymbol{\Phi})$ and since $R(\boldsymbol{\Phi}) = R_z(\psi)R_x(\theta)R_z(\varphi)$, one has $R^{-1}(\boldsymbol{\Phi}) = R_z(-\varphi)R_x(-\theta)\,R_z(-\psi)$. Thus, to convert the result (G.7.25) to the Frame S result, we have to make these changes: (1) $\varphi, \psi, \theta \rightarrow -\psi, -\varphi, -\theta$; (2) add the overall minus sign just noted. We do that right here:

$$\varphi, \psi, \theta \rightarrow -\psi, -\varphi, -\theta$$

$$\begin{aligned}
(\omega)'_1 &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi \equiv \omega_{x'} \\
(\omega)'_2 &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \equiv \omega_{y'} \\
(\omega)'_3 &= \dot{\varphi}\cos\theta + \dot{\psi} \equiv \omega_{z'} \ . \qquad \textit{(G.7.25)}
\end{aligned}$$

$\rightarrow$

$$\begin{aligned}
-(\omega)_1 &= [-\dot{\psi}][-\sin\theta][-\sin\varphi] + [-\dot{\theta}]\cos\varphi \quad \Rightarrow \omega_1 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi \\
-(\omega)_2 &= [-\dot{\psi}][-\sin\theta]\cos\varphi - [-\dot{\theta}][-\sin\varphi] \quad \Rightarrow \omega_2 = -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi \\
-(\omega)_3 &= [-\dot{\psi}]\cos\theta + [-\dot{\varphi}] \quad \Rightarrow \omega_3 = \dot{\psi}\cos\theta + \dot{\varphi} \qquad \text{(G.7.26)}
\end{aligned}$$

and this agrees with (G.6.16).

Plan C: Compute the ω_i directly using the machinery developed earlier in this section, but take Frame S components instead of Frame S' components. We regard this as a "stress test" of the machinery.

First expression for: $[(d\mathbf{e}_n/dt)_{S'}]_i$

Start with (G.7.11)

$$|(d\mathbf{e}_n(\boldsymbol{\Phi}))_{S'}\rangle = \left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)_{nj} |\mathbf{e}_j(\boldsymbol{\Phi})\rangle \ . \qquad \textit{(G.7.11)}$$

Instead of closing on the left with $\langle\mathbf{e}'_i|$ to get Frame S' components, this time close on the left with $\langle\mathbf{e}_i|$ to get Frames S components,

$$[(d\mathbf{e}_n)_{S'}]_i = \langle\mathbf{e}_i|(d\mathbf{e}_n(\boldsymbol{\Phi}))_{S'}\rangle = \left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)_{nj} \langle\mathbf{e}_i(\boldsymbol{\Phi})|\mathbf{e}_j(\boldsymbol{\Phi})\rangle$$

$$= \left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)_{nj} \delta_{ji} \qquad \text{// (1.1.14)}$$

$$= \left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)_{nj} (R^{-1}(\boldsymbol{\Phi})R(\boldsymbol{\Phi}))_{ji} = [\left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1 \right)R^{-1}(\boldsymbol{\Phi})R(\boldsymbol{\Phi})]_{ni}$$

$$= [\left(R^T(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R^T(\boldsymbol{\Phi}) \right) R(\boldsymbol{\Phi})]_{ni} = [R^T(\boldsymbol{\Phi})\left(R(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) - R(\boldsymbol{\Phi}) \right)]_{in}$$

$$= [R^{T}(\mathbf{\Phi})(dR)]_{in} = [R^{T}(\mathbf{\Phi})(dR)\, R^{-1}(\mathbf{\Phi})R(\mathbf{\Phi})]_{in}$$

and dividing by dt,

$$[(d\mathbf{e}_n/dt)_{S'}]_i = [R^{T}(\mathbf{\Phi})(dR/dt)R^{-1}(\mathbf{\Phi})\, R(\mathbf{\Phi})]_{in} = [R^{T}(\mathbf{\Phi})A\, R(\mathbf{\Phi})]_{in} \qquad \text{(G.7.27)}$$

where recall that $A \equiv (dR/dt)R^{-1}(\mathbf{\Phi})$ from (G.7.16).

Second expression for: $[(d\mathbf{e}_n/dt)_{S'}]_i$

Recall (G.7.1),

$$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega} \times \mathbf{e}_n \,. \qquad \textit{(1.7.4)} \qquad \textit{(G.7.1)}$$

We evaluate the above vector equation in Frame S components,

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -\,\varepsilon_{ikc}\omega_k(\mathbf{e}_n)_c = -\,\varepsilon_{ikc}\omega_k\delta_{nc} = -\,\varepsilon_{ikn}\omega_k \,. \qquad \text{(G.7.28)}$$

Equate the two expressions for: $[(d\mathbf{e}_n/dt)_{S'}]_i$

At this point we have shown that doing component evaluations of $(d\mathbf{e}_n/dt)_{S'}$ in Frame S gives,

$$[(d\mathbf{e}_n/dt)_{S'}]_i = [R^{T}(\mathbf{\Phi})A\, R(\mathbf{\Phi})]_{in} \,. \qquad \textit{(G.7.27)}$$

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -\,\varepsilon_{ikn}\omega_k \,. \qquad \textit{(G.7.28)}$$

Setting the right sides equal, we obtain

$$-\,\varepsilon_{ikn}\omega_k = [R^{T}(\mathbf{\Phi})A\, R(\mathbf{\Phi})]_{in} \equiv B_{in}$$

or

$$\varepsilon_{ink}\omega_k = B_{in}$$

or

$$\varepsilon_{ijk}\omega_k = B_{ij} \qquad B = R^{T}(\mathbf{\Phi})A\, R(\mathbf{\Phi}) \,, \qquad \text{(G.7.29)}$$

Thus we arrive at (G.7.16) with $(\omega)'_i$ replaced by ω_i and with A replaced by B,

$$\omega_1 = B_{23}$$
$$\omega_2 = B_{31}$$
$$\omega_3 = B_{12} \,. \qquad \text{(G.7.30)}$$

It remains only to compute the B_{ij}.

Computation of B and Statement of Final Result

Recall from (G.7.22) that

$$A = -[(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)] \quad . \qquad (G.7.22)$$

We may write this as

$$A = a_i J_i$$

where (G.7.31)

$$a_1 = -i(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)$$

$$a_2 = -i(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)$$

$$a_3 = -i(\dot{\psi} + \dot{\varphi}\cos\theta) \; .$$

Entering into Maple,

```
a1 := -I*(td*cos(psi)+phid*sin(theta)*sin(psi));
```

$$a1 := -I\,(td\cos(\psi) + phid\sin(\theta)\sin(\psi))$$

```
a2 := -I*(td*sin(psi)-phid*sin(theta)*cos(psi));
```

$$a2 := -I\,(td\sin(\psi) - phid\sin(\theta)\cos(\psi))$$

```
a3 := -I*(psid+phid*cos(theta));
```

$$a3 := -I\,(psid + phid\cos(\theta)) \quad . \qquad (G.7.32)$$

Then the matrix B in (G.7.29) can be written,

$$B = R^{-1}(\mathbf{\Phi})A\,R(\mathbf{\Phi}) \;=\; R^{-1}(\mathbf{\Phi})[a_i J_i]\,R(\mathbf{\Phi})$$

$$= a_i\,[R^{-1}(\mathbf{\Phi})J_i R(\mathbf{\Phi})[\;=\; a_i R(\mathbf{\Phi})_{ij} J_j \qquad \text{// Theorem 1 of (G.4.1) with } R \to R^{-1}$$

$$= a_i Q_i \qquad \text{where } Q_i \equiv R(\mathbf{\Phi})_{ij} J_j \; . \qquad (G.7.33)$$

Using our "temporary" R(**Φ**) in (G.7.3) we compute the vector Q as follows (a vector of matrices),

```
Q := evalm(Rz(psi) &* Rx(theta) &* Rz(phi) &* matrix(3,1,[J1,J2,J3]));
```

$$Q := \begin{bmatrix} (\cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi))\,J1 + (-\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi))\,J2 + \sin(\psi)\sin(\theta)\,J3 \\ (\sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi))\,J1 + (-\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi))\,J2 - \cos(\psi)\sin(\theta)\,J3 \\ \sin(\theta)\sin(\phi)\,J1 + \sin(\theta)\cos(\phi)\,J2 + \cos(\theta)\,J3 \end{bmatrix}$$

(G.7.34)

The quantity $B = a_i Q_i$ is then,

```
B := evalm(matrix(1,3,[a1,a2,a3]) &* Q):  B := simplify(B);
```

$$B := [-I\,psid\cos(\theta)\,J3 - I\,phid\,J3 - I\,td\,J1\cos(\phi) + I\,td\,J2\sin(\phi) - I\,psid\sin(\theta)\sin(\phi)\,J1 - I\,psid\sin(\theta)\cos(\phi)\,J2]$$

```
B := collect(B[1,1],[J1,J2,J3]);
```

$$B := (-I\,td\cos(\phi) - I\,psid\sin(\theta)\sin(\phi))\,J1 + (I\,td\sin(\phi) - I\,psid\sin(\theta)\cos(\phi))\,J2 + (-I\,psid\cos(\theta) - I\,phid)\,J3$$

(G.7.35)

We move the factors of i (Maple I) next to the J_k and transcribe the last line above:

$$B = (-\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi)(iJ_1) + (\dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi)(iJ_2) + (-\dot{\psi}\cos\theta - \dot{\varphi})(iJ_3) \quad \text{(G.7.36)}$$

Using (G.1.3) that $(iJ_k)_{ij} = \varepsilon_{kij}$ the matrix elements of B are then,

$$B_{ij} = (-\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi)\varepsilon_{1ij} + (\dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi)\varepsilon_{2ij} + (-\dot{\psi}\cos\theta - \dot{\varphi})\varepsilon_{3ij} \ . \quad \text{(G.7.37)}$$

Therefore,

$$B_{23} = -\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi$$
$$B_{31} = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$$
$$B_{12} = -\dot{\psi}\cos\theta - \dot{\varphi} \ . \quad \text{(G.7.38)}$$

Then from (G.7.30),

$$\omega_1 = B_{23} = -\dot{\theta}\cos\varphi - \dot{\psi}\sin\theta\sin\varphi$$
$$\omega_2 = B_{31} = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$$
$$\omega_3 = B_{12} = -\dot{\psi}\cos\theta - \dot{\varphi} \ . \quad \text{// angles still negated}$$

We now undo the temporary negation of the Euler angles ψ,θ,φ enacted below (G.7.2). The velocities and sines then negate. Our final result for the Frame S components of **ω** is then,

$$\omega_1 = B_{23} = \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi$$
$$\omega_2 = B_{31} = \dot{\theta}\sin\varphi - \dot{\psi}\sin\theta\cos\varphi$$
$$\omega_3 = B_{12} = \dot{\psi}\cos\theta + \dot{\varphi} \quad \text{(G.7.39)}$$

This is in agreement which (G.6.16) which we now quote,

$$(\omega)_x = \cos\varphi\,\dot{\theta} + \sin\theta\sin\varphi\,\dot{\psi}$$
$$(\omega)_y = \sin\varphi\,\dot{\theta} - \sin\theta\cos\varphi\,\dot{\psi}$$
$$(\omega)_z = \dot{\varphi} + \cos\theta\,\dot{\psi} \quad \textit{(G.6.16)}$$

Review of this Section

Frame S and Frame S' are related by some angular velocity **ω** as shown in Fig 1. The actual rotation relating the two frames is $R = R_z(-\psi)R_x(-\theta)R_z(-\varphi)$ as in (G.7.2) and (G.5.17a). If the three Euler angles are all static, **ω** would be 0. It is the fact that one or more of these Euler angles varies in time which causes **ω** to be non-zero.

Our task was to solve the following equation for $\boldsymbol{\omega}$

$$(d\mathbf{e_n})_{S'} = -\boldsymbol{\omega} \times \mathbf{e_n}\, dt \ . \qquad (1.7.4) \qquad (G.7.1)$$

Acting as an Observer in Frame S', we studied the change in the basis vectors $(d\mathbf{e_n})_{S'}$ caused by the time-varying Euler angles. In (G.7.11) we obtained the following vector equation for this change,

$$(d\mathbf{e_n})_{S'} = [\, R^{T}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1\,]_{nj}\, \mathbf{e_j} \ . \qquad (G.7.11)$$

Equating the right sides of the above two equations, we obtained the following vector equation,

$$-\boldsymbol{\omega} \times \mathbf{e_n}\, dt = (\, R^{T}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1\,)_{nj}\, \mathbf{e_j} \ . \qquad (G.7.40)$$

We found that, to lowest order, the quantity $(R^{T}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})R(\boldsymbol{\Phi}) - 1\,)_{nj}$ was linear in the differential angles $d\psi$,$d\theta$ and $d\varphi$ and when both sides were divided by dt, these became rates $\dot{\psi}$, $\dot{\theta}$ and $\dot{\varphi}$.

Although we were observing things from Frame S', we were allowed to take components of any vector equation in either Frame S or in Frame S'.

By taking the Frame S' components of (G.7.40) we obtained the Frame S' components of $\boldsymbol{\omega}$ as in (G.7.25).

By taking the Frame S components of (G.7.40) we obtained the Frame S components of $\boldsymbol{\omega}$ as in (G.7.39).

Along the way we got to exercise some earlier results of this Appendix: the sandwich formulas (G.3.6) and (G.3.8) in the computation of A, Theorem 1 of (G.4.1) saying $RJ_iR^{-1} = R^{-1}{}_{ij}J_j$ in the computation of B, and the rotation generator matrix representations of (G.1.3). Finally, we were able to exercise the Dirac notation of Section 1.1. The approach of this section made use of the "linear combination" side of the Basis Theorem (1.1.29) rather than the "operator" side, so matrices appeared right from the get-go.