This is the Title PhL 1.11.15

Note that page numbering is turned on in this template and view is 125%,
located in phil/roaming/microsoft/templates size about 219K.

$\omega_\varphi = \dot{\varphi}\,\hat{z}$ // Goldstein also says this

$\omega_\theta = \dot{\theta}\,\hat{\xi}'$

$\omega_\psi = \dot{\psi}\,\hat{z}'$

$\hat{\xi}' = \hat{\xi}$ = "line of nodes" // ξ (xi = "zeye")

$\hat{\zeta}' = \hat{z}'$ // ζ (zeta)

$\hat{z}' = \hat{\zeta}' = R_z(\varphi)\,R_x(\theta)\,\hat{z}$ // this seems right to me!!!

$\hat{x}' = R_x(\theta)\,R_z(\varphi+\psi)\,\hat{x}$

Comment: If I put in my usual rotation matrices and say $\hat{z} = (0,0,1)^T$, the resulting vector gives the components of $\hat{z}'$ in (x,y,z)-space coordinates. I am doing an active rotation. Let's do it right here:

```
evalm(Rz(phi) &* Rx(theta) &* matrix(3,1,[0,0,1]) );
```

$$\begin{bmatrix} \sin(\theta)\sin(\phi) \\ -\sin(\theta)\cos(\phi) \\ \cos(\theta) \end{bmatrix}$$

Now write the inverse equation

$\hat{z} = R_x(-\theta)\,R_z(-\varphi)\hat{z}'$

In x'-space I can say that $\hat{z}' = (0,0,1)^T$ and this equation then reads

```
evalm(Rx(-theta) &* Rz(-phi) &* matrix(3,1,[0,0,1]));
```

$$\begin{bmatrix} 0 \\ \sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

But this is the wrong answer! Why is it the wrong answer. I really need to start with

$\hat{z}' = \hat{\zeta}' = R_z(\varphi)\,R_x(\theta)\,R_z(\psi)\,\hat{z}$

Then the inverse would be

$$\hat{\mathbf{z}} = R_z(-\psi)R_x(-\theta)R_z(-\varphi)\hat{\mathbf{z}}'$$

and then I would get

```
evalm(Rz(-phi) &* Rx(-theta) &* Rz(-phi) &* matrix(3,1,[0,0,1]));
```

$$\begin{bmatrix} \sin(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) \\ \cos(\theta) \end{bmatrix}$$

I need to understand IN WORDS why these results are different.

Looking at the pictures, where does the product like $R_z(\varphi)\, R_x(\theta)\, R_z(\psi)$ "fit in" ?

Start with this conjecture:

$$\mathbf{e}_i = R\, \mathbf{e}'_i \qquad \text{using my standard notation for R}$$

What is R ? On example would be

$$\hat{\mathbf{z}} = R\, \hat{\mathbf{z}}' \quad \text{or equivalently} \quad \hat{\mathbf{z}}' = R^{-1}\hat{\mathbf{z}}$$

But from the pictures I know that

$$\hat{\mathbf{z}}' = R_z(\varphi)\, R_x(\theta)\, R_z(\psi)\, \hat{\mathbf{z}} \qquad \text{from doing active rotations.}$$

This applies to ALL the basis vectors, so write

$$\hat{\mathbf{x}}' = R_z(\varphi)\, R_x(\theta)\, R_z(\psi)\, \hat{\mathbf{x}} \qquad \text{from doing active rotations.}$$
$$\hat{\mathbf{y}}' = R_z(\varphi)\, R_x(\theta)\, R_z(\psi)\, \hat{\mathbf{y}} \qquad \text{from doing active rotations.}$$
$$\hat{\mathbf{z}}' = R_z(\varphi)\, R_x(\theta)\, R_z(\psi)\, \hat{\mathbf{z}} \qquad \text{from doing active rotations.}$$

Then right off the bat I know that

$$\boldsymbol{\omega}_\varphi = \dot{\varphi}\, \hat{\mathbf{z}} = \dot{\varphi}\, R_z(-\psi)\, R_x(-\theta)\, R_z(-\varphi)\, \hat{\mathbf{z}}'$$

Maple then says in x'-space

```
evalm(Rz(-psi) &* Rx(-theta) &* Rz(-phi) &* matrix(3,1,[0,0,1]));
```

$$\begin{bmatrix} \sin(\psi)\sin(\theta) \\ \cos(\psi)\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

and therefore,

$$\boldsymbol{\omega_{\varphi}} = \dot{\varphi}\,\sin\theta\sin\psi\;\hat{\mathbf{x}}' + \dot{\varphi}\,\sin\theta\cos\psi\;\hat{\mathbf{x}}' + \dot{\varphi}\cos\theta\;\hat{\mathbf{z}}'$$

This is the **correct answer** on page 134. I also know that

$$\boldsymbol{\omega_{\psi}} = \dot{\psi}\;\hat{\mathbf{z}}'$$

and that is **correct answer** also.

So I only one the middle term to figure out somehow.

$$\hat{\xi} = R_z(\varphi)\,\hat{\mathbf{x}}$$

$$\hat{\xi} = R_z(\varphi)\,\hat{\mathbf{x}} = R_z(\varphi)\,[R_z(-\psi)\,R_x(-\theta)\,R_z(-\varphi)]\;\hat{\mathbf{x}}'$$

Maple says

```
evalm(Rz(phi-psi) &* Rx(-theta) &* Rz(-phi) &* matrix(3,1,[1,0,0]));
```

$$\begin{bmatrix} \cos(-\phi+\psi)\cos(\phi) - \sin(-\phi+\psi)\cos(\theta)\sin(\phi) \\ -\sin(-\phi+\psi)\cos(\phi) - \cos(-\phi+\psi)\cos(\theta)\sin(\phi) \\ \sin(\theta)\sin(\phi) \end{bmatrix}$$

This then says

$\boldsymbol{\omega_{\theta}} = \dot{\theta}\,\hat{\xi}' = \dot{\theta}\,\hat{\xi} = \dot{\theta}$ times the above mess.

This is the WRONG answer for this term!

On the other hand, looking at the pictures I could also say

$$\hat{\mathbf{x}}' = R_{z'}(\psi)\,\hat{\xi}'$$

But I don't know the meaning of $R_{z'}(\psi)$. If I blindly use it I get

$$\hat{\xi}' = R_{z'}(-\psi)\,\hat{\mathbf{x}}'$$

```
evalm(Rz(-psi) &* matrix(3,1,[1,0,0]));
```

$$\begin{bmatrix} \cos(\psi) \\ -\sin(\psi) \\ 0 \end{bmatrix}$$

and then I would get

$\boldsymbol{\omega_\theta} = \dot{\theta}\,\hat{\xi}' = \dot{\theta}\,\hat{\xi}$ = $\dot{\theta}$ times the above

$= \dot{\theta}\cos\psi\,\hat{\mathbf{x}}' - \dot{\theta}\sin\psi\,\hat{\mathbf{y}}'$

and that is the correct answer!

Invert these to get

$\hat{\mathbf{z}} = R_{\mathbf{x}}(-\theta)\,R_{\mathbf{z}}(-\varphi)\hat{\mathbf{z}}'$ // this is just the inverse of the above!

$\hat{\mathbf{x}} = R_{\mathbf{z}}(-\varphi-\psi)R_{\mathbf{x}}(-\theta)\hat{\mathbf{x}}'$

Note that

$\hat{\xi} = R_{\mathbf{z}}(\varphi)\,\hat{\mathbf{x}}$

$\hat{\zeta}' = R_{\mathbf{z}}(\varphi)\,R_{\mathbf{x}}(\theta)\,\hat{\mathbf{z}} = \hat{\mathbf{z}}'$

Therefore

$\hat{\mathbf{z}} = R_{\mathbf{x}}(-\theta)\,R_{\mathbf{z}}(-\varphi)\hat{\mathbf{z}}'$

$\hat{\xi}' = \hat{\xi} = R_{\mathbf{z}}(\varphi)\,\hat{\mathbf{x}} = R_{\mathbf{z}}(\varphi)R_{\mathbf{z}}(-\varphi-\psi)R_{\mathbf{x}}(-\theta)\hat{\mathbf{x}}' = R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(-\theta)\,\hat{\mathbf{x}}'$

Then we find that

$\boldsymbol{\omega_\varphi} = \dot{\varphi}\,\hat{\mathbf{z}} = \dot{\varphi}\,R_{\mathbf{x}}(-\theta)R_{\mathbf{z}}(-\varphi)\hat{\mathbf{z}}'$ // just plugging in!

$\boldsymbol{\omega_\theta} = \dot{\theta}\,\hat{\xi}' = \dot{\theta}\,R_{\mathbf{z}}(-\psi)R_{\mathbf{x}}(-\theta)\,\hat{\mathbf{x}}'$

$\boldsymbol{\omega_\psi} = \dot{\psi}\,\hat{\mathbf{z}}'$

Now it is time for Maple.

```
evalm(Rx(-theta) &* Rz(-phi));
```

$$\begin{bmatrix} \cos(\phi) & \sin(\phi) & 0 \\ -\cos(\theta)\sin(\phi) & \cos(\theta)\cos(\phi) & \sin(\theta) \\ \sin(\theta)\sin(\phi) & -\sin(\theta)\cos(\phi) & \cos(\theta) \end{bmatrix}$$

```
zph := matrix(3,1,[0,0,1]);
```

$$zph := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

```
evalm(Rx(-theta) &* Rz(-phi) &* zph);
```

$$\begin{bmatrix} 0 \\ \sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

Therefore,

$$\begin{pmatrix} (\omega_\varphi)_{x'} \\ (\omega_\varphi)_{y'} \\ (\omega_\varphi)_{z'} \end{pmatrix} = \dot{\varphi}\, R_x(-\theta)R_z(-\varphi)\hat{z}' = \dot{\varphi}\begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix}$$

STOP. This is completely wrong because the correct $\dot{\varphi}$ terms have $\sin\theta\sin\psi$ type products!

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Next,

```
evalm(Rz(-psi) &* Rx(-theta));
```

$$\begin{bmatrix} \cos(\psi) & \sin(\psi)\cos(\theta) & \sin(\psi)\sin(\theta) \\ -\sin(\psi) & \cos(\psi)\cos(\theta) & \cos(\psi)\sin(\theta) \\ 0 & -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Therefore

$$\begin{pmatrix} (\omega_\theta)_{x'} \\ (\omega_\theta)_{y'} \\ (\omega_\theta)_{z'} \end{pmatrix} = \dot{\theta}\begin{pmatrix} \cos\psi \\ -\sin\psi \\ -0 \end{pmatrix}$$

and finally

$$\boldsymbol{\omega}_{\psi} = \dot{\psi}\,\hat{\mathbf{z}}' = \dot{\psi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Add to get

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\varphi} + \boldsymbol{\omega}_{\theta} + \boldsymbol{\omega}_{\psi} = \dot{\varphi}\begin{pmatrix} 0 \\ \sin\theta \\ \cos\theta \end{pmatrix} + \dot{\theta}\begin{pmatrix} \cos\psi \\ -\sin\psi \\ 0 \end{pmatrix} + \dot{\psi}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= [\,\dot{\theta}\cos\psi\,]\,\hat{\mathbf{x}}' + [\dot{\varphi}\sin\theta - \dot{\theta}\sin\psi]\,\hat{\mathbf{y}}' + [\dot{\varphi}\cos\theta + \dot{\psi}]\,\hat{\mathbf{z}}'$$

and therefore

$$\omega_{x'} = \dot{\theta}\cos\psi$$
$$\omega_{y'} = \dot{\varphi}\sin\theta - \dot{\theta}\sin\psi$$
$$\omega_{z'} = \dot{\varphi}\cos\theta + \dot{\psi}$$

My answer in Appendix G however is this:

$$(\omega)'_1 = -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi$$
$$(\omega)'_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$(\omega)'_3 = -\dot{\varphi}\cos\theta - \dot{\psi} \quad . \quad \text{// our result} \quad (G.4.22)$$

So everything is wrong, but that was my first attempt! Now go back and fix the errors!

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Go back to

$$\hat{\mathbf{x}}' = R_z(\varphi)\,R_x(\theta)\,R_z(\psi)\,\hat{\mathbf{x}} \quad \text{from doing active rotations.}$$
$$\hat{\mathbf{y}}' = R_z(\varphi)\,R_x(\theta)\,R_z(\psi)\,\hat{\mathbf{y}} \quad \text{from doing active rotations.}$$
$$\hat{\mathbf{z}}' = R_z(\varphi)\,R_x(\theta)\,R_z(\psi)\,\hat{\mathbf{z}} \quad \text{from doing active rotations.}$$

I am quite confident despite all the problems that these three equations are correct.

Can I take these "part-way" somehow ?

$$= R_x(\theta)\,R_z(\psi)\,\hat{\mathbf{x}}$$