**junk1** PhL 4.3.17

This was tacked onto the end of Section H.6, it seems now like total junk and wrong to boot, so removing it today

Example 1:

For a kinematic vector, we can think of transformations in two ways (ignoring the prime name overload problem). The first line below is (H.3.9) which is the Passive View transformation giving the coordinates of vector **V** in Frame S', while the second line is the Active View transformation giving a new vector **V'** in Frame S. Rotation $R^{-1} = R_z(\varphi)R_x(\theta)R_z(\psi)$ from (H.1.2), so

$$(\mathbf{V})' = R\,\mathbf{V} \qquad \text{// passive}$$
$$\mathbf{V}' = R^{-1}\mathbf{V}\,. \qquad \text{// active} \tag{H.6.14}$$

In particular, we can apply these rules to a position vector **r**,

$$(\mathbf{r})' = R\,\mathbf{r} \qquad \text{// passive}$$
$$\mathbf{r}' = R^{-1}\,\mathbf{r}\,. \qquad \text{// active} \tag{H.6.15}$$

Specifically, we have Maple generate the last equation $\mathbf{r}' = R^{-1}\mathbf{r} = R_z(\varphi)R_x(\theta)R_z(\psi)\,\mathbf{r}$ :

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RINV := evalm(Rz(phi) &* Rx(theta) &* Rz(psi));
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$$RINV := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) \\ \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\sin(\theta)\cos(\phi) \\ \sin(\psi)\sin(\theta) & \cos(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

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r := matrix(3,1,[x,y,z]);
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$$r := \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

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rp := evalm(RINV &* r): rp := simplify(rp);
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$$rp := \begin{bmatrix} x\cos(\psi)\cos(\phi) - x\sin(\psi)\cos(\theta)\sin(\phi) - y\sin(\psi)\cos(\phi) - y\cos(\psi)\cos(\theta)\sin(\phi) + \sin(\theta)\sin(\phi)\,z \\ x\cos(\psi)\sin(\phi) + x\sin(\psi)\cos(\theta)\cos(\phi) - y\sin(\psi)\sin(\phi) + y\cos(\psi)\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\phi)\,z \\ \sin(\psi)\sin(\theta)\,x + \cos(\psi)\sin(\theta)\,y + \cos(\theta)\,z \end{bmatrix}$$

Transcribing the result, one obtains

$$x' = (\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi)x + (-\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi)y + (\sin\theta\sin\varphi)\,z$$

$$y' = (\cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi)x + (-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi)y + (-\sin\theta\cos\varphi)\,z$$

$$z' = (\sin\psi\,\sin\theta)x + (\cos\psi\sin\theta)y + (\cos\theta)z\,. \tag{H.6.16}$$

Applying this to $\hat{\mathbf{z}} = (0,0,1)$ gives

$x' = \sin\theta\sin\varphi$
$y' = -\sin\theta\cos\varphi$
$z' = \cos\theta$ . (H.6.15)

Then applying the rules (H.6.13) one finally gets, in terms of spherical coordinate angles θ and φ,

$x' = \sin\theta\cos\varphi$
$y' = \sin\theta\sin\varphi$
$z' = \cos\theta$ (H.6.17)

This one recognizes as the $\hat{\mathbf{r}}$ vector (E.2.6) of spherical coordinates. These last equations are the same as

$$\hat{\mathbf{r}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{z}} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \quad . \qquad \textit{(E.2.2)}$$

One might ask why we are making a big deal about the minor issue of $\varphi = \varphi + \pi/2$ ? The reason is that when one does calculations for some problems, this can make a big difference.

Example 2:

Recall the equations describing the **ω** vector in Frame S coordinates:

$\omega_1 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi$
$\omega_2 = -\dot{\psi}\sin\theta\cos\varphi + \dot{\theta}\sin\varphi$
$\omega_3 = \dot{\psi}\cos\theta + \dot{\varphi}$ . // Frame S *(H.4.4)*

If one is using spherical coordinates r,θ,φ to describe a rigid body, these equations must be rewritten as

$\omega_1 = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi$
$\omega_2 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi$
$\omega_3 = \dot{\psi}\cos\theta + \dot{\varphi}$ . // Frame S (H.6.18)

For example, if some object with ψ = constant is whirling around in the xz plane of Fig (H.6.2) (so φ = 0), this last result says

$\boldsymbol{\omega} = (0,\dot{\theta},0) \quad \Rightarrow \quad \omega_2 = \dot{\theta}$

which is the correct result. If one inadvertently set $\varphi = \varphi$ in (H.4.4), one would instead get $\boldsymbol{\omega} = (\dot{\theta},0,0)$ which is not the correct result.

Example 3:

Recall the equations describing the **ω** vector in Frame S' coordinates (body frame):

$$(\omega)'_1 = \dot{\varphi}\,\sin\theta\sin\psi + \dot{\theta}\,\cos\psi$$

$$(\omega)'_2 = \dot{\varphi}\,\sin\theta\cos\psi - \dot{\theta}\,\sin\psi$$

$$(\omega)'_3 = \dot{\varphi}\,\cos\theta + \dot{\psi} \qquad \text{// Frame S'} \qquad \textit{(H.4.7)}$$

For these equations, the fact that $\varphi = \varphi+\pi/2$ merely says $\dot{\varphi} = \dot{\varphi}$ so the $\pi/2$ adder makes no difference and one then has,

$$(\omega)'_1 = \dot{\varphi}\,\sin\theta\sin\psi + \dot{\theta}\,\cos\psi$$

$$(\omega)'_2 = \dot{\varphi}\,\sin\theta\cos\psi - \dot{\theta}\,\sin\psi$$

$$(\omega)'_3 = \dot{\varphi}\,\cos\theta + \dot{\psi} \qquad \text{// Frame S'} \qquad \text{(H.6.19)}$$