Constants of motion for the symmetric top PhL 3.25.17

Here I flail around trying to understand the ideas that L_z and L'_z are constants of the motion for the top. I know how to see this with the Lagrangian, but how to show without? I resolve that in some document other than this one, and the derivations are now shown in frames doc. So I don't think what is below has much value.

In my top thing I say that because torque has no components in the z or z' directions, it follows that both L_z and $L_{z'}$ are constants of the motion. How do I justify that claim? For L_z we have $N_z = 0$ at all times, and we are in an inertial frame, so seems it is OK there. But all this tells me is that

$$(L)_z = [\sin^2\theta I'_1 + \cos^2\theta I'_3]\,\dot{\varphi} + \cos\theta\, I'_3\dot{\psi} = bI'_1$$

so this is what says that "b" is a constant of the motion.

But what about $L_{z'}$? Now z' is a moving axis. And what exactly to I mean by $L_{z'}$?

If **L** is the Frame S **L**, then I can write

$$\mathbf{L} = (L)'_x\,\hat{\mathbf{x}}' + (L)'_y\,\hat{\mathbf{y}}' + (L)'_z\,\hat{\mathbf{z}}'$$

What is $(d\mathbf{L}/dt)_S$? I would have to first write out

$$\begin{aligned}
\hat{\mathbf{x}}' &= (\cos\psi\cos\varphi - \sin\psi\sin\varphi\cos\theta)\,\hat{\mathbf{x}} + (\cos\psi\sin\varphi + \sin\psi\cos\varphi\cos\theta)\hat{\mathbf{y}} + (\sin\theta\sin\psi)\,\hat{\mathbf{z}} \\
\hat{\mathbf{y}}' &= (-\sin\psi\cos\varphi - \cos\psi\sin\varphi\cos\theta)\,\hat{\mathbf{x}} + (-\sin\psi\sin\varphi + \cos\psi\cos\varphi\cos\theta)\hat{\mathbf{y}} + (\sin\theta\cos\psi)\,\hat{\mathbf{z}} \\
\hat{\mathbf{z}}' &= (\sin\varphi\sin\theta)\hat{\mathbf{x}} + (-\cos\varphi\sin\theta)\hat{\mathbf{y}} + (\cos\theta)\,\hat{\mathbf{z}}\ .
\end{aligned} \qquad \text{(H.3.19)}$$

Then I have to compute the dots of all these objects which is a mess involving $\dot{\theta}$ and $\dot{\varphi}$.

So instead, what is $(d\mathbf{L}/dt)_{S'}$. I guess that would be

$$(d\mathbf{L}/dt)_{S'} = (\dot{L})'_x\,\hat{\mathbf{x}}' + (\dot{L})'_y\,\hat{\mathbf{y}}' + (\dot{L})'_z\,\hat{\mathbf{z}}'$$

From this I could indeed compute using the G Rule,

$$(d\mathbf{L}/dt)_S = (d\mathbf{L}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{L}$$

OK, that is fine, but what is $\boldsymbol{\omega} \times \mathbf{L}$? Well I do know this much

$$\mathbf{L} = I\boldsymbol{\omega} \quad \Rightarrow \quad (L)'_i = (I')_{ij}(\omega')_j = (I')_i(\omega')_i \qquad \text{no implied sum}$$

But I don't know that $(\omega')_3$ = constant without using the Lagrangian and saying p_ψ = constant which begs the question.

So at this point, the ONLY way I can say $(\dot{\phi}\cos\theta + \dot{\psi})I'_3 \equiv aI'_1$ is to show the Lagrangian is cyclic in ψ. There must be some direct way to show this.

What about

$$\boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\omega} \times (I\boldsymbol{\omega})$$

$$[\boldsymbol{\omega} \times \mathbf{L}]_i = \varepsilon_{ijk}\,(\omega)'_j\,[I\boldsymbol{\omega}]'_k = \varepsilon_{ijk}\,(\omega)'_j\,(I)'_{kn}\,(\omega)'_n = \varepsilon_{ijk}\,(\omega)'_j\,(I)'_k\delta_{nk}\,(\omega)'_n$$

$$= \varepsilon_{ijk}\,(\omega)'_j\,(I)'_k\,(\omega)'_k = \varepsilon_{ijk}\,(\omega)'_j\,(\omega)'_k(I)'_k$$

Not sure this helps, put this on hold and go back to the known equations of motion!

$$(N)'_1 = I'_1\,(\dot{\omega})'_1 - (\omega)'_2(\omega)'_3\,(I'_2 - I'_3)$$
$$(N)'_2 = I'_2\,(\dot{\omega})'_2 - (\omega)'_3(\omega)'_1\,(I'_3 - I'_1)$$
$$(N)'_3 = I'_3\,(\dot{\omega})'_3 - (\omega)'_1(\omega)'_2\,(I'_1 - I'_2)\ . \qquad \text{(I.4.3)}$$

For the symmetric top this becomes

$$(N)'_1 = I'_1\,(\dot{\omega})'_1 - (\omega)'_2(\omega)'_3\,(I'_2 - I'_3)$$
$$(N)'_2 = I'_2\,(\dot{\omega})'_2 - (\omega)'_3(\omega)'_1\,(I'_3 - I'_1)$$
$$(N)'_3 = I'_3\,(\dot{\omega})'_3\ . \qquad \text{(I.4.3)}$$

Now what do I know about the torque components of the Frame S torque N?

$\mathbf{N} = r_{cms} \times (-m\mathbf{g})$ = something entirely in the ξ' direction.

I know that

$$\hat{\xi}' = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}\ . \qquad \text{// this particular fact is obvious from Fig (H.1.5) top} \qquad \text{(H.4.2)}$$

But I also know that $\hat{\xi}' = \hat{\xi}$ and that

$$\hat{\xi}' = \hat{\xi} = \cos\psi\,\hat{\mathbf{x}}' - \sin\psi\,\hat{\mathbf{y}}'\ . \qquad \text{(H.4.5)}$$

So I then have,

$$\mathbf{N} = (\text{stuff})\,[\,\cos\psi\,\hat{\mathbf{x}}' - \sin\psi\,\hat{\mathbf{y}}'\,]$$

This then says clearly that

$(N)'_3 = 0$

Now I don't want at this point to try and say that this makes $(L)'_3$ be constant, because Newton is not valid in Frame S' ! But I can claim from the third EOM that $(\omega)'_3$ = constant. What do I do next? I can certainly show that $(L)'_z = I'_3 (\omega')_3$ and THEN since $(\omega')_3$ = constant, I know that $(L)'_z$ = constant.

Somehow it must also be true that in Frame S' there are no fictitious torques in the z' direction. For Special Case #4 I do know that

$$\mathbf{N'}_{fict} = \mathbf{r'} \times \mathbf{F'}_{fict} = - \mathbf{r'} \times [\, m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2m\, \boldsymbol{\omega} \times \mathbf{v'} + m\dot{\boldsymbol{\omega}} \times \mathbf{r'}]$$

But this is a mess since have to sum over points in the body and do all the cross products! So that seems a bad pathway. I suppose at leave $\mathbf{v}' = 0$.

So I need to change my wording, here is an attempt.

OK I have figured out a simple way to explain that $(L)'_3$ = constant, and it does not involve using Newton's Law. It involves the fact that the inertia tensor is diagonal and axisymmetric! Case closed, but it was good to be careful about sloppy assumptions.