Dumbbell Satellite as Rigid Body Problem v1 PhL 3.26.17

do not read below, see summary doc

In Appendix F I use swap notation, so Frame S is the body frame.

It seems clear that θ,φ are just Euler angles.

Before starting, note that for such a satellite

$$GM_E/r_0^2 = \omega^2 r_0$$

$$\omega^2 = GM_E/r_0^3 \approx GM_E/b^3$$

in the far approximation, so ω is determined by outside factors. It is not something you solve for. It is the ω which relates the rotation of the two Frames of reference.

What are the moments of a dumbbell?

In this rigid body analysis I will assume that the two masses are spheres of radius R_1 and R_2. In this case, there is inertial about the long axis. It will be

$$I_3 = (2/5)m_1R_1^2 + (2/5)m_2R_2^2$$

Since the spheres could have any radii we want, I will just maintain I_3 as a constant in the problem.

The other moments in the body frame are a bit complicated since I would have to integrate over each sphere or I could maybe use the axis-shift theorem which I really might add somewhere *****. So let's just say there is some I_1 which I could compute, and $I_2 = I_1$ because the rigid object is axisymmetric. For a point-mass dumbbell we would know that

$$I_1 = m_1r_1^2 + m_2r_2^2 \qquad \text{with} \qquad m_1r_1 = m_2r_2$$

so

$$I_1 = (1/m_2)\,[\, m_1m_2r_1^2 + m^2{}_2r_2^2 \,] = (1/m_2)\,[\, m_1m_2r_1^2 + m^2{}_1r_1^2 \,]$$

$$= (m_1/m_2)r_1^2\,[\, m_2 + m_1 \,] = (m_1/m_2)r_1^2M = (m_1/\mu_2)r_1^2$$

I am going to say the R_1 and R_2 are very small and then this expression for I_1 is valid.

Side question: Is the torque on a sphere the same as on a point mass at its center?

For me this is a very hard question. I think the answer is NO. Therefore, if I am going to use the results of Appendix F for torque, I will have to assume that the $R_1,R_2 << r_1,r_2$. Consider that done.

What are the equations of motion

I start with the equations of motion in the body frame, changed to swap notation:

$$N_1 = I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3)$$
$$N_2 = I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1)$$
$$N_3 = I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) \quad . \qquad \text{// Frame S = body frame} \qquad \text{(I.4.3)}$$

and with comments above this becomes

$$N_1 = I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_1 - I_3)$$
$$N_2 = I_1 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1)$$
$$N_3 = I_3 \dot{\omega}_3 \quad . \qquad \text{// Frame S = body frame} \qquad \text{(I.4.3)}$$

Getting pretty simple it would seem. Remember that the vector equation was

$$\mathbf{N} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) \qquad \text{(I.2.6)}$$

and maybe that will be helpful, but ignore for now.

I show in (F.3.10) the torque on the satellite (with point masses) to be

$$\mathbf{N'^{(b)}} = (GM_E m_1 b r_1 \sin\theta) [1/r'_1{}^3 - 1/r'_2{}^3] \hat{\boldsymbol{\varphi}}$$

and this is about "b" which is the center of Frame S. In the far approximation I write this as

$$\mathbf{N'^{(b)}} = -3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \hat{\boldsymbol{\varphi}} \ .$$

What are the Cartesian Frame S components of this torque? Use

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\ \hat{\mathbf{x}} + \cos\varphi\ \hat{\mathbf{y}} \qquad \text{(E.2.6)}$$

So then

$$\mathbf{N'^{(b)}} = -3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ [\ -\sin\varphi\ \hat{\mathbf{x}} + \cos\varphi\ \hat{\mathbf{y}}\ ]$$

all in body Frame S. So I then know that true torque is same in both frames, so

$$N_1 = 3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \sin\varphi$$
$$N_2 = -3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \cos\varphi$$

So now I have these three equations of motion in the far approximation

$$3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta \sin\varphi = I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_1 - I_3)$$
$$-3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta \cos\varphi = I_1 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1)$$
$$0 = I_3 \dot{\omega}_3$$

So at once we know that ω_3 = constant. Now as usual we are worrying about the $\boldsymbol{\omega}$ vector and we have to think about what vector that really is. In body frame $\mathbf{L} = I\boldsymbol{\omega}$, so

$$L_1 = I_1\omega_1$$
$$L_2 = I_1\omega_2$$
$$L_3 = I_3\omega_3$$

Lets set $\omega_3 = K$ for now.

I am now going to assume that $I_3 << I_1$ to make these equations more tractable, so rewrite as

$$3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta \sin\varphi = I_1 \dot{\omega}_1 - \omega_2 \omega_3 I_1$$
$$-3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta \cos\varphi = I_1 \dot{\omega}_2 + \omega_3 \omega_1 I_1$$

and divide by I_1 so get

$$3GM_E(m_1/\mu_2)\, I_1^{-1}\, b^{-3} r_1^2 \sin\theta\cos\theta \sin\varphi = \dot{\omega}_1 - K\omega_2$$
$$-3GM_E(m_1/\mu_2)\, I_1^{-1} b^{-3} r_1^2 \sin\theta\cos\theta \cos\varphi = \dot{\omega}_2 + K\omega_1$$

Now define

$$C \equiv 3GM_E(m_1/\mu_2)\, I_1^{-1}\, b^{-3} r_1^2$$

What do I know about C ? I claim above that

$$C = 3GM_E(m_1/\mu_2)\, I_1^{-1}\, b^{-3} r_1^2$$

$$I_1 = (m_1/\mu_2) r_1^2 \qquad \Rightarrow \qquad I_1^{-1} = (\mu_2/m_1) r_1^{-2}$$

and then

$$C = 3GM_E(m_1/\mu_2)(\mu_2/m_1) r_1^{-2} b^{-3} r_1^2$$
$$= 3GM_E b^{-3}$$

which is at least simpler than it was. Moreover, recall from above that

$$\omega^2 = GM_E/b^3$$

so then we can write

$$C = 3\omega^2$$

Then equations of motion are

$$3\omega^2 \sin\theta\cos\theta \sin\varphi = \dot{\omega}_1 - K\omega_2$$
$$-3\omega^2 \sin\theta\cos\theta \cos\varphi = \dot{\omega}_2 + K\omega_1$$

Recall that I have assumed that $0 < I_3 << I_1$ and $R_1, R_2 << r_1, r_2$ for the two "balls" .

Offhand, does not look too bad. Now would be a good time to roll out ω expressed in body frame Euler angles (I have many weapons now! Removing primes since in swap notation now

$$\omega_1 = \dot{\varphi} \sin\theta\sin\psi + \dot{\theta} \cos\psi$$
$$\omega_2 = \dot{\varphi} \sin\theta\cos\psi - \dot{\theta} \sin\psi$$
$$\omega_3 = \dot{\varphi} \cos\theta + \dot{\psi} \qquad \text{(H.4.7)}$$

So then

$$\dot{\omega}_1 = \dot{\varphi} \sin\theta\cos\psi\dot{\psi} + \dot{\varphi}\cos\theta\dot{\theta} \sin\psi + \ddot{\varphi} \sin\theta\sin\psi - \dot{\theta} \sin\psi\dot{\psi} + \ddot{\theta} \cos\psi$$
$$\dot{\omega}_2 = -\dot{\varphi} \sin\theta\sin\psi\dot{\psi} + \dot{\varphi}\cos\theta\dot{\theta} \cos\psi - \ddot{\varphi} \sin\theta\cos\psi - \dot{\theta}\cos\psi\dot{\psi} - \ddot{\theta} \sin\psi$$

Very interesting so far. Now install these into the equations along with the above

$$3\omega^2 \sin\theta\cos\theta \sin\varphi = [\dot{\varphi} \sin\theta\cos\psi\dot{\psi} + \dot{\varphi}\cos\theta\dot{\theta} \sin\psi + \ddot{\varphi} \sin\theta\sin\psi - \dot{\theta} \sin\psi\dot{\psi} + \ddot{\theta} \cos\psi]$$
$$- K[\dot{\varphi} \sin\theta\cos\psi - \dot{\theta} \sin\psi]$$

$$-3\omega^2 \sin\theta\cos\theta \cos\varphi = [-\dot{\varphi} \sin\theta\sin\psi\dot{\psi} + \dot{\varphi}\cos\theta\dot{\theta} \cos\psi - \ddot{\varphi} \sin\theta\cos\psi - \dot{\theta}\cos\psi\dot{\psi} - \ddot{\theta} \sin\psi]$$
$$+ K [\dot{\varphi} \sin\theta\sin\psi + \dot{\theta} \cos\psi]$$

Earlier I wanted to just set $\psi \equiv 0$, but I have $I_3 > 0$ and there will be some $\psi(t)$ and I do NOT want to try and throw that out. So we are stuck then with the above fairly messy equations for now.

Recall now where I want to end up somehow :

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi (\omega \cos\theta \cos\varphi - 2\sin\theta \dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$
$$\ddot{\varphi} + 2\dot{\theta} \dot{\varphi} \cot\theta - \omega\cos\varphi (\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}} \qquad \text{(F.5.7)}$$

My rigid body equations of THIS doc do not include the symbol ω, so that is one big difference.

Is ω a constant of the motion? Well it is determined by the satellite circular orbit, so it is a constant! So there is another equation I have so far ignored. Write again

$$\omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} \qquad \text{(H.4.7)}$$

Then

$$\omega_1^2 = \dot{\varphi}^2\sin^2\theta\sin^2\psi + \dot{\theta}^2\cos^2\psi + 2\dot{\varphi}\sin\theta\sin\psi\dot{\theta}\cos\psi$$

$$\omega_2^2 = \dot{\varphi}^2\sin^2\theta\cos^2\psi + \dot{\theta}^2\sin^2\psi - 2\dot{\varphi}\sin\theta\cos\psi\dot{\theta}\sin\psi$$

$$\omega_3^2 = \dot{\varphi}^2\cos^2\theta + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$

Add the first two to get

$$\omega_1^2 + \omega_2^2 = \dot{\varphi}^2\sin^2\theta + \dot{\theta}^2$$

Then add the third to get

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = \dot{\varphi}^2\sin^2\theta + \dot{\theta}^2 + \dot{\varphi}^2\cos^2\theta + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$

$$= \dot{\varphi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$

I also have another equation,

$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} = K$$

Question: How do you interpret the fact that $\omega_1^2 + \omega_2^2$ has no dependence on ψ ? (defer)

Now if you are enforcing ω as a constant from Earth rotation, we then have lots of equations that I have developed in this appendix

$$K = \dot{\varphi}\cos\theta + \dot{\psi}$$
$$\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$
$$3\omega^2\sin\theta\cos\theta\sin\varphi = \ddot{\theta} - K\dot{\varphi}\sin\theta \qquad (1)$$

$$-3\omega^2\sin\theta\cos\theta\cos\varphi = \dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta + K\dot{\theta} \qquad (2)$$

Insert the K results to have then only three equations to deal with

$$\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$

$$3\omega^2\sin\theta\cos\theta\sin\varphi = \ddot{\theta} - (\dot{\varphi}\cos\theta + \dot{\psi})\dot{\varphi}\sin\theta$$

$$-3\omega^2\sin\theta\cos\theta\cos\varphi = \dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta + (\dot{\varphi}\cos\theta + \dot{\psi})\dot{\theta}$$

or

$$\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$

$$3\omega^2\sin\theta\cos\theta\sin\varphi = \ddot{\theta} - \dot{\varphi}\cos\theta\,\dot{\varphi}\sin\theta - \dot{\psi}\,\dot{\varphi}\sin\theta$$

$$-3\omega^2\sin\theta\cos\theta\cos\varphi = \dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta + \dot{\varphi}\,\dot{\theta}\cos\theta + \dot{\psi}\dot{\theta}$$

Somehow you must be able to show that these three equations are the same as

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$

$$\ddot{\varphi}\sin\theta + 2\dot{\theta}\,\dot{\varphi}\cos\theta - \omega\cos\varphi\sin\theta(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}} \qquad (F.5.7)$$

One problem is that now ψ has entered the problem and it was not present before.

Can I somehow eliminate $\dot{\psi}$ from the set of three equations ?

Plan A . Take a look at the first of the three

$$\dot{\psi}^2 + (2\dot{\varphi}\cos\theta)\,\dot{\psi} + (\omega^2 - \dot{\varphi}^2 - \dot{\theta}^2) = 0$$

Think of this as

$$x^2 + bx + c = 0$$

Then

$$2x = -b \pm \sqrt{b^2 - 4ac}$$

Now

$$b^2 - 4ac = (2\dot{\varphi}\cos\theta)^2 - 4\,(\omega^2 - \dot{\varphi}^2 - \dot{\theta}^2)$$

$$= 4[\,\dot{\varphi}^2\cos^2\theta - \omega^2 + \dot{\varphi}^2 + \dot{\theta}^2]$$

But recall from above that

$$\omega^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\cos\theta\dot{\psi}$$

but this does not make the square root go away. So my quadratic solution does not pay off.

Plan B . Treat K as the basic constant. Have

$$K = \dot{\phi}\cos\theta + \dot{\psi}$$
$$K^2 = \dot{\phi}^2\cos^2\theta + \dot{\psi}^2 + 2\dot{\phi}\cos\theta\dot{\psi}$$
$$\omega^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\cos\theta\dot{\psi}$$

$$\omega^2 - K^2 = \dot{\theta}^2 + \dot{\phi}^2\sin^2\theta = \omega^2 - \omega_3^2 = \omega_\perp^2 = \text{constant}$$ // agrees with earlier $\omega_\perp^2$

So now differentiate the above to get

$$2\dot{\theta}\ddot{\theta} + 2\dot{\phi}\ddot{\phi}\sin^2\theta + \dot{\phi}^2 2\sin\theta\cos\theta\dot{\theta} = 0$$

or

$$\dot{\theta}\ddot{\theta} + \dot{\phi}\ddot{\phi}\sin^2\theta + \dot{\phi}^2\dot{\theta}\sin\theta\cos\theta = 0.$$

I don't think this fact ever appeared in Appendix F. Meanwhile we have another fact

$$\omega^2 = \dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\cos\theta\dot{\psi}$$

$$0 = \dot{\phi}\ddot{\phi} + \dot{\theta}\ddot{\theta} + \dot{\psi}\ddot{\psi} + \ddot{\phi}\cos\theta\dot{\psi} - \dot{\phi}\sin\theta\dot{\theta}\dot{\psi} + \dot{\phi}\cos\theta\ddot{\psi} = 0$$

This seems a useless entanglement of ψ into things when I am trying to remove ψ.

So I now have several equations of possible interest

$$\dot{\phi}\cos\theta + \dot{\psi} = K = \omega_3$$
$$\dot{\phi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\phi}\cos\theta\dot{\psi} = \omega^2$$
$$\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta = \omega^2 - \omega_3^2 = \omega_\perp^2$$
$$\dot{\theta}\ddot{\theta} + \dot{\phi}\ddot{\phi}\sin^2\theta + \dot{\phi}^2\dot{\theta}\sin\theta\cos\theta = 0$$
$$3\omega^2\sin\theta\cos\theta\sin\varphi = \ddot{\theta} - \dot{\phi}\cos\theta\,\dot{\phi}\sin\theta - \dot{\psi}\,\dot{\phi}\sin\theta$$
$$-3\omega^2\sin\theta\cos\theta\cos\varphi = \dot{\phi}\cos\theta\dot{\theta} - \ddot{\phi}\sin\theta + \dot{\phi}\,\dot{\theta}\cos\theta + \dot{\psi}\dot{\theta}$$

I still don't know how to "eliminate" ψ in my point mass limit of interest. I am trying to get here:

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$

$$\ddot{\varphi}\sin\theta + 2\dot{\theta}\,\dot{\varphi}\cos\theta - \omega\cos\varphi\sin\theta(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}} \qquad \text{(F.5.7)}$$

Plan C: OK, let's just try this to eliminate $\dot{\psi}$

$$\dot{\psi} = \omega_3 - \dot{\varphi}\cos\theta$$

Then my two big equations become

$$3\omega^2\sin\theta\cos\theta\sin\varphi = \ddot{\theta} - \dot{\varphi}\cos\theta\,\dot{\varphi}\sin\theta - \dot{\varphi}\sin\theta\,(\omega_3 - \dot{\varphi}\cos\theta)$$

$$-3\omega^2\sin\theta\cos\theta\cos\varphi = \dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta + \dot{\varphi}\,\dot{\theta}\cos\theta + \dot{\theta}\,(\omega_3 - \dot{\varphi}\cos\theta)$$

which compare

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$

$$\ddot{\varphi}\sin\theta + 2\dot{\theta}\,\dot{\varphi}\cos\theta - \omega\cos\varphi\sin\theta(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}} \qquad \text{(F.5.7)}$$

Things are NEVER going to match as long as ω_3 is sitting in my equations.

Plan D. It is possible that maybe $\dot{\psi}$ is very small? If that were the case, maybe my rigid equations are

$$3\omega^2\sin\theta\cos\theta\sin\varphi = \ddot{\theta} - \dot{\varphi}\cos\theta\,\dot{\varphi}\sin\theta$$

$$-3\omega^2\sin\theta\cos\theta\cos\varphi = \dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta + \dot{\varphi}\,\dot{\theta}\cos\theta$$

But no linear combining of these can give (F.5.7) because nothing can develop a linear ω factor!