Dumbbell Satellite as Rigid Body Problem v2 PhL 3.26.17

do not read below, see summary doc

Part I : The issue of Special Case #4

Status: I tried in v1 of this doc to solve the dumbbell satellite problem using these body frame equations for rigid body motion.

$$(N)'_1 = I'_1 (\dot{\omega})'_1 - (\omega)'_2(\omega)'_3 (I'_2 - I'_3)$$
$$(N)'_2 = I'_2 (\dot{\omega})'_2 - (\omega)'_3(\omega)'_1 (I'_3 - I'_1)$$
$$(N)'_3 = I'_3 (\dot{\omega})'_3 - (\omega)'_1(\omega)'_2 (I'_1 - I'_2) \ . \qquad \text{(I.4.3)}$$

Things were not working out very well, and now I think I realize why that is the case ??

Looking just above (I.1.1) I have stated that what follows applies in Special Case #4 where $\mathbf{b}=0$. But this is NOT the situation for the dumbbell satellite problem, where we have $\mathbf{b} \neq 0$! In particular, the equations of motion above do not apply!! So no wonder they led to problems. [but see conclusion at the end]

Is it possible to "fix this up" in some simple way. I can track through Section I.1 and see how things would have to be generalized. The satellite problem in fact is Special case #1 which has ω axis passing through the Frame S origin (non swap notation, Frame S us the inertial frame). In that case we know that

$$\dot{\mathbf{b}}_S \equiv (d\mathbf{b}/dt)_S = \boldsymbol{\omega} \times \mathbf{b} \ . \qquad \text{// Special Case \#1} \qquad \text{(4.4.3)}$$

Now staying in non-swap notation, we can try to update Section I.1 for Special Case #1

$$\mathbf{v}_\alpha = \mathbf{v}'_\alpha + \boldsymbol{\omega} \times \mathbf{r}'_\alpha + \dot{\mathbf{b}}_S \qquad \textit{(12.1.2b)} \qquad \text{(I.1.1)}$$
$$= 0 + \boldsymbol{\omega} \times \mathbf{r}'_\alpha + \boldsymbol{\omega} \times \mathbf{b}$$
$$= 0 + \boldsymbol{\omega} \times (\mathbf{r}'_\alpha + \mathbf{b})$$
$$= 0 + \boldsymbol{\omega} \times \mathbf{r}_\alpha \qquad \textit{(12.1.2a)}$$
$$= \boldsymbol{\omega} \times \mathbf{r}_\alpha$$

Then the total Frame S angular momentum of the rigid body is

$$\mathbf{L} = \Sigma_\alpha \ \mathbf{r}_\alpha \times \mathbf{p}_\alpha = \Sigma_\alpha \ \mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha = \Sigma_\alpha \ m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)$$
$$= \Sigma_\alpha \ m_\alpha [r_\alpha^2 \boldsymbol{\omega} - (\mathbf{r}_\alpha \bullet \boldsymbol{\omega}) \mathbf{r}_\alpha] \ . \qquad \text{//} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C} \qquad \text{(I.1.2)}$$

So this result survives! Hmmm. The following results also survive,

$$L_i = \Sigma_\alpha\ m_\alpha [\ r_\alpha^2 \omega_i - (\mathbf{r}_\alpha \bullet \boldsymbol{\omega})\ (\mathbf{r}_\alpha)_i\ ]$$

$$= \Sigma_\alpha\ m_\alpha [\ r_\alpha^2 \Sigma_j \omega_j \delta_{ij} - \Sigma_j (\mathbf{r}_\alpha)_j \omega_j\ (\mathbf{r}_\alpha)_i\ ]$$

$$= \Sigma_\alpha\ m_\alpha\ \Sigma_j [\ r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j\ ]\ \omega_j$$

$$= \Sigma_j\ \{\ \Sigma_\alpha m_\alpha [\ r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j\ ]\ \}\ \omega_j$$

$$= \Sigma_j\ I_{ij} \omega_j$$

where

$$I_{ij} \equiv \Sigma_\alpha m_\alpha [\ r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j\ ] \qquad = \text{the inertia tensor} \qquad \text{(I.1.3)}$$

or $\quad I_{ij} \equiv \int dx_1 dx_2 dx_3\ \rho(\mathbf{x})[\ r^2 \delta_{ij} - x_i x_j]\ .$ $\qquad$ // continuum notation

We rewrite (I.1.3) as a vector equation,

$$\mathbf{L} = I\boldsymbol{\omega}\ . \qquad \text{(I.1.4)}$$

Next, how about

$$T = \Sigma_\alpha (1/2) m_\alpha v_\alpha^2 = \Sigma_\alpha (1/2) m_\alpha\ \mathbf{v}_\alpha \bullet\ (\boldsymbol{\omega} \times \mathbf{r}_\alpha)\ = \Sigma_\alpha (1/2) m_\alpha\ \boldsymbol{\omega} \bullet\ (\mathbf{r}_\alpha \times \mathbf{v}_\alpha) \quad \text{// cyclic rule}$$

$$= \Sigma_\alpha\ (1/2)\ \boldsymbol{\omega} \bullet\ (\mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha)\ = (1/2)\ \boldsymbol{\omega} \bullet \Sigma_\alpha\ \mathbf{r}_\alpha \times \mathbf{p}_\alpha\ =\ (1/2)\ \boldsymbol{\omega} \bullet \mathbf{L}$$

$$= (1/2)\ \boldsymbol{\omega} \bullet (I\boldsymbol{\omega})\ . \qquad \text{// (I.1.4)} \qquad \text{(I.1.7)}$$

This too seems OK.

$$L_i = I_{ij}\ (\omega)_j \qquad I_{ij} = \Sigma_\alpha m_\alpha [\ r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j\ ] \qquad \text{// } r_\alpha^2 = (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_i$$

$$(L)'_i = (I)'_{ij}\ (\omega)'_j \qquad (I)'_{ij} = \Sigma_\alpha m_\alpha [\ r'^2_\alpha \delta_{ij} - (\mathbf{r}_\alpha)'_i (\mathbf{r}_\alpha)'_j\ ]\ . \quad \text{// } r'^2_\alpha = (\mathbf{r}_\alpha)'_i (\mathbf{r}_\alpha)'_i \qquad \text{(I.1.6)}$$

So maybe I should relax my condition to be Special Case #1 and not just Special Case #4 !!!!

Let's continue into Section I.2 It looks totally fine! And I.3 looks fine, and I.4 looks fine.

So I was wrong with my conjecture above! These equations SHOULD work for the satellite. But now have to pay attention that $r \neq \mathbf{r}'$.

Part II : Are Euler Angles defined for Special Case #1 ??

Go back to their definition. In Fig (H.1.5) I show Frame S and Frame S' having a common origin, which means Special Case #4. But is this significant? I can always move the basis vectors to have a common

origin even if there is some long b vector. The discussion of Section H seems OK through (H.3.5) where I am really only dealing with basis vectors.

Now what about the transformation of Kinematic Vectors in (H.3.6) ? Consider this claim,

$$(\mathbf{r})' = R\,\mathbf{r} = R_z(-\psi)R_x(-\theta)R_z(-\varphi)\,\mathbf{r} \qquad \text{(H.3.10)}$$

where I do NOT have anything called vector **r'**. My notation x' I think should be $(r)'_1$ to be clear. Other than this sloppy notation, things seem OK through end of Section H.3.

What about Section H.4 ? It just seems OK through (H.4.4). Everything is just using unit vectors, and origin location just does not matter! So I think this is after all OK

$$\begin{aligned}
(\omega)'_x &= \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi && \equiv \omega_{x'} \\
(\omega)'_y &= \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi && \equiv \omega_{y'} \\
(\omega)'_z &= \dot{\varphi}\cos\theta + \dot{\psi}\ . && \equiv \omega_{z'} \qquad \text{// Frame S'}
\end{aligned} \qquad \text{(H.4.7)}$$

That is, I think this is OK for Special Case #1 situations.

Conclusion: I don't see any reason why the Euler angle stuff is restricted to Special Case #4.

Part III : For the dumbbell satellite problem, what do I know about ω ?

Looking at Fig (F.1.1) I would say that

$$\boldsymbol{\omega} = \omega\hat{\mathbf{x}}' = \omega\hat{\mathbf{x}}$$

at all times. I thought I wrote this fact down somewhere in Appendix F ?

***** Maybe I should add the above equation somewhere useful!

I just don't see it written anywhere, but the picture makes it obvious.

Implication: In the body frame we have $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$ which says

$$\begin{aligned}
\omega_1 &= \omega \\
\omega_2 &= 0 \\
\omega_3 &= 0
\end{aligned}$$

This certainly simplifies the development in a way I did not take advantage of!! I will start a new v3 with this simplification installed from the start.

Part IV : For the dumbbell satellite problem, body frame?

In my dumbbell satellite treatment I have Frame S' at the Earth, and I have Frame S at the satellite. But the dumbbell moves freely within Frame S! So it is wrong to identify this Frame S with a "body frame" of rigid body motion theory!! So maybe this is the Big Error I have been making in this series of docs!