Dumbbell Satellite as Rigid Body Problem PhL 3.26.17

do not read below, see summary doc

In Appendix F I use swap notation, so Frame S is the body frame.

It seems clear that θ,φ are just Euler angles and that ψ probably plays no role since it's axis has no inertia.

I start with the equations of motion in the body frame, changed to swap notation:

$$N_1 = I_1 \dot{\omega}_1 - \omega_2 \omega_3 (I_2 - I_3)$$
$$N_2 = I_2 \dot{\omega}_2 - \omega_3 \omega_1 (I_3 - I_1)$$
$$N_3 = I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) \quad . \qquad \text{// Frame S = body frame} \qquad (I.4.3)$$

For my particular problem, $I_3 = 0$ since just two point masses, so then have

$$N_1 = I_1 \dot{\omega}_1 - \omega_2 \omega_3 I_2$$
$$N_2 = I_2 \dot{\omega}_2 + \omega_3 \omega_1 I_1$$
$$N_3 = I_3 \dot{\omega}_3 - \omega_1 \omega_2 (I_1 - I_2) \quad . \qquad \text{// Frame S = body frame} \qquad (I.4.3)$$

If I take the rigid body lined up on z' say, then clearly $I_1 = I_2$ so then have

$$N_1 = I_1 \dot{\omega}_1 - \omega_2 \omega_3 I_1$$
$$N_2 = I_2 \dot{\omega}_2 + \omega_3 \omega_1 I_1$$
$$N_3 = I_3 \dot{\omega}_3 \quad . \qquad \text{// Frame S = body frame} \qquad (I.4.3)$$

Getting pretty simple it would seem. Remember that the vector equation was

$$\mathbf{N} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) \qquad (I.2.6)$$

and maybe that will be helpful, but ignore for now. For the dumbbell we know that

$$I_1 = m_1 r_1^2 + m_2 r_2^2 \qquad \text{with} \qquad m_1 r_1 = m_2 r_2$$

So

$$I_1 = (1/m_2) [\ m_1 m_2 r_1^2 + m^2_2 r_2^2\ ] = (1/m_2) [\ m_1 m_2 r_1^2 + m^2_1 r_1^2\ ]$$

$$= (m_1/m_2) r_1^2 [\ m_2 + m_1\ ] = (m_1/m_2) r_1^2 M = (m_1/\mu_2) r_1^2 .$$

I show in (F.3.10) the torque on the satellite to be

$$\mathbf{N'^{(b)}} = (GM_E m_1 b r_1 \sin\theta)\,[1/r'^3_1 - 1/r'^3_2]\,\hat{\boldsymbol{\varphi}}$$

and this is about "b" which is the center of Frame S. In the far approximation I write this as

$$\mathbf{N'^{(b)}} = -3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \hat{\boldsymbol{\varphi}}\ .$$

What are the Cartesian Frame S components of this torque? Use

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\ \hat{\mathbf{x}} + \cos\varphi\ \hat{\mathbf{y}} \qquad \text{(E.2.6)}$$

So then

$$\mathbf{N'^{(b)}} = -3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \left[\ -\sin\varphi\ \hat{\mathbf{x}} + \cos\varphi\ \hat{\mathbf{y}}\ \right]$$

all in body Frame S. So I then know that true torque is same in both frames, so

$$N_1 = 3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \sin\varphi$$
$$N_2 = -3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \cos\varphi$$

So now I have these three equations of motion in the far approximation

$$3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \sin\varphi = I_1\,\dot{\omega}_1 - \omega_2\,\omega_3\,I_1$$
$$-3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \cos\varphi = I_1\,\dot{\omega}_2 + \omega_3\,\omega_1\,I_1$$
$$0 = I_3\,\dot{\omega}_3$$

So at once we know that ω_3 = constant. Now as usual we are worrying about the $\boldsymbol{\omega}$ vector and we have to think about what vector that really is. In body frame $\mathbf{L} = I\boldsymbol{\omega}$. On scratch I write that out as a matrix equation and it tells me this

$$L_1 = I_1\omega_1$$
$$L_2 = I_1\omega_2$$
$$L_3 = 0$$

Does this make sense for the dumbbell? Why is $L_3 = 0$, that just seems wrong. Ah, but this is all in the body frame remember, so it does make sense. So we then have a very simple connection in the body frame between $\mathbf{L}$ and $\boldsymbol{\omega}$. Lets set $\omega_3 = K$ for now and divide by I and rewrite our equations

$$3GM_E(m_1/\mu_2)\, I_1^{-1}\, b^{-3} r_1^2 \sin\theta\cos\theta\ \sin\varphi = \dot{\omega}_1 - K\,\omega_2$$
$$-3GM_E(m_1/\mu_2)\, I_1^{-1} b^{-3} r_1^2 \sin\theta\cos\theta\ \cos\varphi = \dot{\omega}_2 + K\,\omega_1$$

Now define

$$C \equiv 3GM_E(m_1/\mu_2)\, I_1^{-1}\, b^{-3} r_1^2$$

What do I know about C ? I claim above that

$$C = 3GM_E(m_1/\mu_2)\, I_1^{-1}\, b^{-3} r_1^2$$

$$I_1 = (m_1/\mu_2) r_1^2 \qquad \Rightarrow \qquad I_1^{-1} = (\mu_2/m_1) r_1^{-2}$$

and then

$$C = 3GM_E(m_1/\mu_2)(\mu_2/m_1) r_1^{-2} b^{-3} r_1^2$$

$$= 3GM_E b^{-3}$$

which is at least simpler than it was.

Then equations of motion are

$$C\sin\theta\cos\theta\,\sin\varphi = \dot{\omega}_1 - K\omega_2$$
$$-C\sin\theta\cos\theta\,\cos\varphi = \dot{\omega}_2 + K\omega_1 \qquad\qquad C = 3GM_E b^{-3}$$

Offhand, does not look too bad. Now would be a good time to roll out ω expressed in body frame Euler angles (I have many weapons now! Removing primes since in swap notation now

$$\omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} \qquad\qquad (H.4.7)$$

Can I remove ψ completely from the problem? Let's be safe and maintain it, and later with larger sphere masses I will want to maintain it anyway. So then

$$\dot{\omega}_1 = \dot{\varphi}\sin\theta\cos\psi\dot{\psi} + \dot{\varphi}\cos\theta\dot{\theta}\sin\psi + \ddot{\varphi}\sin\theta\sin\psi - \dot{\theta}\sin\psi\dot{\psi} + \ddot{\theta}\cos\psi$$
$$\dot{\omega}_2 = -\dot{\varphi}\sin\theta\sin\psi\dot{\psi} + \dot{\varphi}\cos\theta\dot{\theta}\cos\psi - \ddot{\varphi}\sin\theta\cos\psi - \dot{\theta}\cos\psi\dot{\psi} - \ddot{\theta}\sin\psi$$

Very interesting so far. Now install these into the equations along with the above

$$C\sin\theta\cos\theta\,\sin\varphi = [\dot{\varphi}\sin\theta\cos\psi\dot{\psi} + \dot{\varphi}\cos\theta\dot{\theta}\sin\psi + \ddot{\varphi}\sin\theta\sin\psi - \dot{\theta}\sin\psi\dot{\psi} + \ddot{\theta}\cos\psi] - K[\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]$$

$$-C\sin\theta\cos\theta\,\cos\varphi = [-\dot{\varphi}\sin\theta\sin\psi\dot{\psi} + \dot{\varphi}\cos\theta\dot{\theta}\cos\psi - \ddot{\varphi}\sin\theta\cos\psi - \dot{\theta}\cos\psi\dot{\psi} - \ddot{\theta}\sin\psi] + K[\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]$$

NOW let's get rid of ψ somehow. Assume ψ ≡ 0 at this point, so sinψ = 0 and cosψ = 1,

$$C\sin\theta\cos\theta\,\sin\varphi = [\ddot{\theta}]$$
$$- K[\dot{\varphi}\sin\theta]$$

$$-C\sin\theta\cos\theta\,\cos\varphi = [\,+\dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta]$$
$$+ K\,[\,+\dot{\theta}\,]$$

which rewrite as

$$C\sin\theta\cos\theta\,\sin\varphi = \ddot{\theta} - K\,\dot{\varphi}\sin\theta$$
$$-C\sin\theta\cos\theta\,\cos\varphi = \dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta + K\dot{\theta}$$

or

$$\ddot{\theta} - K\,\dot{\varphi}\sin\theta - C\sin\theta\cos\theta\,\sin\varphi = 0$$

$$\ddot{\varphi}\sin\theta - K\dot{\theta} - \dot{\varphi}\cos\theta\dot{\theta} - C\sin\theta\cos\theta\,\sin\varphi = 0 \qquad C = 3GM_{E}b^{-3}$$

How do these EOM compare with my results in Appendix F ??

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad //\ \hat{\boldsymbol{\varphi}}$$
$$\ddot{\varphi} + 2\dot{\theta}\,\dot{\varphi}\cot\theta - \omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad //\ \hat{\boldsymbol{\theta}} \qquad \text{(F.5.7)}$$

My rigid body equations do not include the symbol ω, so that is one big difference.

Is ω a constant of the motion? Well it is determined by the satellite circular orbit, so it is a constant! So there is another equation I have so far ignored. Write again

$$\omega_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} \qquad \text{(H.4.7)}$$

Then

$$\omega_1^2 = \dot{\varphi}^2\sin^2\theta\sin^2\psi + \dot{\theta}^2\cos^2\psi + 2\,\dot{\varphi}\sin\theta\sin\psi\dot{\theta}\cos\psi$$

$$\omega_2^2 = \dot{\varphi}^2\sin^2\theta\cos^2\psi + \dot{\theta}^2\sin^2\psi - 2\,\dot{\varphi}\sin\theta\cos\psi\dot{\theta}\sin\psi$$

$$\omega_3^2 = \dot{\varphi}^2\cos^2\theta + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$

Add the first two to get

$$\omega_1^2 + \omega_2^2 = \dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2$$

Then add the third to get

$$\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = \dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2 + \dot{\varphi}^2 \cos^2\theta + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$

$$= \dot{\varphi}^2 + \dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\cos\theta\dot{\psi}$$

And now set $\dot{\psi} = 0$ to get

$$\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2$$

simpler than I thought it would be. I also have another equation,

$$\omega_3 = \dot{\varphi}\cos\theta + \dot{\psi} = K$$

which then says

$K = \dot{\varphi}\cos\theta$ for our simple case.

Now if you are enforcing ω as a constant from Earth rotation, we then have lots of equations that I have developed in this appendix

$K = \dot{\varphi}\cos\theta$

$\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2$ = constant which implies the next line

$\dot{\varphi}\ddot{\varphi} + \dot{\theta}\ddot{\theta} = 0$ // I have never seen this equation before!

$C\sin\theta\cos\theta\sin\varphi = \ddot{\theta} - K\dot{\varphi}\sin\theta$

$-C\sin\theta\cos\theta\cos\varphi = \dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta + K\dot{\theta}$ $C = 3GM_E b^{-3}$

But this still looks a far cry from my Appendix F equations of motion which are these

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \quad // \hat{\boldsymbol{\varphi}}$$

$$\ddot{\varphi} + 2\dot{\theta}\,\dot{\varphi}\cot\theta - \omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0 \quad // \hat{\boldsymbol{\theta}} \qquad \text{(F.5.7)}$$

I am now worried about the terms in (F.5.7) that are linear in ω since $\omega = \sqrt{\dot{\varphi}^2 + \dot{\theta}^2}$.

So here are our four rigid body equations again

$$K = \dot{\varphi}\cos\theta \qquad\qquad K = \omega_3$$

$$\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2 = \text{constant}$$

$$3GM_E b^{-3}\sin\theta\cos\theta\sin\varphi = \ddot{\theta} - K\dot{\varphi}\sin\theta \qquad (1)$$

$$-3GM_E b^{-3}\sin\theta\cos\theta\cos\varphi = \dot{\varphi}\cos\theta\dot{\theta} - \ddot{\varphi}\sin\theta + K\dot{\theta} \qquad (2)$$

which again I am comparing to

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$

$$\ddot{\varphi} + 2\dot{\theta}\dot{\varphi}\cot\theta - \omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}} \qquad (F.5.7)$$

We still seem to be in separate worlds!

Let's work on (1) and (2) this way,

$$3GM_E b^{-3}\sin\theta\cos\theta\sin\varphi\cos\varphi = \ddot{\theta}\cos\varphi - K\dot{\varphi}\sin\theta\cos\varphi \qquad (1)\cos\varphi$$

$$-3GM_E b^{-3}\sin\theta\cos\theta\cos\varphi\sin\varphi = \dot{\varphi}\cos\theta\dot{\theta}\sin\varphi - \ddot{\varphi}\sin\theta\sin\varphi + K\dot{\theta}\sin\varphi \qquad (2)\sin\varphi$$

Add these to get

$$\ddot{\theta}\cos\varphi - K\dot{\varphi}\sin\theta\cos\varphi + \dot{\varphi}\cos\theta\dot{\theta}\sin\varphi - \ddot{\varphi}\sin\theta\sin\varphi + K\dot{\theta}\sin\varphi = 0$$

Now insert $K = \dot{\varphi}\cos\theta$ to get

$$\ddot{\theta}\cos\varphi - \dot{\varphi}\cos\theta\,\dot{\varphi}\sin\theta\cos\varphi + \dot{\varphi}\cos\theta\dot{\theta}\sin\varphi - \ddot{\varphi}\sin\theta\sin\varphi + \dot{\varphi}\cos\theta\dot{\theta}\sin\varphi = 0$$

or

$$\ddot{\theta}\cos\varphi - \dot{\varphi}^2\cos\theta\sin\theta\cos\varphi + \dot{\varphi}\dot{\theta}\cos\theta\sin\varphi - \ddot{\varphi}\sin\theta\sin\varphi + \dot{\varphi}\,\dot{\theta}\cos\theta\sin\varphi = 0$$

or

$$\ddot{\theta}\cos\varphi - \dot{\varphi}^2\cos\theta\sin\theta\cos\varphi - \ddot{\varphi}\sin\theta\sin\varphi = 0$$

We are still operating in different worlds it seems to me. I need to assert the constraint that ω is constant somehow

$$\omega^2 = \dot{\varphi}^2 + \dot{\theta}^2 \qquad \text{and} \qquad \dot{\varphi}\ddot{\varphi} + \dot{\theta}\,\ddot{\theta} = 0$$