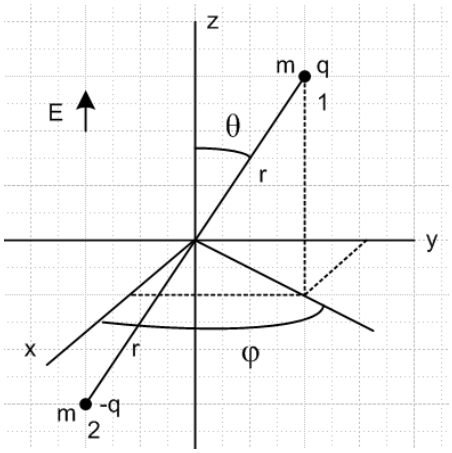
This led to a 8 hours paradox which turned out to be confusion of θ.φ as angles of vector **L** with *different* angles θ.φ as components of vector **r**. Go now to version 1 and try again! I did get to test many foundational facts from App E and found nothing wrong there.

I.10 Motion of an electric dipole dumbbell in a uniform E field.

The kinematic setup is similar to that for the dumbbell satellite shown in Fig (F.1.1), but here we simplify so that $r_1 = r_2 = r$ and $m_1 = m_2 = m$, so the picture is this,



The two equal masses m are separated by a massless stick of length 2r which has a universal pivot point at the origin, allowing the dumbbell to rotate freely in θ and φ. One mass has charge q and the other -q. The pivot point won't really be needed but it helps to think about the problem. Instead of the gravitational field of the Earth, this dumbbell is immersed in a uniform electric field **E** in the z direction.

Force and Torque

In the dumbbell satellite problem both masses were attracted to the Earth, but here one mass is pushed up and the other is pushed down. In fact, letting T be the tension in the stick,

$$\mathbf{F}_1 = q\mathbf{E} + T\hat{\mathbf{r}} = qE\hat{\mathbf{z}} + T\hat{\mathbf{r}}$$

$$\mathbf{F}_2 = (-q)\mathbf{E} - T\hat{\mathbf{r}} = -qE\hat{\mathbf{z}} - T\hat{\mathbf{r}} = -\mathbf{F}_1 \qquad \mathbf{r}_2 = -\mathbf{r}_1 = -\mathbf{r}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = 0$$

so the center of mass does not move (hence no pivot needed).

The torques on the two masses about the origin are,

$$\mathbf{N}_1 = \mathbf{r}_1 \times \mathbf{F}_1 = [r\hat{\mathbf{r}}] \times [\, qE\hat{\mathbf{z}}] = rqE\, \hat{\mathbf{r}} \times \hat{\mathbf{z}} = rqE\, [\, -\sin\theta\, \hat{\boldsymbol{\varphi}}\,] = -rqE\sin\theta\, \hat{\boldsymbol{\varphi}}$$

$$\mathbf{N}_2 = \mathbf{r}_2 \times \mathbf{F}_2 = (-\mathbf{r}_1) \times (-\mathbf{F}_1) = \mathbf{N}_1$$

Here we ignored the tension forces since $\mathbf{r_1} \times \mathbf{r} = \mathbf{r_2} \times \mathbf{r} = 0$.

The total torque is then,

$$\mathbf{N} = \mathbf{N_1} + N_2 = 2\mathbf{N_1} = -2rqE\sin\theta\, \hat{\boldsymbol{\varphi}}$$

Now define the dipole moment **p** as

$$p \equiv 2rq \qquad\qquad \mathbf{p} \equiv 2rq\hat{\mathbf{r}}$$

Then using (E.2.15) that $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \sin\theta\, \hat{\boldsymbol{\varphi}}$ we may write,

$$\mathbf{N} = -2rqE\sin\theta\, \hat{\boldsymbol{\varphi}} = -2rqE\sin\theta\, [\, \hat{\mathbf{z}} \times \hat{\mathbf{r}}/\sin\theta\,] = 2rqE\, \hat{\mathbf{r}} \times \hat{\mathbf{z}}$$

$$= [\, 2rq\, \hat{\mathbf{r}}] \times [E\hat{\mathbf{z}}\,] = \mathbf{p} \times \mathbf{E}$$

which is the classic result for the torque on a dipole in an electric field.

Newton's Angular Law

Newton tells us that

$$\mathbf{N} = \dot{\mathbf{L}}$$

so we have the following vector equation of motion for the dipole dumbbell,

$$\dot{\mathbf{L}} = \mathbf{p} \times \mathbf{E} = -2rqE\sin\theta\, \hat{\boldsymbol{\varphi}} = -pE\sin\theta\, \hat{\boldsymbol{\varphi}}\ .$$

We now consider the following generic problem where **a** is some generic vector,

$$\dot{\mathbf{a}} = K(\theta)\, \hat{\boldsymbol{\varphi}}\ . \qquad \text{// } \textit{this } \theta \text{ is the physical } \mathbf{r} \text{ vector's } \theta$$

If we place the tail of vector **a** at the origin and write $\mathbf{a} = a\hat{\mathbf{r}}$ to describe is position, then

[Note: "a,θ_a,φ_a" below are the coordinates of vector **a**, not of the physical vector **r** in the Figure, so don't confuse two meanings of θ,φ. The notation $\hat{\mathbf{r}}$ really means $\hat{\mathbf{a}}$.]

$$\mathbf{a} = a\hat{\mathbf{r}}_a \qquad \textbf{ok}$$

$$\dot{\mathbf{a}} = \dot{a}\, \hat{\mathbf{r}}_a + a\, \partial_t \hat{\mathbf{r}}_a$$

$$= \dot{a}\, \hat{\mathbf{r}}_a + a\, [\ \dot{\theta}_a\, \hat{\boldsymbol{\theta}}_a + \dot{\varphi}_a \sin\theta_a\, \hat{\boldsymbol{\varphi}}_a\,] \qquad \text{// (E.2.11)}$$

$= \dot{a}\,\hat{\mathbf{r}}_a + a\,\dot{\theta}_a\,\hat{\boldsymbol{\theta}}_a + a\dot{\varphi}_a \sin\theta_a\,\hat{\boldsymbol{\varphi}}_a$. **ok**

Therefore, since $\dot{\mathbf{a}} = K(\theta)\,\hat{\boldsymbol{\varphi}}$, one has

$\dot{a}\,\hat{\mathbf{r}}_a + a\,\dot{\theta}_a\,\hat{\boldsymbol{\theta}}_a + a\dot{\varphi}_a \sin\theta_a\,\hat{\boldsymbol{\varphi}}_a = K(\theta)\,\hat{\boldsymbol{\varphi}}$ **ok**

So here is my error! On the left $\hat{\boldsymbol{\varphi}}_a$ is the azimuthal unit vector for vector **a** , whereas $\hat{\boldsymbol{\varphi}}$ on the right is for the vector **r**.

One may then conclude that

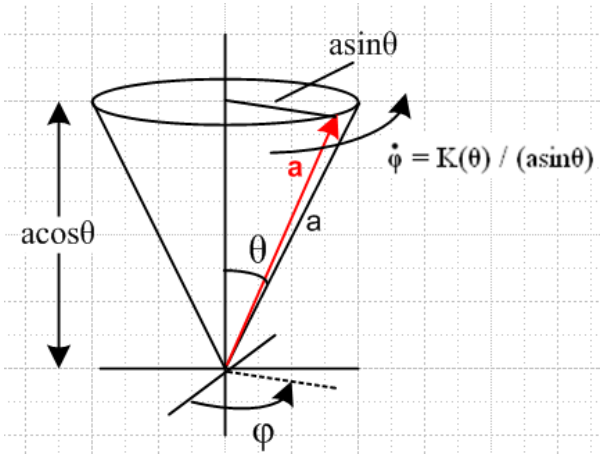
$\dot{a} = 0 \quad \Rightarrow$ length of vector a is fixed

$a\,\dot{\theta} = 0 \quad \Rightarrow$ θ is a constant

$a\dot{\varphi}\sin\theta = K(\theta) \quad \Rightarrow \quad \dot{\varphi} = K(\theta) / (a\sin\theta)$ **ok**

[But θ is not a coordinate of **r**, it is a coordinate of **a**, don't forget!]

Thus the vector **a** precesses around a cone with half-angle θ at constant rate $\dot{\varphi} = a\sin\theta / K(\theta)$:



ok

In our application we had

$\dot{\mathbf{L}} = -pE\sin\theta\,\hat{\boldsymbol{\varphi}}$ compare to $\dot{\mathbf{a}} = K(\theta)\,\hat{\boldsymbol{\varphi}}$

[But THIS θ is a coordinate of **r** !]

In our application we then have **a** = **L** and a = L and $K(\theta) = -pE\sin\theta$, so

$\dot{\varphi} = K(\theta) / (a\sin\theta) = K(\theta) / (L\sin\theta) = -pE\sin\theta / (L\sin\theta) = -pE/L$

Comment: This seems to me to be a strange result. Very large L is associated with very small $\dot{\varphi}$.

Dimension check: pE/L

pE = torqe = rxF = L M L T^{-2}

L = r x p = L M L T^{-1}

then pE/L = L M L T^{-2} / L M L T^{-1} = T^{-1} = correct

Question: What about a solution where it spins like mad in perp to the plane of paper. I think this is a viable solution, and there would be a torque, so the plane would rotate. But this is not a θ = constant solution, so something is wrong with my cone theory which says θ = constant for any solution! Maybe I am finally onto something here!

Thus, the **L** vector maintains its length and precesses at rate $\dot{\varphi} = -pE/L$ about a cone of half-angle θ which we don't yet know.

As shown in (F.2.2) we know that

$$\mathbf{L} = 2m r^2 (\dot{\theta}\, \hat{\boldsymbol{\varphi}} - \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\theta}})$$

$$\dot{\mathbf{L}} = 2mr^2 [(\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta)\, \hat{\boldsymbol{\varphi}} - (2\dot{\theta}\, \dot{\varphi} \cos\theta + \ddot{\varphi}\sin\theta)\, \hat{\boldsymbol{\theta}}]$$

But since θ = constant and $\dot{\varphi}$ = constant, one can write these as

$$\mathbf{L} = 2m r^2 (- \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\theta}}) = - 2m r^2\, \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\theta}}$$

$$\dot{\mathbf{L}} = 2mr^2 [(- \dot{\varphi}^2 \sin\theta \cos\theta)\, \hat{\boldsymbol{\varphi}}] = - 2mr^2\dot{\varphi}^2\sin\theta \cos\theta\, \hat{\boldsymbol{\varphi}}$$

Inserting $\dot{\varphi} = - pE/L$ gives

$$\mathbf{L} = - 2m r^2\, \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\theta}} = 2mr^2\, [pE/L] \sin\theta\, \hat{\boldsymbol{\theta}} = L\, \hat{\mathbf{L}}$$

$$\dot{\mathbf{L}} = - 2mr^2\dot{\varphi}^2\sin\theta \cos\theta\, \hat{\boldsymbol{\varphi}} = - 2mr^2[pE/L]^2\sin\theta \cos\theta\, \hat{\boldsymbol{\varphi}}$$

The first line says

$$\hat{\mathbf{L}} = \hat{\boldsymbol{\theta}} \qquad \text{and} \qquad L^2 = 2mr^2 pE\sin\theta \qquad (*)$$

STOP. This says that **L** points down, but Marion p 366 says it points up. That is because in our example here, $\dot{\varphi} < 0$, not what Marion shows in his picture.

Now see if this is consistent with the second line which says

$$\dot{\mathbf{L}} = -2mr^2[pE/L]^2 \sin\theta \cos\theta \; \hat{\boldsymbol{\varphi}}$$

Recall that we found earlier that

$$\dot{\mathbf{L}} = -pE\sin\theta \; \hat{\boldsymbol{\varphi}}$$

This then requires that

$$2mr^2[pE/L]^2 \sin\theta \cos\theta = pE\sin\theta$$

or

$$2mr^2 pE \sin\theta \cos\theta = \sin\theta \, L^2$$

or

$$2mr^2 pE\cos\theta = L^2$$

or

$$L^2 = 2mr^2 pE\cos\theta$$

STOP! This conflicts with result (*) above! Or it requires that $\sin\theta = \cos\theta$ so $\theta = \pi/4$ fixed.
Paradox!! Am now blocked until this is resolved.

Paradox: Two conflicting results above.

Look at the DC situation. No motion, the stick just lines up on the z axis so $\theta = 0$. In this case, we expect that $L = 0$ since nothing is happening with the two point masses on a stick. But the second formula in this case says that $L^2 = 2mr^2 pE\cos\theta = 2mr^2 pE > 0$ which must be wrong! The other formula gives the correct answer of 0. So something is wrong with my second result (debugging in progress!)

So let's go through all the steps leading to the $\cos\theta$ result.

(-1). I derive the fact that $\mathbf{N} = -pE\sin\theta \; \hat{\boldsymbol{\varphi}} = \mathbf{p} \times \mathbf{E}$ based on $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \sin\theta \; \hat{\boldsymbol{\varphi}}$. I know the $\mathbf{p} \times \mathbf{E}$ result is correct. Newton in inertial frame says $\dot{\mathbf{L}} = \mathbf{N}$ so I end up with $\dot{\mathbf{L}} = -pE\sin\theta \; \hat{\boldsymbol{\varphi}}$.

0. I treat the problem $\dot{\mathbf{L}} = -pE\sin\theta \; \hat{\boldsymbol{\varphi}}$ generically and I conclude that L=const and θ = const and finally that $\dot{\varphi} = -pE/L$ = const. This relies on the claim that $\partial_t \hat{\mathbf{r}} = \dot{\theta} \; \hat{\boldsymbol{\theta}} + \dot{\varphi} \sin\theta \; \hat{\boldsymbol{\varphi}}$ in (E.2.11). This in turn relies on claims about $(\partial_r \hat{\mathbf{r}})$ and $(\partial_\theta \hat{\mathbf{r}})$ and $(\partial_\varphi \hat{\mathbf{r}})$.

1. I have a formula (F.2.2) which might be suspect. It says this for masses constrained to sphere:

$$\dot{\mathbf{L}} = 2mr^2 \left[(\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta) \; \hat{\boldsymbol{\varphi}} - (2\dot{\theta} \; \dot{\varphi} \cos\theta + \ddot{\varphi}\sin\theta) \; \hat{\boldsymbol{\theta}} \right]$$

I was, however, able to verify that, for θ = constant, my hand calculation of $\dot{\mathbf{L}}$ agrees with the above for θ = constant, and in particular, the sinθcosθ term really is present. This hand calculation however relies on the claim that $\partial_t \hat{\boldsymbol{\theta}} = -\dot{\theta}\,\hat{\mathbf{r}} + \dot{\phi}\cos\theta\,\hat{\boldsymbol{\phi}}$ and THAT is the source of the cosθ factor. [I review this in detail below step 6 below and it seems right]

2. I accept that $\dot{\theta} = 0$ from my generic precession calculation, so that leaves me with

$$\dot{\mathbf{L}} = 2mr^2 [-\dot{\phi}^2 \sin\theta \cos\theta)\,\hat{\boldsymbol{\phi}} - (\ddot{\phi}\sin\theta)\,\hat{\boldsymbol{\theta}}]$$

$$= -2mr^2[\,\dot{\phi}^2 \sin\theta \cos\theta\,\hat{\boldsymbol{\phi}} + \ddot{\phi}\sin\theta\,\hat{\boldsymbol{\theta}}\,]$$

3. But the generic calculation also says $\dot{\phi} = -\,pE/L$ and since I know from generic cone that L is a constant, I know that $\dot{\phi}$ = constant and therefore $\ddot{\phi} = 0$. I am then left with

$$\dot{\mathbf{L}} = -2mr^2\,\dot{\phi}^2 \sin\theta \cos\theta\,\hat{\boldsymbol{\phi}}$$

4. Now insert $\dot{\phi} = -\,pE/L$ to get

$$\dot{\mathbf{L}} = -2mr^2\,(pE/L)^2 \sin\theta \cos\theta\,\hat{\boldsymbol{\phi}}$$

5. Meanwhile, from early on I show that

$$\mathbf{N} = -2rqE\sin\theta\,\hat{\boldsymbol{\phi}} = \mathbf{p} \times \mathbf{E}$$

6. Frame S of my picture is inertial, so should have

$$\mathbf{N} = \dot{\mathbf{L}}$$

7. Therefore have

$$\dot{\mathbf{L}} = -2rqE\sin\theta\,\hat{\boldsymbol{\phi}}$$

8. Step 6 contradicts step 4 because two expressions have different θ dependence. ***UNLESS*** of course the quantity L is L(θ). So let's look into that next.

9. What is L ? This quantity is not mentioned in any of the previous steps!

9a. I accept the claim of (F.2.2) that $\mathbf{L} = 2m\,r^2\,(\dot{\theta}\,\hat{\boldsymbol{\phi}} - \dot{\phi}\sin\theta\,\hat{\boldsymbol{\theta}})$ and this clearly says that

$$L^2 = (2mr^2)^2\,[\,\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\,]$$

9b. But by step (0) $\dot{\theta} = 0$ and this then says $L^2 = (2mr^2)^2\, \dot{\phi}^2 \sin^2\theta$

from which we conclude that: $L = (2mr^2)\, |\, \dot{\phi}\, |\, \sin\theta$

9c. Using the generic cone result that $\dot{\phi} = -\, pE/L$ we then get

$L = (2mr^2)\, (pE/L)\sin\theta \quad \equiv \quad L^2 = (2mr^2)\, (pE)\, \sin\theta$

This L does in fact have θ dependence. This result seems correct because as $\theta \rightarrow 0$, we expect to have $L \rightarrow 0$ as it approaches the static situation.

Review of contradiction of Step 8. We had at this point

step 4 says: $\dot{\mathbf{L}} = -2mr^2\, (pE/L)^2\, \sin\theta \cos\theta\, \hat{\boldsymbol{\phi}}$

step 6 says: $\dot{\mathbf{L}} = -2rqE\sin\theta\, \hat{\boldsymbol{\phi}}$

If these are both correct, then comparing them says

$2mr^2\, (pE)^2 \cos\theta = 2rqE\, L^2$

We now install the result for L^2 from step 9c to get

$2mr^2\, (pE)^2 \cos\theta = 2rqE\, (2mr^2)\, (pE)\, \sin\theta$

or

$(pE)^2 \cos\theta = pE\, (pE)\, \sin\theta$

or

$\cos\theta = \sin\theta$

and this is a contradiction unless θ is a fixed angle of π/4 which seems unlikely to me. For example, in the DC limit we need to have θ = 0.

So something is wrong with Step 4 or Step 6 or Step 9a. The paradox survives well and healthy after 1.5 hours my trying to make it go away.

Which of these three steps is "unreasonable" in some problem limit? Everything interacts with everything else, so hard to debug!

What about (F.2.2) for L ? (F.2.2) is claiming that

$\mathbf{L} = 2m\, r^2\, (\, \dot{\theta}\, \hat{\boldsymbol{\phi}} - \dot{\phi} \sin\theta\, \hat{\boldsymbol{\theta}})$

Where does this come from? It comes from an 8 line calculation near (F.2.1). The main idea is that you have

$$\mathbf{L} = (2mr)\,\hat{\mathbf{r}} \times \mathbf{v}$$

Apart from the constant factor, this is hard to argue with from the definition of L. Each mass contributes the same amount, so each is doing : = **r** x **p**. I then depend on Appendix E which says

$$\mathbf{v} = v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}} \qquad \text{// velocity is tangent to the sphere : } \mathbf{r}\bullet\mathbf{v} = 0$$

$$v_\theta = r\,\dot{\theta}$$

$$v_\varphi = r\,\dot{\varphi}\sin\theta \qquad \text{(E.3.5)}$$

and then I can write

$$\mathbf{L} = (2mr)\ \hat{\mathbf{r}} \times [\ v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}}] = (2mr)\,v_\theta\ \hat{\boldsymbol{\varphi}} + (2mr^2)\,v_\varphi\,[-\hat{\boldsymbol{\theta}}\ ] = (2mr)\,[\ v_\theta\ \hat{\boldsymbol{\varphi}} - v_\varphi\ \hat{\boldsymbol{\theta}}\ ]$$

Now I install the (E.3.5) expressions and get

$$\mathbf{L} = (2mr)\,[\ v_\theta\ \hat{\boldsymbol{\varphi}} - v_\varphi\ \hat{\boldsymbol{\theta}}\ ] = (2mr)\,[\ r\dot{\theta}\ \hat{\boldsymbol{\varphi}} - r\dot{\varphi}\sin\theta\ \hat{\boldsymbol{\theta}}\ ] = (2mr^2)\ [\ \dot{\theta}\ \hat{\boldsymbol{\varphi}} - \dot{\varphi}\sin\theta\ \hat{\boldsymbol{\theta}}\ ]$$

and this confirms the claim of (F.2.2).

Examine Step 1 in more detail

As noted, it assumes that $\partial_t\hat{\boldsymbol{\theta}} = -\dot{\theta}\ \hat{\mathbf{r}} + \dot{\varphi}\cos\theta\ \hat{\boldsymbol{\varphi}}$ and this is the source of the problematic cosθ factor. Let's go derive this result using Appendix E as an aid. Let's look specifically at this $\partial_t\hat{\boldsymbol{\theta}}$ claim:

$$\partial_t\hat{\boldsymbol{\theta}} = (\partial_r\hat{\boldsymbol{\theta}})\,\dot{r} + (\partial_\theta\hat{\boldsymbol{\theta}})\,\dot{\theta} + (\partial_\varphi\hat{\boldsymbol{\theta}})\,\dot{\varphi}$$

$$= \quad 0 \quad + (-\hat{\mathbf{r}}\ )\,\dot{\theta} + [\ \cos\theta\ \hat{\boldsymbol{\varphi}}\ ]\dot{\varphi} = -\dot{\theta}\ \hat{\mathbf{r}} + \dot{\varphi}\cos\theta\hat{\boldsymbol{\varphi}}$$

But now let's back up some more and verify the claim that $\partial_\varphi\hat{\boldsymbol{\theta}} = \cos\theta\ \hat{\boldsymbol{\varphi}}$: My App E method is this

$$\partial_\varphi\hat{\boldsymbol{\theta}} = \partial_\varphi \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} -\cos\theta\sin\varphi \\ \cos\theta\cos\varphi \\ 0 \end{pmatrix} = -\cos\theta\sin\varphi\ \hat{\mathbf{x}} + \cos\theta\cos\varphi\ \hat{\mathbf{y}}$$

$$= -\cos\theta\sin\varphi\ [\ \sin\theta\cos\varphi\ \hat{\mathbf{r}} + \cos\theta\cos\varphi\ \hat{\boldsymbol{\theta}} - \sin\varphi\ \hat{\boldsymbol{\varphi}}\ ]$$

$$+ \cos\theta\cos\varphi\ [\ \sin\theta\sin\varphi\ \hat{\mathbf{r}} + \cos\theta\sin\varphi\ \hat{\boldsymbol{\theta}} + \cos\varphi\ \hat{\boldsymbol{\varphi}}\ ]$$

The $\hat{\mathbf{r}}$ terms cancel.
The $\hat{\boldsymbol{\theta}}$ terms cancel.

Result is: $= +\cos\theta\sin^2\varphi\,\hat{\boldsymbol{\varphi}} + \cos\theta\cos^2\varphi\,\hat{\boldsymbol{\varphi}} = \cos\theta\,\hat{\boldsymbol{\varphi}}$.

So I have verified that this cosθ factor is present in $\partial_{\varphi}\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\varphi}}$ and then in $\partial_t\hat{\boldsymbol{\theta}} = -\dot{\theta}\,\hat{\mathbf{r}} + \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\varphi}}$ and finally in $\dot{\mathbf{L}}$ as computed both in (F.2.2) and also by hand on scratch. So this is not the problem.

Post lunch debug session: There are maybe 30 underlying moving parts here. I need to get something physical to tell me better where to look.

Question: are there any constants of this motion? We have $\mathbf{N} = -pE\sin\theta\,\hat{\boldsymbol{\varphi}}$. Now write

$$\mathbf{N} = N_r\hat{\mathbf{r}} + N_\theta\hat{\boldsymbol{\theta}} + N_\varphi\hat{\boldsymbol{\varphi}}$$

This seems to say that $N_r = 0$ and $N_\theta = 0$. Now consider inertial frame claim

$$\mathbf{N} = \dot{\mathbf{L}}$$

This seems to say that

$$(\dot{\mathbf{L}})_\theta = 0 \qquad \text{and} \qquad (\dot{\mathbf{L}})_r = 0$$

Now I claim above that

$$\dot{\mathbf{L}} = -2mr^2\dot{\varphi}^2\sin\theta\cos\theta\,\hat{\boldsymbol{\varphi}}$$

and I learn nothing. Compare the two expressions for $\dot{\mathbf{L}}$

$$\dot{\mathbf{L}} = -2mr^2\dot{\varphi}^2\sin\theta\cos\theta\,\hat{\boldsymbol{\varphi}}$$
$$\dot{\mathbf{L}} = -pE\sin\theta\,\hat{\boldsymbol{\varphi}}$$

This is what gives

$$-2mr^2\dot{\varphi}^2\sin\theta\cos\theta = -pE\sin\theta \quad ?$$
$$-2mr^2\dot{\varphi}^2\cos\theta = -pE \quad ?$$
$$2mr^2\dot{\varphi}^2\cos\theta = pE \quad ?$$

$$\dot{\varphi}^2 = pE / [\,2mr^2\cos\theta\,]$$

Is this a physically reasonable result? It says that as $\theta \rightarrow \pi/2$, $\dot{\varphi} \rightarrow \infty$. That might be OK. In that case, centrifugal force swamps the electric aligning force allowing you to have $\theta = \pi/2$.

If you accept this answer as well as the cone result that $\dot{\varphi} = -pE/L$, you then have

$$pE / [\, 2mr^2 \cos\theta \,] = (pE)^2/L^2$$

or

$$1 / [\, 2mr^2 \cos\theta \,] = (pE)/L^2$$

or

$$L^2 = pEmr^2\cos\theta$$

But at $\theta = \pi/2$ you would expect a HUGE L since $\dot{\varphi} = \infty$, but here we are getting instead L = 0. So this result is NOT physical.

Now I think something is wrong with the claim $\dot{\varphi} = -pE/L$.