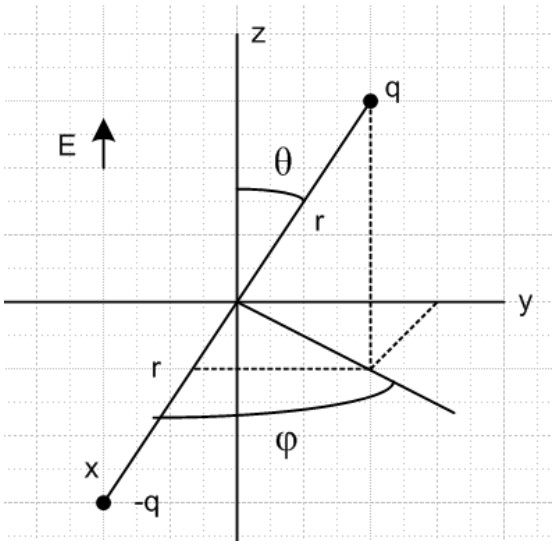
Does an electric dipole in a uniform E field precess? Version 1 PhL 3.24.17

Basically no web hits on this subject, lots on mag dipole in B field.

Statement of Problem: You have a dumbbell of two charges q and -q, each of mass m, connected by a rigid stick. There is a uniform external field E in the z direction. Find equations of motion for the dumbbell.

Note that the total force on the system is 0, so the CMS does not move, so we take the CMS as the origin of Frame S.

1. Use spherical coordinates to set things up, similar to the dumbbell satellite in its Frame S.



Upper right is 1, lower left is 2. Then φ

$\mathbf{F_1} = q\mathbf{E} = qE\hat{\mathbf{z}}$

$\mathbf{F_2} = (-q)\mathbf{E} = -qE\hat{\mathbf{z}} = -\mathbf{F_1}$ $\qquad\qquad$ $\mathbf{r_2} = -\mathbf{r_1}$

$\mathbf{N_1} = \mathbf{r_1} \times \mathbf{F_1} = [r\hat{\mathbf{r}}] \times [qE\hat{\mathbf{z}}] = rqE\,\hat{\mathbf{r}} \times \hat{\mathbf{z}} = rqE[-\sin\theta\,\hat{\boldsymbol{\varphi}}] = -rqE\sin\theta\,\hat{\boldsymbol{\varphi}}$

$\mathbf{N_2} = \mathbf{r_2} \times \mathbf{F_2} = (-\mathbf{r_1}) \times (-\mathbf{F_1}) = \mathbf{N_1}$

We have a massless stick connecting the charges. The total torque on this dumbbell is (about origin)

$\mathbf{N} = \mathbf{N_1} + \mathbf{N_2} = 2\mathbf{N_1} = -2rqE\sin\theta\,\hat{\boldsymbol{\varphi}}$

Right hand shows that this torque is trying to align the dipole. Good.

Now define the dipole moment **p** as

$p \equiv 2rq \qquad\qquad \mathbf{p} \equiv 2rq\hat{\mathbf{r}}$

Then

$\mathbf{N} = 2rqE\,\hat{\mathbf{r}} \times \hat{\mathbf{z}} = [\,2rq\,\hat{\mathbf{r}}] \times [E\hat{\mathbf{z}}\,] = \mathbf{p} \times \mathbf{E}$

which is our first major result! You can imagine increasing q and decreasing r in a way that the product is constant, then you could have **p** very close to the origin.

So we know that there is a torque on a dipole in an electric field given by $\mathbf{N} = \mathbf{p} \times \mathbf{E}$. This will be true in general, not just for the above picture.

2. Now we add into the above situation some "mechanics" by giving each charge a mass m. What is the angular momentum of the dumbbell?

$\mathbf{L}_1 = \mathbf{r}_1 \times m\mathbf{v}_1$

We know that v_1 is constrained to a sphere, so we know that

$\mathbf{v}_1 = v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}} = -\mathbf{v}_2$

Then we have

$\mathbf{L}_1 = \mathbf{r}_1 \times m\mathbf{v}_1 = m\,[r\hat{\mathbf{r}}] \times [v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}}\,]$

$= mr(\,v_\theta\,\hat{\boldsymbol{\varphi}} - v_\varphi\hat{\boldsymbol{\theta}})$

$\mathbf{L}_2 = \mathbf{r}_2 \times m\mathbf{v}_2 = (-\mathbf{r}_1) \times (-m\mathbf{v}_1) = \mathbf{L}_1$

Then the dumbbell has total **L** of

$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = 2\mathbf{L}_1 = 2mr(\,v_\theta\,\hat{\boldsymbol{\varphi}} - v_\varphi\hat{\boldsymbol{\theta}})$ // agrees with below (F.2.1)

Note added: What is L^2 ?

$L^2 = (2mr)^2(v_\theta^2 + v_\varphi^2) = (2mr)^2\,(\,[r\,\dot{\theta}]^2 + [\,r\,\dot{\varphi}\sin\theta]^2)$

$= (2mr^2)^2\,(\,\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi})^2\,.$

Now back up and just write

$$\mathbf{L} = 2\mathbf{r}_1 \text{x } m\mathbf{v}_1$$

Then we know that

$$\dot{\mathbf{L}} = 2\mathbf{r}\text{x } m\dot{\mathbf{v}} = 2\mathbf{r}\text{x } m\mathbf{a} = 2mr\,\hat{\mathbf{r}} \text{ x } \mathbf{a}$$

and we read off for our constrained case,

$$\mathbf{a} = a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}} \qquad \text{// acceleration}$$

$$a_r = -r\dot{\theta}^2 - r\,\dot{\varphi}^2 \sin^2\theta$$

$$a_\theta = r\ddot{\theta} - r\,\dot{\varphi}^2 \sin\theta\cos\theta$$

$$a_\varphi = 2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta \ . \qquad \text{(E.3.6)}$$

Now do the cross product :

$$\hat{\mathbf{r}} \text{ x } \mathbf{a} = \hat{\mathbf{r}} \text{ x } [a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}}] = a_\theta\,\hat{\boldsymbol{\varphi}} - a_\varphi\,\hat{\boldsymbol{\theta}}$$

So then we have

$$\dot{\mathbf{L}} = 2mr(a_\theta\,\hat{\boldsymbol{\varphi}} - a_\varphi\,\hat{\boldsymbol{\theta}}) \qquad \text{// agrees with above (F.2.2)}$$

Now comes Newton's Law in our inertial Frame S

$$\mathbf{N} = \dot{\mathbf{L}}$$

and here then is our dipole dumbbell equation of motion

$$-2rqE\sin\theta\,\hat{\boldsymbol{\varphi}} = 2mr(a_\theta\,\hat{\boldsymbol{\varphi}} - a_\varphi\,\hat{\boldsymbol{\theta}}) \qquad \text{ok}$$

The first thing we learn is

$$a_\varphi = 0 \quad \Rightarrow \quad 2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta = 0 \qquad \text{ok} \qquad \text{// (E.3.6)}$$

so the particle 1 is not accelerating in the azimuthal direction (consistent with uniform precession!) And then we have

$$-2rqE\sin\theta = 2mr\,a_\theta$$

$$-qE\sin\theta = ma_\theta = m\,[\,r\ddot{\theta} - r\,\dot{\varphi}^2 \sin\theta\cos\theta]$$

So we arrive then at these two equations of motion

$$2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta = 0$$

$$-qE\sin\theta = m\left[r\ddot{\theta} - r\,\dot{\varphi}^2\sin\theta\cos\theta\right]$$

or

$$2\,\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta = 0$$

$$-(qE/mr)\sin\theta = \ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta]$$

somewhat similar to the fact that the mass dumbbell had two equations of motion. This suggests generally complicated solution motions. Rewrite

$$\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta + (qE/mr)\sin\theta = 0$$
$$\ddot{\varphi} + 2\,\dot{\theta}\,\dot{\varphi}\cot\theta = 0$$

Suppose I were to turn off the E field, so then no torque, so should have a trivial solution. Get

$$\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta = 0$$
$$\ddot{\varphi}\sin\theta + 2\,\dot{\theta}\,\dot{\varphi}\cos\theta = 0 \qquad\qquad E = 0$$

What solution do I expect? The dumbell just swings around in some plane. Suppose I assume that $\varphi = 0$ so in that plane. Then get

$$\ddot{\theta} = 0$$
$$0 = 0$$

and just says $\theta = At$, so that would be circular motion in the plane $\varphi = 0$.

Suppose instead $\theta = \pi/2$. Then

$$\dot{\varphi}^2\sin\theta\cos\theta = 0$$
$$\ddot{\varphi}\sin\theta = 0$$

These both say $\dot{\varphi} = 0$ so $\varphi = At$ and just goes around in THAT circle.

It would take me some work to figure out for a general plane. consider motion where the cross section of the circle is the -q to q line segment in the above figure. How do you characterize that plane? Well there is no single such plane. I am thinking a plane though origin and the two particles, but that only specifies a

line and there would be lots of planes, so let's forget this. In sphericals, the equation of an arbitrary circle is complicated.

Go back now to the general result with E on:

$$\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta + (qE/mr)\sin\theta = 0$$

$$\ddot{\varphi} + 2\,\dot{\theta}\,\dot{\varphi}\cot\theta = 0$$

I now seek a precession solution where θ = constant. Then we need to have

$$-\dot{\varphi}^2 \sin\theta \cos\theta + (qE/mr)\sin\theta = 0$$

$$\ddot{\varphi} = 0$$

The first equation says

$$-\dot{\varphi}^2 \cos\theta + (qE/mr) = 0$$

$$\dot{\varphi}^2 = (qE/mr\cos\theta)$$

So there is your precession solution! For any θ you specify, there is a solution. For $\theta = 0$ you get

$$\dot{\varphi}^2 = (qE/mr)$$

and it just spins around on the z axis. That seems weird. What makes it spin? Start over:

$$\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta + (qE/mr)\sin\theta = 0$$

$$\ddot{\varphi}\sin\theta + 2\,\dot{\theta}\,\dot{\varphi}\cos\theta = 0$$

Set $\theta \equiv 0$ and look for a solution

$$0 - 0 + 0 = 0$$
$$0 + 0 = 0$$

So I guess this says a static $\theta = 0$ is in fact a solution.

Now try a static $\theta = \theta_0$

$$-\dot{\varphi}^2 \cos\theta_0 + (qE/mr) = 0 \quad \Rightarrow \quad \dot{\varphi}^2 = (qE/mr\cos\theta_0)$$

$$\ddot{\varphi}\sin\theta_0 = 0 \quad \Rightarrow \quad \dot{\varphi} = \text{constant}$$

$$\dot{\varphi} = \sqrt{qE/(mr\cos\theta_0)} \qquad p \equiv 2rq$$

$\varphi(t) = \sqrt{qE/(mr\cos\theta_0)}\, t + \varphi_0$

If you try for $\theta_0 = \pi/2$, it has to spin infinitely fast to not deviate??? If it spun slowly, the torque would affect it.

So I think I have shown that there is a precession solution.

Dimension check

$\dot{\varphi}^2 = (qE/mr\cos\theta)$ RHS = force/ML = $MLT^{-2}/ML = T^{-2}$ correct

Express the result in terms of p:

$\dot{\varphi}^2 = (qE/mr\cos\theta)$ $p \equiv 2rq$ $q = p/2r$

$\dot{\varphi}^2 = (pE/2mr^2\cos\theta)$

so it still depends on r. But recall that

$L^2 = (2mr^2)^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\varphi}^2)$.

so in our simple case we get

$L^2 = (2mr^2)^2 \sin^2\theta\, \dot{\varphi}^2$

so

$L = (2mr^2) \sin\theta\, \dot{\varphi}$ // if $\dot{\varphi} > 0$

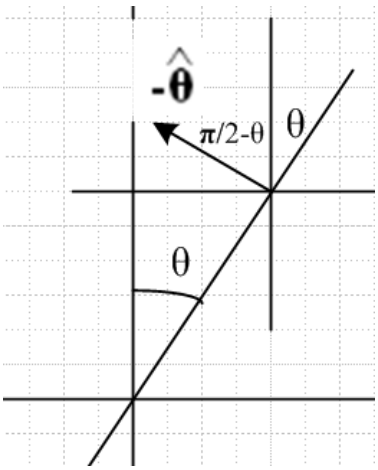
This L would be a constant of this precession motion.

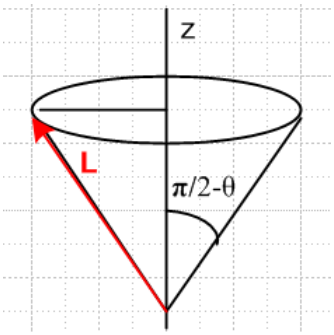
What does **L** do in this precession case?

$\mathbf{L} = 2mr(v_\theta\, \hat{\boldsymbol{\varphi}} - v_\varphi \hat{\boldsymbol{\theta}}) = 2mr([r\, \dot{\theta}]\, \hat{\boldsymbol{\varphi}} - [r\, \dot{\varphi} \sin\theta] \hat{\boldsymbol{\theta}})$

$= - 2mr^2\, \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\theta}}$

Now that seems a weird direction for $\hat{\boldsymbol{\theta}}$**,** but looking at r x p that is correct. This direction is moving, so we are not zero torque. So this **L** is going on a cone! What do we know about this cone?





So since

$$\mathbf{L} = 2mr^2 \dot{\varphi} \sin\theta_0 (-\hat{\boldsymbol{\theta}})$$

I conclude that **L** moves on a cone whose half angle is π/2-θ, shown on the right above. I can compute then that

$$L_z = \mathbf{L} \bullet \hat{\mathbf{z}} = 2mr^2 \dot{\varphi} \sin\theta_0 (-\hat{\boldsymbol{\theta}}) \bullet \hat{\mathbf{z}} = 2mr^2 \dot{\varphi} \sin\theta_0 (\sin\theta_0) = 2mr^2 \dot{\varphi} \sin^2\theta_0$$

$$L_x = \mathbf{L} \bullet \hat{\mathbf{x}} = 2mr^2 \dot{\varphi} \sin\theta_0 (-\hat{\boldsymbol{\theta}}) \bullet \hat{\mathbf{x}} = -2mr^2 \dot{\varphi} \sin\theta_0 (\cos\theta\cos\varphi) =$$

$$L_y = \mathbf{L} \bullet \hat{\mathbf{y}} = 2mr^2 \dot{\varphi} \sin\theta_0 (-\hat{\boldsymbol{\theta}}) \bullet \hat{\mathbf{y}} = -2mr^2 \dot{\varphi} \sin\theta_0 (\cos\theta\sin\varphi) =$$

$$L_\rho = 2mr^2 \dot{\varphi} \sin\theta\cos\theta = mr^2\dot{\varphi} \sin(2\theta)$$

So the dimensions on the cone are these

$$\text{height} = L_z = 2mr^2 \dot{\varphi} \sin^2\theta$$

$$\text{radius} = L_\rho = mr^2\dot{\varphi} \sin(2\theta)$$

I also know that

$$L^2 = [2mr^2 \dot{\varphi} \sin\theta_0]^2 = (2mr^2)^2 \dot{\varphi}^2 \sin^2\theta_0$$

$$L = 2mr^2\dot{\varphi}\sin\theta$$

So this is new to me really.

What do I know about $\boldsymbol{\omega}$? What is the definition of $\boldsymbol{\omega}$? You would think that would be a simple question for someone who just wrote 402 pages on frames of reference! See page 366 of Marion. If I think of Frame S' as embedded in the rotating sticks, then $\boldsymbol{\omega}$ is how Frame S' moves relative to Frame S. The Frame S' axes $\mathbf{e'_n}$ are rotating about the z axis. So ω must be along z. This is what Marion claims on page 366. So let's try that

$$\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$$

Then what happens to $\mathbf{L} = I\boldsymbol{\omega}$? Consider time instant where position as shown in my first picture above. Then we have a mass at distance rsinθ from the rotation axis, so $I = mr^2\sin^2\theta$. But what is the inertia tensor? For the upper mass, it is at location

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Then

$$I_{ij} \equiv \Sigma_\alpha m_\alpha [\, r_\alpha^2\delta_{ij} - (\mathbf{r}_\alpha)_i(\mathbf{r}_\alpha)_j \,]$$

$$I_{11} = m(r^2 - x^2) = m(r^2 - r^2\sin^2\theta\cos^2\varphi) = mr^2(1 - \sin^2\theta\cos^2\varphi)$$

$$I_{22} = mr^2(1 - \sin^2\theta\sin^2\varphi)$$

$$I_{33} = mr^2(1 - \cos^2\theta)$$

$$I_{12} = m(-xy) = -\,m(\, r\sin\theta\cos\varphi\; r\sin\theta\sin\varphi\,) = -mr^2\sin^2\theta\,\cos\varphi\sin\varphi$$

$$I_{13} = m(-xz) = -\,m(\, r\sin\theta\cos\varphi\; r\cos\theta\,) = -mr^2\sin\theta\cos\theta\cos\varphi$$

$$I_{23} = m(-yz) = -mr^2\sin\theta\cos\theta\sin\varphi$$

Wow. So then we have (for one of the two particles)

$$I = mr^2\begin{pmatrix} 1-\sin^2\theta\cos^2\varphi & -\sin^2\theta\,\cos\varphi\sin\varphi & -\sin\theta\cos\theta\cos\varphi \\ -\sin^2\theta\,\cos\varphi\sin\varphi & 1-\sin^2\theta\sin^2\varphi & -\sin\theta\cos\theta\sin\varphi \\ -\sin\theta\cos\theta\cos\varphi & -\sin\theta\cos\theta\sin\varphi & 1-\cos^2\theta \end{pmatrix}$$

For particle #2, located at the mirror image point, we know x,y,z→-x,-y,-z so I_{ij} for that particle is the same as for particle 1. Then

$$I = 2mr^2\begin{pmatrix} 1-\sin^2\theta\cos^2\varphi & -\sin^2\theta\,\cos\varphi\sin\varphi & -\sin\theta\cos\theta\cos\varphi \\ -\sin^2\theta\,\cos\varphi\sin\varphi & 1-\sin^2\theta\sin^2\varphi & -\sin\theta\cos\theta\sin\varphi \\ -\sin\theta\cos\theta\cos\varphi & -\sin\theta\cos\theta\sin\varphi & 1-\cos^2\theta \end{pmatrix}$$

So let's then try to compute L from $\mathbf{L} = I\boldsymbol{\omega}$:

$$\mathbf{L} = 2mr^2\omega \begin{pmatrix} 1-\sin^2\theta\cos^2\varphi & -\sin^2\theta\cos\varphi\sin\varphi & -\sin\theta\cos\theta\cos\varphi \\ -\sin^2\theta\cos\varphi\sin\varphi & 1-\sin^2\theta\sin^2\varphi & -\sin\theta\cos\theta\sin\varphi \\ -\sin\theta\cos\theta\cos\varphi & -\sin\theta\cos\theta\sin\varphi & 1-\cos^2\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= 2mr^2\omega \begin{pmatrix} -\sin\theta\cos\theta\cos\varphi \\ -\sin\theta\cos\theta\sin\varphi \\ \sin^2\theta \end{pmatrix}$$

Now I think $\omega = \dot{\varphi}$. So here is the first thing to check:

$$L^2 = (2mr^2)^2\dot{\varphi}^2 \left[(\sin\theta\cos\theta\cos\varphi)^2 + (\sin\theta\cos\theta\sin\varphi)^2 + (\sin^2\theta)^2 \right]$$

$$= (2mr^2)^2\dot{\varphi}^2[\sin^2\theta\cos^2\theta + (\sin^2\theta)^2]$$

$$= (2mr^2)^2\dot{\varphi}^2 \sin^2\theta [\cos^2\theta + \sin^2\theta]$$

$$= (2mr^2)^2\dot{\varphi}^2 \sin^2\theta$$

Earlier I got

$$L^2 = [2mr^2 \dot{\varphi} \sin\theta_0]^2 = (2mr^2)^2 \dot{\varphi}^2 \sin^2\theta_0$$

so this works!!! Now having done this check, write out in more detail

$$\mathbf{L} = 2mr^2\omega[-\sin\theta\cos\theta\cos\varphi\hat{\mathbf{x}} - \sin\theta\cos\theta\sin\varphi\hat{\mathbf{y}} + \sin^2\theta\,\hat{\mathbf{z}}]$$

$$= 2mr^2\omega\sin\theta [-\cos\theta\cos\varphi\hat{\mathbf{x}} - \cos\theta\sin\varphi\hat{\mathbf{y}} + \sin\theta\,\hat{\mathbf{z}}]$$

$$= -2mr^2\omega\sin\theta [\cos\theta\cos\varphi\hat{\mathbf{x}} + \cos\theta\sin\varphi\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}}]$$

Now quote from App E

$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\,\hat{\mathbf{x}} + \cos\theta\sin\varphi\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \qquad \text{(E.2.6)}$$

Therefore

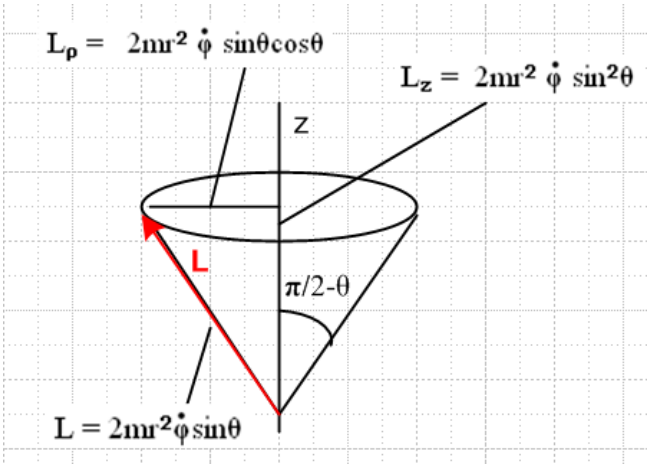
$$\mathbf{L} = -2mr^2\omega\sin\theta\hat{\boldsymbol{\theta}}$$

and above I got

$$\mathbf{L} = 2mr^2\,\dot{\varphi}\,\sin\theta_0\,(-\hat{\boldsymbol{\theta}})$$

So I have verified that $\mathbf{L} = I\boldsymbol{\omega}$ actually works in this sample problem. Constant checking!!

Meanwhile, here is more about the L cone



The precession motion of **L** is then

$$L_x = 2mr^2\sin\theta\,\dot{\varphi}\,\cos\theta\,\sin(\dot{\varphi}t) \qquad \dot{\varphi}^2 = (qE/mr\cos\theta)$$
$$L_y = 2mr^2\sin\theta\,\dot{\varphi}\,\cos\theta\,\cos(\dot{\varphi}t)$$
$$L_x = 2mr^2\sin\theta\,\dot{\varphi}\,\sin\theta\,\sin(\dot{\varphi}t)$$

Verify the cone half-angle

$$\sin(\pi/2-\theta) = L_\rho/L = 2mr^2\,\dot{\varphi}\,\sin\theta\cos\theta\ /\ 2mr^2\dot{\varphi}\sin\theta = \cos\theta \qquad \text{// yes!}$$

So this is a very strange problem. The figure axis is on a different cone of angle θ.

Here is one conclusion

$$\dot{\varphi} = \omega = \sqrt{qE/mr\cos\theta}$$

Note that for this problem we have

$$\mathbf{L} = 2mr^2\,\dot{\varphi}\,\sin\theta_0\,(-\hat{\boldsymbol{\theta}})$$

$\mathbf{p} = 2qr\,\hat{\mathbf{r}}$ = dipole moment.

Note that **L** and **p** are NOT in the same direction.

What is the corresponding magnetic problem?

The force is different in nature going from E to B. And there are no magnetic monopoles to put on the end of a stick. So I guess I have to put a bar magnet of some sort. It really seems to be a totally different problem. What exactly is the problem?

We know from Bleaneys that

$$\mathbf{N} = \boldsymbol{\mu} \times \mathbf{B}$$

where $\boldsymbol{\mu}$ is the magnetic moment of the dipole I guess made of two magnetic monopoles, but probably you could make a real magnet with a thin waist to do this?? The thing would precess like so

$$\dot{\phi}^2 = (\mu B/2mr^2\cos\theta)$$

$$L = (2mr^2)\ \sin\theta\ \dot{\phi}$$

But here is what we learn on the web

Larmor frequency [edit]

The angular momentum vector $\vec{J}$ precesses about the external field axis with an angular frequency known as the **Larmor frequency**,

$$\omega = -\gamma B$$

where ω is the angular frequency,[1] and B is the magnitude of the applied magnetic field. γ is (for a particle of charge $-e$) the gyromagnetic ratio,[2] equal to $-\frac{eg}{2m}$, where m is the mass of the precessing system, while g is the g-factor of the system. The g-factor is the unit-less proportionality factor relating the system's angular momentum to the intrinsic magnetic moment; in classical physics it is just 1.

They are saying that

$$\omega = (q/2m)B$$

But my model above is giving

$$\omega = \sqrt{qE/mr\cos\theta}$$

These results are not even close!!! How do you explain that ???

Site also says

$$\vec{\tau} = \vec{\mu} \times \vec{B} = \gamma \vec{J} \times \vec{B}$$

suggesting that

$$\boldsymbol{\mu} = \gamma\, \mathbf{J}$$

Magnetic Model Start Over

OK, we have a sphere of mass m and Frame S and Frame S' have their origins at the sphere center. The sphere has a magnetic moment $\boldsymbol{\mu}$ fixed to itself, it is a little round magnet at the origin. It is like the Earth hanging in space. Without a **B** field this sphere would do torque-free precession as it rotates, as the Earth does. Is that possible with all inertia moments the same? [but if round, no precession]

Look at the Section I.6 solution if a sphere. What happens in this case?

1. $\Omega' \equiv (\omega)'_3\ (I'_3-I'_1)/I'_1 = 0$. So here is what ω does in Frame S'

$$\begin{aligned} (\omega)'_1 &= \omega\sin\alpha \qquad\qquad \Omega' = \omega\cos\alpha\ (I'_3-I'_1)/I'_1 = 0 \\ (\omega)'_2 &= 0 \\ (\omega)'_3 &= \omega\cos\alpha\ . \qquad\qquad \alpha = \text{cone half-angle},\ 0 \le \alpha \le \pi \end{aligned} \tag{I.6.7}$$

It just sits in one spot. Next,

$$\begin{aligned} (L)'_1 &= I'_1\omega\sin\alpha \\ (L)'_2 &= 0 \\ (L)'_3 &= I'_1\omega\cos\alpha\ . \end{aligned} \tag{I.6.8}$$

Obviously we have $\mathbf{L} = I'_1\boldsymbol{\omega}$.

It just sits there too (of course). You can see that $L^2 = I'^2_1\omega^2$ here and $2T = I'_1\omega^2$. Euler angles do this

$$\begin{aligned} &\theta = \tan^{-1}[\ (I'_1/I'_3)\tan\alpha\ ] = \alpha \qquad\qquad // \ \theta(t) = \text{constant} \\ &\psi(t) = \pi/2 \quad \dot{\psi} = 0 \\ &\varphi(t) = (L/I'_1)t \qquad\qquad \dot{\varphi} = L/I'_1\ . \end{aligned}$$

I guess this means that it does not spin on its axis, but it does still precess. Is that possible?? What happens in Frame S components?

$$\begin{aligned} \omega_1 &= \omega\ \sin(\theta-\alpha)\ \cos[(L/I'_1)t\ ] \\ \omega_2 &= \omega\ \sin(\theta-\alpha)\ \sin[(L/I'_1)t\ ] \\ \omega_3 &= \omega\ \cos(\theta-\alpha)\ . \end{aligned} \tag{I.6.29}$$

What is this sphere doing? Look at Fig (I.6.30). Since $\theta-\alpha = 0$, both ω and L are straight up.

Question: If you slowly take $I'_3 \to I'_1$, what happens in (I.6.30)?

1. The **ω** cone narrows and approaches nothing, just ω vertical. [yes]

2. $\theta \rightarrow \alpha$. and $\Omega' \rightarrow 0$, so no rotation in the body frame. The ω vector does not move on the body cone, so you get a version of (I.6.30) where ω and L are aligned vertically, you can draw the body cone but it does nothing, **ω** just points up. The sphere just sits there and spins about the vertical axis.

Fact: a perfectly round sphere does not have any torque-free precession, it just sits and spins in some direction, as you would expect.

SO, I don't have to worry about such precession in my little spherical magnet deal.

Go back to dipole problem again.

Suppose you have a little sphere of mass M with a dipole moment **p** that is fixed. I think this is a different problem than my original electric dipole problem above. How are things different? Now the torque acts on a sphere, not on a dumbbell. I could integrate over differential dumbbells I suppose. But suppose we just say that **N** = **p** x **E** and that torque acts on the sphere of mass M. I think we get similar result,

$$\mathbf{N} = -pE\sin\theta\,\hat{\boldsymbol{\varphi}}$$

if **p** is mounted in my picture of the dumbbell. Equations of motion (I.4.3) seem pretty simple now

$$(N)'_1 = I'_1\,(\dot{\omega})'_1$$
$$(N)'_2 = I'_2\,(\dot{\omega})'_2$$
$$(N)'_3 = I'_3\,(\dot{\omega})'_3$$

But inertia is diagonal right in Frame S, so why not just write

$$N_1 = I_1\,\dot{\omega}_1$$
$$N_2 = I_1\,\dot{\omega}_2$$
$$N_3 = I_1\,\dot{\omega}_3 \quad \Rightarrow \quad \mathbf{N} = I_1\,\dot{\boldsymbol{\omega}} \qquad I_1 = (2/5)MR^2 \text{ I think. (yes, wiki)}$$

Then equation of motion is this

$$\mathbf{p} \times \mathbf{E} = I_1\,\dot{\boldsymbol{\omega}}$$

or

$$-pE\sin\theta\,\hat{\boldsymbol{\varphi}} = I_1\,\dot{\boldsymbol{\omega}}$$ [note that θ and $\hat{\boldsymbol{\varphi}}$ describe vector **r**, the spin axis]

That seems to say that

$$\dot{\boldsymbol{\omega}} = -(pE\sin\theta/I_1)\hat{\boldsymbol{\varphi}}$$

I think this alone means conical motion, but let's get it into Cartesian: [be careful!]

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}$$

So then have

$$\dot{\boldsymbol{\omega}} = -(pE\sin\theta/I_1)\,[\,-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}\,] = d\boldsymbol{\omega}/dt$$

Write

$$\boldsymbol{\omega} = \omega_x\hat{\mathbf{x}} + \omega_y\hat{\mathbf{y}}$$

$$\dot{\boldsymbol{\omega}} = \dot{\omega}_x\hat{\mathbf{x}} + \dot{\omega}_y\hat{\mathbf{y}}$$

Then have

$$\dot{\omega}_x\hat{\mathbf{x}} + \dot{\omega}_y\hat{\mathbf{y}} = -(pE\sin\theta/I_1)\,[\,-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}\,]$$

from which conclude that

$$\dot{\omega}_x = (pE\sin\theta/I_1)\sin\varphi$$
$$\dot{\omega}_y = -(pE\sin\theta/I_1)\cos\varphi$$
$$\dot{\omega}_z = 0$$

This does say that

$$\ddot{\omega}_x = (pE\sin\theta/I_1)\cos\varphi\,\dot{\varphi} = -\dot{\omega}_y\dot{\varphi} \qquad \text{[how do you know } \theta.\varphi \text{ are constants?]}$$
$$\ddot{\omega}_y = (pE\sin\theta/I_1)\sin\varphi\,\dot{\varphi} = \dot{\omega}_x\dot{\varphi}$$

Why am I having so much trouble integrating a trivial equation? The problem is that I don't really know what the symbol $\boldsymbol{\omega}$ means. For Euler angles I know that

$$\omega_1 = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi$$
$$\omega_2 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi$$
$$\omega_3 = \dot{\psi}\cos\theta + \dot{\varphi}\,. \qquad \text{// Frame S} \qquad (H.6.18)$$

I think in this problem that $\dot{\psi} = 0$ so we then have

$$(\omega)_x = \sin\varphi\,\dot{\theta} = 0$$
$$(\omega)_y = \cos\varphi\,\dot{\theta} = 0 \qquad \theta = \text{constant ansatz}$$

$(\omega)_z = \dot{\varphi}$ (H.4.12)

So maybe I can combine that with

So I end up with nothing but $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ and then contradiction:

$\dot{\omega}_x = (pE\sin\theta/I_1)\sin\varphi$

$\dot{\omega}_y = -(pE\sin\theta/I_1)\cos\varphi$

but

$(\omega)_x = 0$

$(\omega)_y = 0$

Back up and leave $\dot{\psi}$ in there to get

$(\omega)_x = \sin\theta\sin\varphi\,\dot{\psi} =$

$(\omega)_y = -\sin\theta\cos\varphi\,\dot{\psi}$

$(\omega)_z = \dot{\varphi} + \cos\theta\,\dot{\psi}$. (H.4.12)\

I am flailing away here!!! Go back to

$\dot{\boldsymbol{\omega}} = -(pE\sin\theta/I_1)\hat{\boldsymbol{\varphi}}$

$\dot{\omega}_x = (pE\sin\theta/I_1)\sin\varphi$

$\dot{\omega}_y = -(pE\sin\theta/I_1)\cos\varphi$

Just assume the $\varphi = \alpha t$ and see what happens

$\dot{\omega}_x = (pE\sin\theta/I_1)\sin(\alpha t)$

$\dot{\omega}_y = -(pE\sin\theta/I_1)\cos(\alpha t)$

$\omega_x = -(pE\sin\theta/I_1)(1/\alpha)\cos(\alpha t)$

$\omega_y = -(pE\sin\theta/I_1)(1/\alpha)\sin(\alpha t)$

$\omega_z = \omega_z$ = some constant.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.07: Electromagnetism II — November 21, 2012
Prof. Alan Guth

QUIZ 2 SOLUTIONS

QUIZ DATE: NOVEMBER 15, 2012

PROBLEM 1: THE MAGNETIC FIELD OF A SPINNING, UNIFORMLY CHARGED SPHERE *(25 points)*

This problem is based on Problem 1 of Problem Set 8.

A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) *(10 points)* What is the magnetic dipole moment of the sphere?

And here is the answer:

The total magnetic dipole moment is then

$$\vec{m} = \int \omega\rho r^2 \sin\theta\,(\pi r^2 \sin^2\theta)\,dr\,d\theta\,\hat{z}$$

$$= \pi\omega\rho \int_0^R r^4\,dr \int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta\,\hat{z}$$

$$= \pi\omega \frac{Q}{\frac{4}{3}\pi R^3}\frac{R^5}{5}\frac{4}{3} = \boxed{\frac{1}{5}Q\omega R^2\,\hat{z}\ .}$$

So if you model an electron in this manner, you get

$\mu = (1/5)\, q\, \omega\, R^2$

I have shown on the board that the precession rate is

$\dot{\phi} = -\mu B/I_1 = -B[(1/5)Q\omega R^2]/[(2/5)mR^2] = -(1/2)(Q/m)B$

Wiki is saying that

$\dot{\phi} = -\gamma B$ where γ = gyromagratio $= -(Q/2m)g = -(Q/2m)$

where g = 1 for classical calculation. So I think I am finally onto it!

Pieces are slowly assembling now.

[I don't think anything below is right, start over on this problem!]

Sub problem: Suppose you are told that

$\dot{\boldsymbol{\omega}} = K\hat{\boldsymbol{\varphi}}$ [be careful !!!]

How do you solve this equation for $\boldsymbol{\omega}$? This was puzzling me for quite a while, but now I know how to do it.

First, you think about spherical coordinates in ω-space. The radius vector usually called **r** is $\boldsymbol{\omega}$. Then

$\boldsymbol{\omega} = \omega\, \hat{\mathbf{r}}$ [but this assumes that ω points in the **r** direction!]

$\dot{\boldsymbol{\omega}} = \dot{\omega}\, \hat{\mathbf{r}} + \omega \partial_t \hat{\mathbf{r}} = \dot{\omega}\, \hat{\mathbf{r}} + \omega (\dot{\theta}\, \hat{\boldsymbol{\theta}} + \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\varphi}})$

So now you just equate

$\dot{\boldsymbol{\omega}} = K\hat{\boldsymbol{\varphi}} = \dot{\omega}\, \hat{\mathbf{r}} + \omega (\dot{\theta}\, \hat{\boldsymbol{\theta}} + \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\varphi}})$

and you conclude that

$\dot{\omega} = 0 \qquad \dot{\theta} = 0 \qquad K = \omega\dot{\varphi}\sin\theta$

So the length of the ω vector does not change, it sits at a fixed polar angle, and it rotates at

$\dot{\varphi} = K/(\omega\sin\theta).$

This is "cone motion". Now in the case above I have

$\dot{\boldsymbol{\omega}} = -(pE\sin\theta/I_1)\hat{\boldsymbol{\varphi}}$ so $K = -(pE\sin\theta/I_1)$

The solution is then conical motion with

$\dot{\varphi} = K/(\omega\sin\theta) = -(pE\sin\theta/I_1) / (\omega\sin\theta) = -(pE/\omega I_1)$

Now for a sphere we know that $I_1 = (2/5)mR^2$ so you end up with your dipoled sphere doing

$\dot{\varphi} = -pE/ [\, \omega(2/5)mR^2 \,] = -(5/2)pE/(\omega mR^2)$

The precession rate is proportional to E. Units are $Q\, L\, F/Q / (ML^2) = F/(ML) = MLT^{-2}/ ML = T^{-2}$

NOW, for a magnetic dipole you would since torque is **μ** x **B** instead of **p** x **E** you get

$$\dot{\phi} = -(5/2)\mu B/(\omega m R^2)$$

It is easy to show that if you have a uniform sphere of rotating charge you have

$$\mu = (1/5)\omega Q R^2$$

You then get

$$\dot{\phi} = -(5/2)[(1/5)\omega Q R^2]B/(\omega m R^2) = -(1/2)\, QB/m = -(Q/2m)\, B$$

and I think this is the Larmor situation for classical charged particles. This is a famous result and I have finally gotten it right. Recall from above.

Larmor frequency [edit]

The angular momentum vector $\vec{J}$ precesses about the external field axis with an angular frequency known as the **Larmor frequency**,

$$\omega = -\gamma B$$

where ω is the angular frequency,[1] and B is the magnitude of the applied magnetic field. γ is (for a particle of charge $-e$) the gyromagnetic ratio,[2] equal to $-\frac{eg}{2m}$, where m is the mass of the precessing system, while g is the g-factor of the system. The g-factor is the unit-less proportionality factor relating the system's angular momentum to the intrinsic magnetic moment; in classical physics it is just 1.

We also know that

$$\mathbf{L} = I_1\boldsymbol{\omega} = (2/5)mR^2\,\boldsymbol{\omega}$$

so L goes around the cone with **ω** . The $\dot{\phi}$ seems to be independent of θ, so it precesses at this rate no matter what initial θ you give it.

So now in this writeup I have solved three different problems, two being very close.