Electric dipole in a top PhL 3.29.17

I am starting completely over on this problem. First I had a dipole in a sphere, but I think I might just as well use a general symmetric top of any shape as in my drawings.

1. Conjecture: Consider these two systems :

A. Top with a uniform gravity force $\mathbf{F} = -mg\mathbf{z}$ acting on the top's center of mass.

B. Top with no gravity at all, but which contains an embedded electric dipole **p** which is aligned with the symmetry axis. This top is then surrounded by a uniform electric field $\mathbf{E} = -E\hat{\mathbf{z}}$

The conjecture is that these two systems have identical motions if you make parameter equates. For example, the two systems have very similar torques acting on the center of mass

gravity top: $\mathbf{N} = \mathbf{r}_{cms} \times \mathbf{F}$ $\mathbf{F} = -mg\hat{\mathbf{z}}$

dipole top: $\mathbf{N} = \mathbf{p} \times \mathbf{E}$ $\mathbf{E} = -E\hat{\mathbf{z}}$

Write these two torques again:

gravity top: $\mathbf{N} = [r_{cms}\hat{\mathbf{r}}] \times [-mg\hat{\mathbf{z}}] = -r_{cms}mg\ \hat{\mathbf{r}}\times\hat{\mathbf{z}}$

dipole top: $\mathbf{N} = [p\hat{\mathbf{r}}] \times [-E\hat{\mathbf{z}}] = -pE\ \hat{\mathbf{r}}\times\hat{\mathbf{z}}$

The torques have the same form, and are identical if we set

$r_{cms}mg = pE$

The force situation is a bit different. For the gravity top, $\mathbf{F} = 0$ because gravity is balanced by the pivot point force up. For the dipole top, $\mathbf{F} = 0$ without a pivot point.

What is the potential energy situation for these two systems?

$V = mgr_{cms}\cos\theta$	gravity top	max when $\theta = 0$, top vertical
$V = pE\cos\theta$	dipole top	max when $\theta = 0$, top antialigned with E

Again the connection is $mgr_{cms} = pE$.

So I am pretty sure these two systems are isomorphic with $r_{cms}mg = pE$.

2. Recall that the top general solution does not particularly simplify if you make the top be a sphere which is fixed at a pivot point. So if we embed an electric dipole into a spherical top, the motion is still highly complex and looks like the various simulations I could have done and which others did.

3. **Question**: Does the general top have a conical motion solution among its complex solutions? This would be a solution where θ = constant. This means u = constant, and we have

$$\dot{u}^2 = \beta(u-A)(u-B)(u-C) \qquad \text{(I.7.16)}$$

Goldstein discusses the general situation with his figure on page 167. In order to have a fixed θ solution, you have to have the two roots u_1 and u_2 be the same so the hump comes up and just touches the horizontal axis. Then the nutation will just vanish. So yes, a conical solution is possible.

Is there some fast way to find this solution?

1. The solution θ must satisfy this condition

$$\sin^2\theta(\alpha - \beta\cos\theta) - (b-a\cos\theta)^2 = 0$$

or

$$\sin^2\theta(\alpha - \beta\cos\theta) = (b-a\cos\theta)^2$$

Alternatively, we have

$$\dot{u}^2 = f(u) \qquad f(u) = \beta(u-A)(u-B)(u-C) \qquad \beta \equiv (2Mgr_{cms}/I'_1) > 0$$

The solution exists only when f(u) shown Goldstein page 167 has f'(u) = 0 when f(0) = 0. Note that root C is off to the right. I think this merely requires that A = B, In that case we will have f(u) = 0, and since the two roots are the same on page 167, one must have zero slope at that point.

So the condition for a conical solution is A = B where these are the two smallest roots. I know that

$$f(u) = \beta u^3 - (\alpha+a^2)u^2 + (2ab-\beta)u + (\alpha-b^2)$$

Maple chokes on writing down the roots of this thing. and the solution is stated page 32 of Schaum, so basically this is a numeric-only problem. So this is not a profitable way to find the conical solutions!

The top is Section I.7. But suppose you can adjust the four constants so as to have a conical solution at some angle θ. In the case, you would know that

$$\dot{\varphi} = \frac{b-a\cos\theta}{\sin^2\theta}$$

$$\dot{\psi} = a\,(I'_1/I'_3) - \cos\theta(\, b - \cos\theta\, a)/\sin^2\theta$$

so you would know the precession rate and you would know the spin rate.

Marion does more on finding this θ_0 point, but he never actually gets a solution.

My conclusion is that this conical solution is just not very interesting and it is hard to select constants so that it exists, and computing its value is a numerical-only problem!

General facts about the general top solution:

1. In general you get precession with some kind of nutation as shown in the pictures.

2. The **ω** vector is given by

$$\omega_1(t) = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi$$
$$\omega_2(t) = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi$$
$$\omega_3(t) = \dot{\psi}\cos\theta + \dot{\varphi} \qquad \text{(H.6.18)}$$

When you install your complicated solutions for θ(t),φ(t) and ψ(t), you will get some complicated motion of **ω**(t) which is NOT going to just be motion on a cone.

3. Although L_z = constant = p_φ , the other **L** components have no simple expressions. Neither **L** nor **ω** is even mentioned in Goldstein's 12 page treatment of the top. Nor are either mentioned in Marion. Nor in T&M. You would have to compute I in Frame S for the top and then get **L** from **L**= I**ω** .
Alternative to get **L** :

1. Compute (**ω**)' from

$$(\omega)'_3 = \text{constant} = (aI'_1/I'_3) \qquad \text{(I.7.7)}$$
$$(\omega)'_1 = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$(\omega)'_2 = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi \qquad \text{(H.6.19)}$$

2. Compute (I)' which should not be too hard for any kind of top. For a spherical top it is a diagonal matrix. But we know how to do this.

3. Then (L)' = (I)'(ω)' so to speak.

4. Then compute **L** = R(L)' using the Euler angle rotation.

You can see what they don't talk about the location of these vectors much. If the top is spinning fast, you would expect **ω** and **L** to be in the general direction of the figure axis.

Conclusions:

1. I *thought* that the "electric dipole embedded in a sphere" was going to be a simple problem. It turns out that it is isomorphic to the general axisymmetric top problem which has very complicated solutions, and the simple precessing solution is hard to identify. I cannot write an expression for the angle θ at which such a non-nutating solution would exist.

2. I thought that this problem would have a simple precession solution which I could then simply transfer over to the magnetic problem of a $\boldsymbol{\mu}$ in the **B** field. Then I was somehow going to come up with a derivation of the Larmor precession frequency which is so simple and basic.

3. Right now, I am confused at how the electric-dipole problem can be so complicated, and the magnetic dipole problem so simple. Offhand I would guess that the magnetic problem is also a full top problem with messy solutions. So at 6 PM 3/29/17 I am totally confused about this question.

The Magnetic Dipole Problem

Let's start with a "ring of sticky charge" of radius r, linear charge density λ and linear mass density κ. The total charge on the ring is $Q = 2\pi r\lambda$ and the mass is $M = 2\pi r\kappa$. Inertia moment is $I = Mr^2$.

The ring is set to rotate in the xy plane, center at origin, at angular rate $\boldsymbol{\omega}$. We then know that

$$\mathbf{L} = I\boldsymbol{\omega} = (Mr^2)\omega\hat{\mathbf{z}}$$

B&B page 131 say the mag moment of this ring is

$$\mu = i\,A = (r\lambda\omega)(\pi r^2) = \pi\lambda\omega r^3 \qquad\qquad Q\,T^{-1}\,L^2$$

so then

$$\boldsymbol{\mu} = \pi\lambda\omega r^3\,\hat{\mathbf{z}}$$

Let's now tip the right so it is at some θ and $\varphi = 0$. Then we can say

$$\boldsymbol{\omega} = \omega\hat{\mathbf{r}}$$
$$\mathbf{L} = (Mr^2)\omega\,\hat{\mathbf{r}}$$
$$\boldsymbol{\mu} = \pi\lambda\omega r^3\,\hat{\mathbf{r}}$$

$$\boldsymbol{\mu} = \pi\lambda\omega r^3\,\hat{\mathbf{r}} = (1/M)r\pi\lambda\,(M\omega r^2\,\hat{\mathbf{r}}) = (r\pi\lambda/M)\,\mathbf{L} \equiv \gamma\,\mathbf{L}\,, \quad \gamma = r\pi\lambda/M = (e/2m)$$

again

$$\boldsymbol{\mu} = \gamma\,\mathbf{L} = (e/2m)\,\mathbf{L}$$

So far, all is very simple.

Now we turn on a **B** field in the $\hat{\mathbf{z}}$ direction. We know this produces this torque on the ring

$$\mathbf{N} = \boldsymbol{\mu} \times \mathbf{B} = [\,\pi\lambda\omega r^3\,\hat{\mathbf{r}}] \times [B\hat{\mathbf{z}}] = \pi\lambda\omega r^3\,B\hat{\mathbf{r}}\times\hat{\mathbf{z}} = \pi\lambda\omega r^3 B\,[-\sin\theta\hat{\boldsymbol{\varphi}}]$$

$= - \pi\lambda\omega r^3 B\sin\theta\, \hat{\boldsymbol{\varphi}}$

I assume this torque is with respect to our CMS ring center which is at the origin. (no force, so origin does not matter)

Can also write

$\mathbf{N} = \boldsymbol{\mu} \times \mathbf{B} = \gamma\, \mathbf{L} \times \mathbf{B}$

The one equation I know is this

$\mathbf{N} = \dot{\mathbf{L}}$ $\qquad$ $\dot{\mathbf{L}} = \gamma\, \mathbf{L} \times \mathbf{B}$ $\qquad$ d**L** is perp to **L** so L = constant

Now when it precesses, assume that is very small somehow compared to the axial spinning. So we write

$\mathbf{L} = (Mr^2)\omega\, \hat{\mathbf{r}}$

$\dot{\mathbf{L}} = (Mr^2)\omega\, \partial_t \hat{\mathbf{r}} = (Mr^2)\omega\, [\, \dot{\theta}\, \hat{\boldsymbol{\theta}} + \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\varphi}}\,]$

where I just assume that ω does not change, magically. Newton's Law then says

$(Mr^2)\omega\, \dot{\theta}\, \hat{\boldsymbol{\theta}} = 0$

$(Mr^2)\omega\, \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\varphi}} = - \pi\lambda\omega r^3 B\sin\theta\, \hat{\boldsymbol{\varphi}}$

The first equation says θ = constant which is cone motion. The second says

$(Mr^2)\omega\, \dot{\varphi} = - \pi\lambda\omega r^3 B$

or

$M\, \dot{\varphi} = - \pi\lambda r B \qquad \gamma = r\pi\lambda/M$

or

$\dot{\varphi} = - \pi\lambda r B/ (M) = - B\gamma$ = "the Larmor frequency"

So my "first cut" here says that this ring of rotating sticky charge precesses on a cone with this rate.

What do others say about this? Goldstein says something page 176.

Suppose my sticky charge is made of particles of charge e and mass m. Then go back to the starting point which I quote,

"Let's start with a "ring of sticky charge" of radius r, linear charge density λ and linear mass density κ. The total charge on the ring is Q = 2πλ and the mass is M = 2πκ. Inertia moment is I = Mr^2. "

I think the implication is that $\lambda/\kappa = e/m = Q/M$ and $\lambda = (e/m)\kappa = (e/m)M/2\pi r$

$\lambda = (e/m)M/(2\pi r)$

This then says (factor of 2 now appears)

$\gamma = r\pi\lambda/M = (r\pi/M)*(e/m)M/(2\pi r) = (e/2m)$

Then the claim is

$\dot{\varphi} = -B\gamma = -B(e/2m)$

This agrees apart from cgs units with 5-87 Gold p 177.

But still, it seems that the fact that my ring of charged particles is precessing means there is a correction to L due to this precession motion, sort of $\omega_z = \dot{\varphi}$ contributing to L_z. Gold does comment on this as I knew he would, and in fact refers to his own article somewhere!

The Classical Motion of a Rigid Charged Body in a Magnetic Field
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The Classical Motion of a Rigid Charged Body in a Magnetic Field

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A frequently used classical analog to the quantum precession of a magnetic moment in a magnetic field is the motion in a uniform static magnetic field of a symmetrical rigid body charged so that the ratio of charge to mass density is uniform. In general, the figure axis of such a magnetic top both precesses and nutates, and the mechanism producing the nutation is qualitatively different from that responsible for the nutation of the familiar gravitational top. Unless the body is spherical the angular momentum vector also nutates in the course of the precession, a phenomenon caused by the deviation of the torque on the body from the customary, but here incorrect, expression $\mathbf{M}\times\mathbf{B}$. The uniform precession frequency of the magnetic top is *not* the Larmor frequency but involves in addition terms depending on higher powers of the magnetic field, a correction which corresponds to the quadratic Zeeman effect.

THE precession of a magnetic moment in a magnetic field is one of the most ubiquitous phenomena of modern physics. From the earliest investigations on the Zeeman effect through the latest nuclear resonance experiment it has played a vital role. In introducing the concept to the student, and even in much of our intuitive thinking about the Larmor precession, comparison is instinctively made to the familiar precession of a heavy top in a gravitational field. The exact classical analog, however, is not the heavy top but rather a charged rigid body in a magnetic field. Such a model shares with the nonclassical intrinsic "spin" of a fundamental particle the important characteristic that the magnetic moment is proportional to the angular momentum. The motion of a charged top differs both qualitatively and quantitatively from that of a heavy top, and affords considerable insight into the nature and genesis of the Larmor precession. A detailed discussion of the motion may therefore not be without interest.

To facilitate comparison of the heavy and charged tops Part I presents a summary of the salient features of the motion of the heavy top with one point fixed. This section also serves to introduce the notation and many of the methods of investigation. Part II discusses the precession and nutation of the figure axis and angular momentum vector of the charged top when released initially with zero precession and nutation velocities. Finally, Part III considers the initial precession velocity needed to insure uniform precession without nutation.

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I. THE HEAVY TOP IN A GRAVITATIONAL FIELD

Consider a symmetrical rigid body of mass M fixed at one point on the figure axis at a distance l from the center of gravity and placed in a uniform gravitational field. The orientation of the top is described by the usual Euler angles, θ giving the inclination of the figure axis to the vertical, φ the azimuth of the figure axis about the vertical, and ψ the rotation angle of the rigid body about its figure axis. Thus φ corresponds to the precessional motion of the top, while the change in θ describes the nutation of the symmetry axis. It will be assumed that at $t=0$ the top is spinning about its figure axis and is released without any initial precession or nutation velocity.

It will be sufficient for our needs merely to indicate the methods used and to quote the re-