Electric dipole in a sphere PhL 3.29.17

Step 1. If the dipole is embedded in a uniform density sphere, I know at once that all three inertia components are the same no matter how you choose the principal axes, so the Frame S' equations of motion are :

$$(N)'_1 = I'_1 (\dot{\omega})'_1 - (\omega)'_2(\omega)'_3 (I'_2 - I'_3)$$
$$(N)'_2 = I'_2 (\dot{\omega})'_2 - (\omega)'_3(\omega)'_1 (I'_3 - I'_1)$$
$$(N)'_3 = I'_3 (\dot{\omega})'_3 - (\omega)'_1(\omega)'_2 (I'_1 - I'_2) \ . \qquad \text{(I.4.3)}$$

and then

$$(N)'_1 = I'_1 (\dot{\omega})'_1$$
$$(N)'_2 = I'_1 (\dot{\omega})'_2$$
$$(N)'_3 = I'_1 (\dot{\omega})'_3 \qquad I'_1 = (2/5)Mr^2 \text{ looking it up on the web (could calc it here)}$$

Let $I'_1 = (2/5)Mr^2 \equiv J$. then the above three equations say

$$(N)'_i = J \ (\dot{\omega})'_i$$

Since this is true for all Frame S' components, it must be true as a vector equation, so

$$\mathbf{N} = J\dot{\boldsymbol{\omega}}$$

and therefore in Frame S we could write

$$N_i = J \ (\dot{\omega})_i$$

Step 2. We know that the torque on the dipole is this

$$\mathbf{N} = \mathbf{p} \times \mathbf{E}$$

based on our I.10 problem work. Thus, in Frame S we have

$$\varepsilon_{ijk} p_j E_k = J \ (\dot{\omega})_i$$

But only $E_3 = E$ exists in Frame S, so set k = 3 to get

$$\varepsilon_{ij3} p_j E = J \ (\dot{\omega})_i$$

Write out these equations

i=1 $\quad \varepsilon_{123} p_2 E = J\,(\dot{\omega})_1 \quad \Rightarrow \quad p_2 E = J\,(\dot{\omega})_1$

i=2 $\quad \varepsilon_{213} p_1 E = J\,(\dot{\omega})_2 \quad \Rightarrow \quad -p_1 E = J\,(\dot{\omega})_2$

i = 3 $\quad \varepsilon_{3j3} p_j E = J\,(\dot{\omega})_3 \quad \Rightarrow \quad 0 = J\,(\dot{\omega})_3$

We then end up with :

ω_3 = constant

$(\dot{\omega})_1 = [p_2 E/J]$

$(\dot{\omega})_2 = -\,[p_1 E/J]$

Now since $\mathbf{p} = p\hat{\mathbf{r}}$ by fiat (I declare the embedded **p** to be in the $\hat{\mathbf{r}}$ direction) , we have

$\mathbf{p} = p\,[\,\sin\theta\cos\varphi,\ \sin\theta\sin\varphi,\cos\theta] = [p_1,p_2,p_3]$

This our equations can be written

ω_3 = constant

$(\dot{\omega})_1 = [pE/J]\,\sin\theta\sin\varphi$

$(\dot{\omega})_2 = -\,[pE/J]\,\sin\theta\cos\varphi$

At this point I know nothing about $\theta(t)$ and $\varphi(t)$ so cannot simply integrate. So what do I do next??

What is **L** ?

$\mathbf{L} = I\boldsymbol{\omega} = J\boldsymbol{\omega}$ $\qquad$ J = scalar = $(2/5)MR^2$

$\dot{\mathbf{L}} = J\,\dot{\boldsymbol{\omega}}$

What does Newton say in Frame S?

$\mathbf{N} = \dot{\mathbf{L}}$

Thus we know that

$\mathbf{p} \times \mathbf{E} = J\,\dot{\boldsymbol{\omega}}$

$\varepsilon_{ijk} p_j E \delta_{k3} = J\,(\dot{\boldsymbol{\omega}})_i$

but I already know this, so nothing new here!

Maybe look for some conserved quantities! Maybe I should be starting by writing force and torque as I did in the electric dipole dumbbell problem.

Plan A: Imagine that the dumbbell is embedded in the sphere so that there are charges q and -q as before, and the stick length is 2r where r is the radius of the mass sphere. . Then p = 2rq. So this is just the dumbbell problem with a different inertia tensor. In this case I have

$$F = 0$$

$$\mathbf{N} = \mathbf{N}_1 + N_2 = 2\mathbf{N}_1 = -2rqE\sin\theta\,\hat{\boldsymbol{\varphi}}$$

Thus the equation of motion can be written

$$\mathbf{p} \times \mathbf{E} = J\,\dot{\boldsymbol{\omega}}$$

$$\mathbf{N} = J\,\dot{\boldsymbol{\omega}} = \dot{\mathbf{L}}$$

$$-2rqE\sin\theta\,\hat{\boldsymbol{\varphi}} = J\,\dot{\boldsymbol{\omega}}$$

$$\dot{\boldsymbol{\omega}} = -\,(2rqE/J)\sin\theta\,\hat{\boldsymbol{\varphi}}$$

or

$$\dot{\boldsymbol{\omega}} = -\,(pE/J)\sin\theta\,\hat{\boldsymbol{\varphi}}$$

So here then is the equations of motion in terms of spherical unit vectors. I have to be careful now not to make my previous mistake in trying to solve this equation! I think I can still claim that

$$(\dot{\mathbf{L}})_r = 0 \qquad (\dot{\boldsymbol{\omega}})_r = 0 \qquad N_r = 0$$

$$(\dot{\mathbf{L}})_\theta = 0 \qquad (\dot{\boldsymbol{\omega}})_\theta = 0 \qquad N_\theta = 0$$

$$(\dot{\mathbf{L}})_\varphi = -pE\sin\theta\ . \qquad (\dot{\boldsymbol{\omega}})_\varphi = -(pE/J)\sin\theta \qquad N_\varphi = -2rqE\sin\theta \qquad \text{(I.10.9)}$$

But now I no longer have (F.2.2) for **L**, maybe I have to complete **L** somehow!! Go back to

$$\omega_3 = \text{constant}$$

$$(\dot{\omega})_1 = [pE/J]\sin\theta\sin\varphi$$

$$(\dot{\omega})_2 = -\,[pE/J]\sin\theta\cos\varphi$$

I have four unknowns $\omega_1(t)$, $\omega_2(t)$, $\theta(t)$ and $\varphi(t)$, but only two equations! I need more equations!

How about these equations,

$$\omega_1 = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi$$
$$\omega_2 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi$$
$$\omega_3 = \dot{\psi}\cos\theta + \dot{\varphi}\,. \qquad \text{// Frame S} \qquad \text{(H.6.18)}$$

Can I use them somehow? Can I assume ψ = constant? Is there torque about the $\hat{r}$ axis? Well above I have written $N_r = 0$ so do I know somehow that $\dot{\psi} = N_r$?

Look at $(\dot{\mathbf{L}})_r = 0$ $\quad \dot{\mathbf{L}} \bullet \hat{\mathbf{r}} = 0$

$(d\mathbf{L}/dt)_S \bullet \hat{\mathbf{r}} = 0$ $\quad [(d\mathbf{L}/dt)_S]_r = 0$

I used this kind of fact in (H.4.1).

$$\boldsymbol{\omega}_\varphi = \dot{\varphi}\,\hat{\mathbf{z}}$$
$$\boldsymbol{\omega}_\theta = \dot{\theta}\,\hat{\boldsymbol{\xi}}'$$
$$\boldsymbol{\omega}_\psi = \dot{\psi}\,\hat{\mathbf{z}}' \quad . \qquad \text{(H.4.1)}$$

How did I get these three equations? How do I know that $\dot{\theta}$ does not contribute to $\boldsymbol{\omega}_\varphi$? // OK, I pondered this and added a new paragraph to frames doc below (H.4.1), I think this was worth pondering and getting stated.

So OK, what does the information $(\dot{\mathbf{L}})_r = 0$ tell me?

Here is what I want to say

$N_r = 0 \qquad \Rightarrow \qquad \psi$ = constant

How do I justify this?

$$\mathbf{N} = \dot{\mathbf{L}}$$

$$N_r = (\dot{\mathbf{L}})_r = 0$$

Can I say this

$(d\mathbf{L}/dt)_r = dL_r/dt$?

If I *could* say this, then I would have

$N_r = dL_r/dt = 0 \quad \Rightarrow L_r$ = constant

Then I have to argue that $L_r = J\omega_r$ and then $\omega_r = 0$ and so then ψ = constant

But this is a commutation theorem issue. I really have

$$\mathbf{L} = L_r\hat{\mathbf{r}} + L_\theta\hat{\boldsymbol{\theta}} + L_\varphi\hat{\boldsymbol{\varphi}}$$

Then

$$\dot{\mathbf{L}} = \dot{L}_r\,\hat{\mathbf{r}} + L_r\,\partial_t\hat{\mathbf{r}} + \dot{L}_\theta\hat{\boldsymbol{\theta}} + L_\theta\,\partial_t\hat{\boldsymbol{\theta}} + \dot{L}_\varphi\hat{\boldsymbol{\varphi}} + L_\varphi\,\partial_t\hat{\boldsymbol{\varphi}}$$

$$= \dot{L}_r\,\hat{\mathbf{r}} + L_r\,[\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}]$$
$$+ \dot{L}_\theta\hat{\boldsymbol{\theta}} + L_\theta\,[-\dot{\theta}\,\hat{\mathbf{r}} + \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\varphi}}\,] + \dot{L}_\varphi\hat{\boldsymbol{\varphi}} + L_\varphi\,[-\dot{\varphi}\sin\theta\,\hat{\mathbf{r}} - \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\theta}}]$$

$$= \{\dot{L}_r - \dot{\theta}L_\theta - \dot{\varphi}\sin\theta L_\varphi\}\hat{\mathbf{r}} + \{L_r\dot{\theta} + \dot{L}_\theta - \dot{\varphi}\cos\theta\,L_\varphi\}\hat{\boldsymbol{\theta}} + \{L_r\,\dot{\varphi}\sin\theta + L_\theta\,\dot{\varphi}\cos\theta + \dot{L}_\varphi\}\hat{\boldsymbol{\varphi}}$$

This tells me that

$$(\dot{\mathbf{L}})_r = \dot{L}_r - \dot{\theta}L_\theta - \dot{\varphi}\sin\theta L_\varphi$$

$$(\dot{\mathbf{L}})_\theta = L_r\dot{\theta} + \dot{L}_\theta - \dot{\varphi}\cos\theta\,L_\varphi$$

$$(\dot{\mathbf{L}})_\varphi = L_r\,\dot{\varphi}\sin\theta + L_\theta\,\dot{\varphi}\cos\theta + \dot{L}_\varphi$$

I then combine these with results from above

$(\dot{\mathbf{L}})_r = 0$	$(\dot{\boldsymbol{\omega}})_r = 0$	$N_r = 0$	
$(\dot{\mathbf{L}})_\theta = 0$	$(\dot{\boldsymbol{\omega}})_\theta = 0$	$N_\theta = 0$	
$(\dot{\mathbf{L}})_\varphi = -pE\sin\theta$.	$(\dot{\boldsymbol{\omega}})_\varphi = -(pE/J)\sin\theta$	$N_\varphi = -2rqE\sin\theta$	(I.10.9)

and I end up with these equations of motion

$$\dot{L}_r - \dot{\theta}L_\theta - \dot{\varphi}\sin\theta L_\varphi = 0$$

$$L_r\dot{\theta} + \dot{L}_\theta - \dot{\varphi}\cos\theta\,L_\varphi = 0$$

$$L_r\,\dot{\varphi}\sin\theta + L_\theta\,\dot{\varphi}\cos\theta + \dot{L}_\varphi = -pE\sin\theta$$

This is very ugly stuff. I suppose I can say

$$\dot{\mathbf{L}} = J\,\dot{\boldsymbol{\omega}}$$

and rewrite this mess in terms of ω components. I hate it! But recall my conjecture from above

$$(d\mathbf{L}/dt)_{\mathbf{r}} = dL_{\mathbf{r}}/dt \qquad ?$$

Well in fact

$$(d\mathbf{L}/dt)_{\mathbf{r}} = \dot{L}_{\mathbf{r}} - \dot{\theta}L_{\boldsymbol{\theta}} - \dot{\varphi}\sin\theta L_{\boldsymbol{\varphi}}$$

so forget that pathway!

I really need to solve things within Frame S'!

Status: Today 3.29.17 I have spent perhaps 4 hours on this problem, and I have made zero progress. Typical for the course!

Is this perhaps the same as the top problem? There the torque is **N** = **r** x [-m**g**] about the center of mass of the top, whereas here the torque is **N** = **p** x **E** = 2q **r** x **E** , sure looks the same. Both torques are about the CMS. Then connection ought then to be 2q**E** = -m**g** or 2qE = -mg.

So I think it really is the top problem, but with a simpler inertia tensor.