Determinant of an inertia matrix PhL 3.27.17

From (I.1.6),

$$L_i = I_{ij}\,(\omega)_j \qquad I_{ij} = \Sigma_\alpha m_\alpha [\, r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j \,] \qquad // \; r_\alpha^2 = (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_i \qquad \textit{(I.1.6)}$$

Since we have done no examples of computing inertia matrices in the space frame, we do it here. This is a special case I where in effect there is only one term in the Σ_α sum. This is for a dumbbell of two equal masses and equal radii. This is based on

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

$$I_{11} = 2m(r^2 - x^2) = 2m(r^2 - r^2\sin^2\theta\cos^2\varphi) = 2mr^2\,(1 - \sin^2\theta\cos^2\varphi)$$

$$I_{22} = 2m(r^2 - y^2) = 2mr^2\,(1 - \sin^2\theta\sin^2\varphi)$$

$$I_{33} = 2m(r^2 - z^2) = 2m(r^2 - r^2\cos^2\theta\,) = 2mr^2\sin^2\theta$$

$$I_{12} = 2m(-xy) = -2mr^2\sin^2\theta\cos\varphi\sin\varphi$$

$$I_{13} = 2m(-xz) = -2mr^2\sin\theta\cos\theta\cos\varphi$$

$$I_{23} = 2m(-yz) = -2mr^2\sin\theta\cos\theta\sin\varphi$$

Then

$$I = 2mr^2 \begin{pmatrix} 1 - \sin^2\theta\cos^2\varphi & -\sin^2\theta\cos\varphi\sin\varphi & -\sin\theta\cos\theta\cos\varphi \\ -\sin^2\theta\cos\varphi\sin\varphi & 1 - \sin^2\theta\sin^2\varphi & -\sin\theta\cos\theta\sin\varphi \\ -\sin\theta\cos\theta\cos\varphi & -\sin\theta\cos\theta\sin\varphi & \sin^2\theta \end{pmatrix}$$

Maple shows this matrix has zero determinant so cannot be inverted!!!!

```
st := sin(theta):
ct := cos(theta):
sp := sin(phi):
cp := cos(phi):
m11 := 1-(st*cp)^2:
m12 := -st^2*cp*sp:
m13 := -st*ct*cp:
m21 := m12:
m22 := 1-(st*sp)^2:
m23 := -st*ct*sp:
m31:= m13:
m32 := m23:
m33 := st^2:
M := matrix(3,3,[m11,m12,m13,m21,m22,m23,m31,m32,m33]);
```

$$M := \begin{bmatrix} 1-\sin(\theta)^2\cos(\phi)^2 & -\sin(\theta)^2\cos(\phi)\sin(\phi) & -\sin(\theta)\cos(\theta)\cos(\phi) \\ -\sin(\theta)^2\cos(\phi)\sin(\phi) & 1-\sin(\theta)^2\sin(\phi)^2 & -\sin(\theta)\cos(\theta)\sin(\phi) \\ -\sin(\theta)\cos(\theta)\cos(\phi) & -\sin(\theta)\cos(\theta)\sin(\phi) & \sin(\theta)^2 \end{bmatrix}$$

```
det(M); simplify(%);
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$$\sin(\theta)^2 - \sin(\theta)^4\sin(\phi)^2 - \sin(\theta)^2\cos(\theta)^2\sin(\phi)^2 - \sin(\theta)^4\cos(\phi)^2 - \sin(\theta)^2\cos(\theta)^2\cos(\phi)^2$$

$$0$$

I was not expecting that. I may have suggested using that method somewhere!!!

Why is the determinant 0? Look at the general single-term expression for I_{ij}

$$I_{ij} = r^2\delta_{ij} - x_i x_j$$

$$\det(I) = \varepsilon_{ijk} I_{1i} I_{2j} I_{3k} = \varepsilon_{ijk}(r^2\delta_{1i} - x_1x_i)(r^2\delta_{2j} - x_2x_j)(r^2\delta_{3k} - x_3x_k)$$

There are many terms. Here is the first term

$$\varepsilon_{ijk} r^2\delta_{1i} r^2\delta_{2j} r^2\delta_{3k} = r^6\,\varepsilon_{ijk}\delta_{1i}\delta_{2j}\delta_{3k} = r^6\,\varepsilon_{123} = r^6$$

Then three terms like this

$$\varepsilon_{ijk}(r^2\delta_{1i})(r^2\delta_{2j})(-x_3x_k) = r^4\,\varepsilon_{12k}(-x_3x_k) = r^4\,\varepsilon_{123}(-x_3x_3) = -r^4x_3^2$$

So I think the sum of those three terms will be $-r^4\ r^2 = -r^6$

Then three terms like this

$$\varepsilon_{ijk} r^2\delta_{1i}(-x_2x_j)(-x_3x_k) = r^2\,x_2x_3\,\varepsilon_{1jk}\,x_jx_k = 0 \text{ by symmetry}$$

The last term is

$$-\varepsilon_{ijk}\,x_1x_i\,x_2x_jx_3x_k = 0 \text{ by symmetry}$$

Therefore the determinant is

$\det(I) = r^6 - r^6 = 0$ so can NEVER invert it!!!

But if diagonal, you CAN invert it, so what gives??

People on the web do invert some inertia tensors, so I think it is my special case that is not invertible where there is only a single particle.

What is the trace of I? For single particle,

$I = r^2\delta_{ij} - x_i x_j$

$tr(I) = tr(r^2\delta_{ij} - x_i x_j) = \Sigma_i (r^2\delta_{ii} - x_i x_i) = 3r^2 - \Sigma_i x_i^2 = 2r^2$??

The diagonal elements are: $r^2 - x^2$, $r^2 - y^2$, $r^2 - z^2$

Their sum is definitely $2r^2$. Also,

$\det(e^I) = e^{tr(I)} = \exp(2r^2)$

This says nothing about det(I).