This appears to be a rewrite of Appendix I where new sections are in red. These changes are all installed, so this doc is just for archiving.

Appendix I: Rigid Body Dynamics

We shall develop here the equations of motion for a rigid body's angular momentum vector $\boldsymbol{\omega}(t)$. This is done using $\boldsymbol{\omega}$ components $(\omega)'_i$ in Frame S', the body frame. It is shown that one might as well choose the Frame S' axes to line up with some set of principal axes of the rigid body since this diagonalizes the inertia tensor, simplifying equations. We then examine the general torque-free solution for $\boldsymbol{\omega}$ using the construction of Poinsot, and then analytically solve for $\boldsymbol{\omega}$ for the special axisymmetric case. The results for $\boldsymbol{\omega}$ are obtained in both Frame S' and inertial Frame S, and appropriate "cone pictures" are displayed. It is noted that the Earth exhibits a tiny torque-free precession called the Chandler wobble. We then consider the torque-present problem of a spinning top. For the Earth, weak torques are applied by the Sun and the Moon causing a slow precession known as the precession of the equinoxes. In the final section we derive an expression for the torque of the Sun or Moon on the slightly oblate Earth.

1.1 The Appearance of the Inertia Tensor

Assume that Frame S' is embedded into a rigid body, so that Frame S' and that rigid body are rotating at angular velocity $\boldsymbol{\omega}$ relative to an inertial Frame S. This is the situation of our "non-swap notation". Goldstein uses the appropriate term "body frame" for such a Frame S'. All sensible authors use "swap notation" for this analysis to avoid an avalanche of prime symbols (or other labels), but we shall be obstinate in using the non-swap notation because it forces us to think carefully about many details. We have to decide whether to put a prime or not put a prime on each entity.

In Frame S' the rigid body is at rest, so the Frame S' velocity $\mathbf{v}'_\alpha$ of any particle α of the rigid body vanishes, $\mathbf{v}'_\alpha = 0$. For particle α we also have $\mathbf{p}'_\alpha = m_\alpha \mathbf{v}'_\alpha = 0$ and $\mathbf{L}'_\alpha = \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha = 0$. In particular, the total Frames S' angular momentum of the rigid body is $\mathbf{L}' = \Sigma_\alpha \mathbf{L}'_\alpha = 0$. These facts are quite obvious but we state them anyway. Note that α here is a label and not a component index. We shall use α in this manner as a particle label, and the usual i,j,k indices as component indices. Obviously, for a continuous rigid body Σ_α is really an integration over the particles of the body.

In Frame S, however, the rigid body has some non-vanishing angular momentum $\mathbf{L} = \Sigma_\alpha \ \mathbf{r}_\alpha \times m\mathbf{v}_\alpha$.

In what follows, we shall be very careful with the use of "primes" to clearly show what objects are the natural objects in what frames, and what frame components are being evaluated. As noted in Section 1.3, in order to mark our "natural" objects with primes or no primes depending on whether they are associated with Frame S' or Frame S, we must use the Passive View of rotation transformations. For vectors this means $(\mathbf{V})' = R\mathbf{V}$ and for rank-2 tensors $(T)' = RTR^{-1}$ (with Frame S' components $(V)'_i$ and $(T)'_{ij}$).

<u>Angular Momentum, the Inertia Tensor, and Kinetic Energy</u>

We shall assume Special Case #1 where the $\boldsymbol{\omega}$ rotation axis passes through the origin of Frame S. In this case recall from (6.11) that

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} \qquad \text{// Special Case \#1 only} \qquad (6.11)$$

Comment: This is valid also for Special Case #4 where the two Frame origins coincide so the $\boldsymbol{\omega}$ axis passes through both origins.

For a rigid body particle α one has, from (6.11) above,

$$\mathbf{v}_\alpha = \mathbf{v}'_\alpha + \boldsymbol{\omega} \times \mathbf{r}_\alpha = \mathbf{0} + \boldsymbol{\omega} \times \mathbf{r}_\alpha = \boldsymbol{\omega} \times \mathbf{r}_\alpha \ . \qquad (I.1.1)$$

Then the total Frame S angular momentum of the rigid body is (with respect to the Frame S origin),

$$\mathbf{L} = \Sigma_\alpha \ \mathbf{r}_\alpha \times \mathbf{p}_\alpha = \Sigma_\alpha \ \mathbf{r}_\alpha \times m_\alpha \mathbf{v}_\alpha = \Sigma_\alpha \ m_\alpha \mathbf{r}_\alpha \times (\boldsymbol{\omega} \times \mathbf{r}_\alpha)$$

$$= \Sigma_\alpha \ m_\alpha [\ r_\alpha^2 \boldsymbol{\omega} - (\mathbf{r}_\alpha \bullet \boldsymbol{\omega})\ \mathbf{r}_\alpha\ ] \ . \qquad \text{//} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C} \qquad (I.1.2)$$

Taking Frame S components,

$$L_i = \Sigma_\alpha \ m_\alpha [\ r_\alpha^2 \omega_i - (\mathbf{r}_\alpha \bullet \boldsymbol{\omega})\ (\mathbf{r}_\alpha)_i\ ]$$

$$= \Sigma_\alpha \ m_\alpha [\ r_\alpha^2 \Sigma_j \omega_j \delta_{ij} - \Sigma_j (\mathbf{r}_\alpha)_j \omega_j\ (\mathbf{r}_\alpha)_i\ ]$$

$$= \Sigma_\alpha \ m_\alpha \Sigma_j [\ r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j\ ]\ \omega_j$$

$$= \Sigma_j \ \{\ \Sigma_\alpha m_\alpha [\ r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j\ ]\ \}\ \omega_j$$

$$= \Sigma_j \ I_{ij} \omega_j$$

where

$$I_{ij} \equiv \Sigma_\alpha m_\alpha [\ r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j\ ] \qquad = \text{the inertia tensor} \qquad (I.1.3)$$

or $\quad I_{ij} \equiv \int dx_1 dx_2 dx_3\ \rho(\mathbf{x})[\ r^2 \delta_{ij} - x_i x_j\ ] \ . \qquad$ // continuum notation

We rewrite (I.1.3) as a vector equation,

$$\mathbf{L} = I\boldsymbol{\omega} \ . \qquad (I.1.4)$$

Exercise: Write an expression for the Dirac inertia operator $\mathcal{I}$ (script I, not T) :

$$\langle \mathbf{e}_i | \ \mathcal{I} \ | \mathbf{e}_j \rangle = I_{ij} = \Sigma_\alpha m_\alpha [\ r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j\ ]$$

$$= \Sigma_\alpha m_\alpha [\ r_\alpha^2 \langle \mathbf{e}_i | \mathbf{e}_j \rangle - \langle \mathbf{e}_i | \mathbf{r}_\alpha \rangle \langle \mathbf{r}_\alpha | \mathbf{e}_j \rangle \ = \ \langle \mathbf{e}_i | \ \{\ \Sigma_\alpha m_\alpha [\ r_\alpha^2 \ \mathbf{1} - |\mathbf{r}_\alpha\rangle\langle\mathbf{r}_\alpha|\ \}\ | \mathbf{e}_j \rangle$$

$$\Rightarrow \ \mathcal{I} = \Sigma_\alpha m_\alpha \ (\ r_\alpha^2 \mathbf{1} - |\mathbf{r}_\alpha\rangle\langle\mathbf{r}_\alpha|\ ) \ . \qquad (I.1.5)$$

In equation (I.1.4) **L**, I and **ω** are all associated with a Frame S observer. **L** is the "natural" angular momentum of the rigid body as seen in Frame S, and **ω** is the angular velocity of Frame S' (and its rigid body) as seen from Frame S. Finally I_{ij} are the inertia tensor components viewed from Frame S.

Although **L** = I**ω** is a Frame S equation, as with any vector equation one can take *components* of the equation in either Frame S or Frame S' :

$$L_i = I_{ij}\,(\omega)_j \qquad I_{ij} = \Sigma_\alpha m_\alpha [\, r_\alpha^2\delta_{ij} - (\mathbf{r}_\alpha)_i(\mathbf{r}_\alpha)_j \,] \qquad // \; r_\alpha^2 = (\mathbf{r}_\alpha)_i(\mathbf{r}_\alpha)_i$$

$$(L)'_i = (I)'_{ij}\,(\omega)'_j \qquad (I)'_{ij} = \Sigma_\alpha m_\alpha [\, r'^2_\alpha\delta_{ij} - (\mathbf{r}_\alpha)'_i(\mathbf{r}_\alpha)'_j \,] . \quad // \; r'^2_\alpha = (\mathbf{r}_\alpha)'_i(\mathbf{r}_\alpha)'_i \tag{I.1.6}$$

Notice in $(I)'_{ij}$ that $(\mathbf{r}_\alpha)'_i = \langle \mathbf{e}'_i | \mathbf{r}_\alpha \rangle$ is constant in time. The components of a vector static in Frame S' do not change in time, although Frame S' moves relative to Frame S. Thus $\partial_t(\mathbf{r}_\alpha)'_i = 0$. The magnitude of the vector $\mathbf{r}_\alpha$ is obviously constant in time, but we verify: $\partial_t r'^2_\alpha = \partial_t[(\mathbf{r}_\alpha)'_i(\mathbf{r}_\alpha)'_i] = (\mathbf{r}_\alpha)'\partial_t[(\mathbf{r}_\alpha)'_i] = 0$. Therefore $\partial_t(I)'_{ij} = 0$ for the entire tensor (I'). In contrast, $\partial_t I_{ij} \neq 0$.

Exercise: Verify that $(I)' = RIR^{-1}$, thus showing that I is a rank-2 tensor (in the Passive View) :

$$(I)'_{ij} = R_{ia}I_{ab}R^{-1}_{bj} = R_{ia}[\,\Sigma_\alpha m_\alpha (\, r_\alpha^2\delta_{ab} - (\mathbf{r}_\alpha)_a(\mathbf{r}_\alpha)_b \,)]R^{-1}_{bj}$$

$$= \Sigma_\alpha m_\alpha (r_\alpha^2\, R_{ia}\delta_{ab}R^{-1}_{bj} - R_{ia}(\mathbf{r}_\alpha)_a R^{-1}_{bj}(\mathbf{r}_\alpha)_b \,)$$

$$= \Sigma_\alpha m_\alpha (r_\alpha^2 R_{ia}R^{-1}_{aj} - [R_{ia}(\mathbf{r}_\alpha)_a][R_{jb}(\mathbf{r}_\alpha)_b] \,)$$

$$= \Sigma_\alpha m_\alpha (r_\alpha^2\, \delta_{ij} - (\mathbf{r}_\alpha)'_i(\mathbf{r}_\alpha)'_j) \; . \qquad \text{QED}$$

In what follows, we ignore kinetic energy associated with motion of the rigid body center of mass. We think of this center of mass being at rest in both Frame S and Frame S'

The total rotational Frame S kinetic energy T of the rigid body is given by, using (I.1.1),

$$T = \Sigma_\alpha(1/2)m_\alpha v_\alpha^2 = \Sigma_\alpha(1/2)m_\alpha\; \mathbf{v}_\alpha \bullet (\boldsymbol{\omega} \times \mathbf{r}_\alpha) = \Sigma_\alpha(1/2)m_\alpha\; \boldsymbol{\omega} \bullet (\mathbf{r}_\alpha \times \mathbf{v}_\alpha) \quad // \text{ cyclic rule}$$

$$= \Sigma_\alpha\; (1/2)\; \boldsymbol{\omega} \bullet (\mathbf{r}_\alpha \times m_\alpha\mathbf{v}_\alpha) = (1/2)\; \boldsymbol{\omega} \bullet \Sigma_\alpha\; \mathbf{r}_\alpha \times \mathbf{p}_\alpha = (1/2)\; \boldsymbol{\omega} \bullet \mathbf{L}$$

$$= (1/2)\; \boldsymbol{\omega} \bullet (\mathrm{I}\boldsymbol{\omega}) \; . \qquad // \text{ (I.1.4)} \tag{I.1.7}$$

We pause to note the various notations one can use for this dot product :

$$\boldsymbol{\omega} \bullet (\mathrm{I}\boldsymbol{\omega}) = \boldsymbol{\omega}^{\mathrm{T}}\mathrm{I}\boldsymbol{\omega} \text{ (matrix notation)} = \omega_i I_{ij}\omega_j = \omega_i\omega_j I_{ij} \text{ (all Frame S components)}$$

$$= \langle \boldsymbol{\omega} \,|\, \mathcal{I} \,|\, \boldsymbol{\omega} \rangle \quad \text{(Dirac notation, where } \mathcal{I} \text{ is the abstract inertia operator}$$

$= \langle \boldsymbol{\omega} | \mathbf{e}_i \rangle\langle \mathbf{e}_i | \mathcal{I} | \mathbf{e}_j \rangle\langle \mathbf{e}_j | \boldsymbol{\omega} \rangle = \omega_i \, I_{ij} \, \omega_j$ // completeness (1.1.20)

$= \boldsymbol{\omega} \bullet \mathrm{I} \bullet \boldsymbol{\omega}$ (Goldstein p 149 5-15) $= \boldsymbol{\omega} \overset{\leftrightarrow}{\mathrm{I}} \boldsymbol{\omega} = \boldsymbol{\omega} \mathrm{I} \boldsymbol{\omega}$. (I.1.8)

Comments: We feel it is useful to keep stressing these notational issues:

• The Frame S' kinetic energy T' = (1/2) $\boldsymbol{\omega}' \bullet \mathrm{I}' \bullet \boldsymbol{\omega}' = 0$ because $\boldsymbol{\omega}' = 0$, meaning $(\boldsymbol{\omega}')_i = 0$. This is so because the rigid body is at rest in Frame S' -- nothing is moving in Frame S' !

• The Frame S kinetic energy T = (1/2) $\boldsymbol{\omega} \bullet \mathrm{I} \bullet \boldsymbol{\omega}$ can be written out in either Frame S or Frame S' components.

$$T = (1/2)\ \boldsymbol{\omega} \bullet \mathrm{I} \bullet \boldsymbol{\omega} = (1/2)\, \omega_i \omega_j I_{ij} = (1/2)\, (\omega)'_i (\omega)'_j (\mathrm{I})'_{ij} \,. \quad \text{(I.1.9)}$$

Proof: In Dirac notation,

$$T = (1/2) \langle \boldsymbol{\omega} | \mathcal{I} | \boldsymbol{\omega} \rangle = (1/2) \langle \boldsymbol{\omega} | \mathbf{e}_i \rangle\langle \mathbf{e}_i | \mathcal{I} | \mathbf{e}_j \rangle\langle \mathbf{e}_j | \boldsymbol{\omega} \rangle = (1/2)\, \omega_i I_{ij} \omega_j$$

$$T = (1/2) \langle \boldsymbol{\omega} | \mathcal{I} | \boldsymbol{\omega} \rangle = (1/2) \langle \boldsymbol{\omega} | \mathbf{e}'_i \rangle\langle \mathbf{e}'_i | \mathcal{I} | \mathbf{e}'_j \rangle\langle \mathbf{e}'_j | \boldsymbol{\omega} \rangle = (1/2) (\omega)'_i (\mathrm{I})'_{ij} (\omega)'_j \,.$$

Using the Goldstein double dot notation, kinetic energy can be rewritten as

$$T = (1/2)\, (\hat{\boldsymbol{\omega}} \bullet \mathrm{I} \bullet \hat{\boldsymbol{\omega}})\, \omega^2 = (1/2)\, I\, \omega^2 \qquad \text{where} \quad I \equiv \hat{\boldsymbol{\omega}} \bullet \mathrm{I} \bullet \hat{\boldsymbol{\omega}} \quad \text{(I.1.10a)}$$

in analogy with $T = (1/2)mv^2$ for linear motion. The scalar object I is called the **moment of inertia** of the rigid body *about* the $\hat{\boldsymbol{\omega}}$ axis,

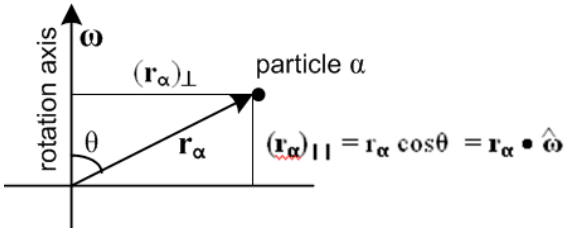
$$I \equiv \hat{\boldsymbol{\omega}} \bullet \mathrm{I} \bullet \hat{\boldsymbol{\omega}} = \hat{\omega}_i \hat{\omega}_j I_{ij} = \hat{\omega}_i \hat{\omega}_j \Sigma_\alpha m_\alpha [\, r_\alpha^2 \delta_{ij} - (\mathbf{r}_\alpha)_i (\mathbf{r}_\alpha)_j \,]$$

$$= \Sigma_\alpha m_\alpha [\, r_\alpha^2 \hat{\omega}_i \hat{\omega}_j \delta_{ij} - (\mathbf{r}_\alpha)_i \, \hat{\omega}_i (\mathbf{r}_\alpha)_j \hat{\omega}_j \,]$$

$$= \Sigma_\alpha m_\alpha [\, r_\alpha^2 - (\mathbf{r}_\alpha \bullet \hat{\boldsymbol{\omega}})^2] = \Sigma_\alpha m_\alpha [\, r_\alpha^2 - (\mathbf{r}_\alpha)_{||}^2]$$

$$= \Sigma_\alpha m_\alpha (\mathbf{r}_\alpha)_\perp^2, \quad \text{(I.1.10b)}$$

where $(\mathbf{r}_\alpha)_\perp$ is the perpendicular distance of particle α from the $\boldsymbol{\omega}$ rotation axis,



(I.1.10c)

I.2 Rigid Body Equations of Motion : Part 1

Up to this point, there has been no requirement that Frame S be an inertial frame, but we now add that requirement. Then Newton's Second Law (11.3.2) (angular motion) applied in Frame S states,

$$\mathbf{N} = \dot{\mathbf{L}} \qquad // \ \dot{\mathbf{L}} = \partial_{\mathtt{S}}\mathbf{L} = (d\mathbf{L}/dt)_{\mathtt{S}} \tag{I.2.1}$$

where we set the torque and angular momentum reference point to the Frame S origin. This law is valid because Frame S is an inertial frame so there are no fictitious torques. $\mathbf{N}$ is the Frame S torque, and $\dot{\mathbf{L}}$ is the natural time derivative of natural $\mathbf{L}$ in Frame S (see (1.8.4) and (1.9.3)).

Using the G Rule (2.1) one finds, in the notation of (1.8.3) where $\partial_{\mathtt{S}} \equiv (d/dt)_{\mathtt{S}}$ and $\partial_{\mathtt{S'}} \equiv (d/dt)_{\mathtt{S'}}$,

$$\dot{\mathbf{L}} \equiv \partial_{\mathtt{S}}\mathbf{L} = \partial_{\mathtt{S'}}\mathbf{L} + \boldsymbol{\omega} \times \mathbf{L} \tag{2.1}$$

so Newton (I.2.1) says

$$(\partial_{\mathtt{S'}}\mathbf{L}) + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N} \ . \tag{I.2.2}$$

Note that $(\partial_{\mathtt{S'}}\mathbf{L})$ is not a "natural" object in the sense of Section 1.8 since $\mathbf{L}$ is a Frame S object, but the time derivative is taken in Frame S', the body frame. Now consider,

$$(\partial_{\mathtt{S'}}\mathbf{L}) = (\partial_{\mathtt{S'}}[\mathrm{I}\boldsymbol{\omega}]) = (\partial_{\mathtt{S'}}\mathrm{I})\boldsymbol{\omega} + \mathrm{I}(\partial_{\mathtt{S'}}\boldsymbol{\omega}) \ . \tag{I.2.3}$$

We noted above that the inertia tensor is constant in Frame S' so we expect that $(\partial_{\mathtt{S'}}\mathrm{I}) = 0$. To make sure, we examine the components of $(\partial_{\mathtt{S'}}\mathrm{I})$ in Frame S',

$$\begin{aligned} [(\partial_{\mathtt{S'}}\mathrm{I})]'_{\mathtt{ij}} &= \partial_{\mathtt{S'}}(\mathrm{I})'_{\mathtt{ij}} && // \text{ commutation theorem (1.11.1) applied to a tensor} \\ &= \partial_{\mathtt{t}}(\mathrm{I}')_{\mathtt{ij}} && // \ (1.10.1) \\ &= 0 \ . && // \text{ shown below (I.1.6)} \end{aligned} \tag{I.2.4}$$

If $[(\partial_{\mathtt{S'}}\mathrm{I})]'_{\mathtt{ij}} = 0$ for all components, then $(\partial_{\mathtt{S'}}\mathrm{I}) = 0$ as a tensor statement. We also know from (2.6) that

$$\partial_{\mathtt{S'}}\boldsymbol{\omega} = \partial_{\mathtt{S}}\boldsymbol{\omega} = \partial_{\mathtt{t}}\boldsymbol{\omega} = \dot{\boldsymbol{\omega}} \ . \tag{2.6}$$

Therefore (I.2.3) reads,

$$(\partial_{\mathtt{S'}}\mathbf{L}) = (\partial_{\mathtt{S'}}[\mathrm{I}\,\boldsymbol{\omega}]) = (\partial_{\mathtt{S'}}\mathrm{I})\,\boldsymbol{\omega} + \mathrm{I}\,(\partial_{\mathtt{S'}}\boldsymbol{\omega}) = 0 + \mathrm{I}\,\dot{\boldsymbol{\omega}} = \mathrm{I}\,\dot{\boldsymbol{\omega}} \ . \tag{I.2.5}$$

Inserting this into (I.2.2) then produces our vector equation of motion for $\boldsymbol{\omega}$,

$$\mathbf{N} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) \,. \qquad \text{(I.2.6)}$$

Taking components in Frame S' (the body frame) one finds

$$(N)'_i = (I)'_{ij}\,(\dot{\omega})'_j + \varepsilon_{ijk}(\omega)'_j(I)'_{ka}\,(\omega)'_a \,. \qquad \text{(I.2.7)}$$

In what follows we shall always solve (I.2.6) in Frame S' components because only in Frame S' can the inertia tensor be diagonalized in a manner that applies for all time t.

I.3 Diagonalization of the Inertia Tensor in Frame S'

So far we have not paid much attention to the manner in which the Frame S' basis vectors are selected, and so far we have called them $\mathbf{e}'_i$. One is of course free to define new orthonormal Frame S' basis vectors $\mathbf{e}''_i$ according to

$$\mathbf{e}''_i \equiv Q_{ij}\,\mathbf{e}'_j \qquad \text{or} \qquad |\,\mathbf{e}''_i\rangle \equiv Q_{ij}\,|\,\mathbf{e}'_j\rangle \qquad \Rightarrow \qquad \langle\mathbf{e}'_k\,|\,\mathbf{e}''_i\rangle = Q_{ik}$$

where Q is some arbitrary rotation matrix ($Q^{-1} = Q^{T}$). Then in this new basis the inertia tensor is

$$(I)''_{ij} = \langle\mathbf{e}''_i|\,\mathcal{I}\,|\,\mathbf{e}''_j\rangle = \langle\mathbf{e}''_i\,|\mathbf{e}'_a\rangle\langle\mathbf{e}'_a|\,\mathcal{I}\,|\mathbf{e}'_b\rangle\langle\mathbf{e}'_b|\,\mathbf{e}''_j\rangle =$$

$$= Q_{ia}(I)'_{ab}Q_{jb} = [Q\,I\,Q^{-1}]_{ij} \qquad // \;\; (I)'' = Q\,I\,Q^{-1}$$

Note that $(I)'_{ij} = \Sigma_\alpha m_\alpha [\, r'^2_\alpha \delta_{ij} - (\mathbf{r}_\alpha)'_i(\mathbf{r}_\alpha)'_j \,]$ is a *symmetric* matrix, $(I)'_{ij} = (I)'_{ji}$.

A staple matrix theorem is that any real symmetric matrix M can be brought to diagonal form by a "similarity transformation" with some real orthogonal matrix S, so $\Lambda = S^{-1}MS$ where Λ is diagonal. There is a specific method for determining a suitable matrix S that does this diagonalization task. If we then select $Q = S^{-1}$, the matrix $(I)'' = Q\,I\,Q^{-1}$ will be diagonal in the $\mathbf{e}''_i$ basis. The basis vectors $\mathbf{e}''_i$ are called "**the principal axes**" of the rigid body in question, and $Q = S^{-1}$ is "**the principal axis transformation**". Note that the principle axes are fixed to a rigid body and do not change relative to the body as the body is reoriented (like the center of mass, and unlike the center of gravity, as both discussed in Appendix D).

Realizing this fact, we might as well start off selecting $\mathbf{e}'_i$ to be a set of principle axes. From now on, then, we shall *assume* that the $\mathbf{e}'_i$ have been so selected and thus the inertia tensor I is diagonal in Frame S' components. (Remember that the Frame S inertia tensor I' is identically zero.)

Comments: In general, any real symmetric matrix M can be diagonalized by some real orthogonal matrix S to become the diagonal matrix $\Lambda = S^{-1}MS$ where then $\Lambda = \mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)$. Suppose the matrix M has some eigenvectors $\mathbf{v}_i$ with eigenvalues μ_i so that $M\mathbf{v}_i = \mu_i\mathbf{v}_i$. Then since $M = S^{-1}MS$ one gets,

$(S\Lambda S^{-1})\mathbf{v}_i = \mu_i \mathbf{v}_i \;\Rightarrow\; \Lambda(S^{-1}\mathbf{v}_i) = \mu_i(S^{-1}\mathbf{v}_i)$

$\Rightarrow \quad \Lambda \mathbf{w}_i = \mu_i \mathbf{w}_i \,. \qquad // \; \mathbf{w}_i \equiv S^{-1}\mathbf{v}_i$

But the eigenvalues of a diagonal matrix Λ *are* those diagonal elements, so we identify $\lambda_i = \mu_i$ and conclude that when M is diagonalized to Λ, the diagonal elements are the eigenvalues of M. The eigenvalues can be found by writing the eigenvalue equation as $(M - \mu_i 1)\mathbf{v}_i = 0$. If $\det(M-\mu_i 1) \neq 0$ one can invert this equation to conclude that $\mathbf{v}_i = 0$ which is wrong, so one must have $\det(M-\mu_i 1) = 0$. This so-called "secular equation" or "characteristic equation" is then a cubic in variable μ_i which can be solved for three values of μ_i (the three roots of the secular equation).

In the complex world, the claim is that any Hermitian matrix H can be diagonalized by a unitary transformation U. A matrix H is Hermitian if $H^{\dagger} = H$($H^{T*} = H$), and a matrix U is unitary if $U^{\dagger}U = 1$ so $U^{\dagger} = U^{-1}$. Then one has $H_d = U^{-1}HU$. As above, the diagonal elements of H_d are the eigenvalues of H. It is easy to show that the eigenvalues of H (which are the diagonal elements of H_d) are real:

$H_d{}^* = U^{*-1}H^*U^* = U^{T*} H^{T*}U^{*-1T} = U^{\dagger} H^{\dagger}U^{-1\dagger} = U^{-1}HU = H_d.$

In quantum mechanics, all observable quantities X are represented by Hermitian operators (matrices) and the diagonal elements of the diagonalized X_d (the eigenvalues of X) are the possible physical values the observable can take.

Given that the $\mathbf{e}'_i$ are selected as a set of principal axes for a rigid body, the diagonal elements of the diagonal inertia tensor can be written I'_i and are called the **principal moments of inertia**. So

$$(I)'_{ij} = I'_i \, \delta_{ij} \quad . \tag{I.3.1}$$

We maintain a prime on I'_i as a reminder that these are associated with Frame S' where the inertia tensor I has been made diagonal by selection of the $\mathbf{e}'_i$ basis vectors of Frame S'.

I.4 Rigid Body Equations of Motion : Part 2

We now insert the diagonal form (I.3.1) for $(I)'_{ij}$ into the equations of motion (I.2.7) to get

$(N)'_i = I'_i \, \delta_{ij} \, (\dot{\omega})'_j + \varepsilon_{ijk}(\omega)'_j \, I'_k \, \delta_{ka} \, (\omega)'_a$

$= I'_i \, (\dot{\omega})'_i + \varepsilon_{ijk}(\omega)'_j \, I'_k(\omega)'_k$

$$= I'_i \, (\dot{\omega})'_i + \varepsilon_{ijk}(\omega)'_j(\omega)'_k \, I'_k \quad . \tag{I.4.1}$$

Setting i =1 gives

$(N)'_1 = I'_1 \, (\dot{\omega})'_1 + \varepsilon_{1jk}(\omega)'_j(\omega)'_k \, I'_k$

$$= I'_1 (\dot{\omega})'_1 + \varepsilon_{123}(\omega)'_2(\omega)'_3 I'_3 + \varepsilon_{132}(\omega)'_3(\omega)'_2 I'_2$$

$$= I'_1 (\dot{\omega})'_1 + (\omega)'_2(\omega)'_3 I'_3 - (\omega)'_3(\omega)'_2 I'_2$$

$$= I'_1 (\dot{\omega})'_1 - (\omega)'_2(\omega)'_3 (I'_2 - I'_3) \,. \qquad \text{(I.4.2)}$$

Examining the other components (or just doing cyclic permutations), we may state the three component rigid-body equations of motion as follows (all components are Frame S' components),

$$\begin{aligned}
(N)'_1 &= I'_1 (\dot{\omega})'_1 - (\omega)'_2(\omega)'_3 (I'_2 - I'_3) \\
(N)'_2 &= I'_2 (\dot{\omega})'_2 - (\omega)'_3(\omega)'_1 (I'_3 - I'_1) \\
(N)'_3 &= I'_3 (\dot{\omega})'_3 - (\omega)'_1(\omega)'_2 (I'_1 - I'_2) \,.
\end{aligned} \qquad \text{(I.4.3)}$$

This is a non-linear system of first-order ODE's and they appear in Goldstein p 158 (5-34) or GPS p 200 (5.39'). At this point Goldstein is using our swap notation so there are no primes on any of the quantities.

For torque-free motion one then has

$$\begin{aligned}
I'_1 (\dot{\omega})'_1 &= (\omega)'_2(\omega)'_3 (I'_2 - I'_3) \\
I'_2 (\dot{\omega})'_2 &= (\omega)'_3(\omega)'_1 (I'_3 - I'_1) \\
I'_3 (\dot{\omega})'_3 &= (\omega)'_1(\omega)'_2 (I'_1 - I'_2) \,.
\end{aligned} \qquad \text{(I.4.4)}$$

Goldstein p 159 (GPS p 201) notes that equations (I.4.4) can in principle be analytically solved in terms of elliptic functions, but that the resulting complicated solutions are not very enlightening. We have seen what such solutions look like for the spherical pendulum in (C.5.13), and the elliptic forms appear again in (I.7.17) for the treatment of the gravity top.