The free precession picture PhL 3.14.17

For a rigid object that is axially symmetric, the symmetry axis is a principal axis and the other two orthonormal principal axes can be selected in any manner to lie in the plane perpendicular to the symmetry axis. For a pancake, this symmetry axis has the largest principal moment, while for a pencil it has the smallest principal moment. The other two principal moments are equal. If $\mathbf{e}'_3$ is the symmetry axis, then $I'_1 = I'_2$ and the last of equations (G.8.18) reads $0 = I'_3(\dot{\omega})'_3$, so $(\omega)'_3$ = constant. The other two equations (G.8.18) are then easily solved. Write them out,

$$(\dot{\omega})'_1 = (\omega)'_3\,(\omega)'_2(I'_1 - I'_3)/ I'_1 = \Omega'(\omega)'_2 \qquad \text{where} \qquad \Omega' \equiv (\omega)'_3\,(I'_1-I'_3)/I'_1$$

$$(\dot{\omega})'_2 = -(\omega)'_3\,(\omega)'_1\,(I'_1 - I'_3)/I'_1 = -\Omega'(\omega)'_1 \qquad \textit{(G.8.18)}$$

Then

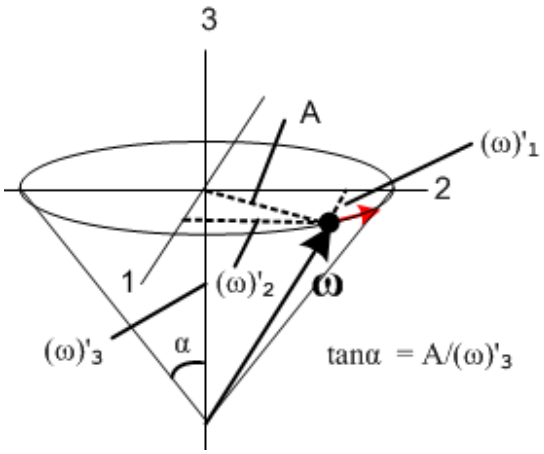
$$(\ddot{\omega})'_1 = \Omega'(\dot{\omega})'_2 = -\,\Omega'^2\,'(\omega)'_1$$

Selecting an arbitrary time origin, we can write the solution as

$$(\omega)'_1 = A\sin(\Omega' t)$$

$$(\omega)'_2 = (\dot{\omega})'_1 / \Omega' = A\cos(\Omega' t)\ . \qquad (G.8.20)$$

The Frame S $\boldsymbol{\omega}$ vector (showing Frame S' components) is thus doing conical motion:



The kinetic energy of this rigid object is (with diagonal I)

$$T = (1/2)\,\boldsymbol{\omega}\bullet I\bullet\boldsymbol{\omega} = (1/2)\,(\omega)'_i(I)'_{ij}(\omega)'_j = (1/2)\ I'_i(\omega)'^2_i$$

$$= (1/2)I'_1[(\omega)'^2_1 + (\omega)'^2_2] + (1/2)I'_1(\omega)'^2_3$$

$$= (1/2)I'_1\,A^2 + (1/2)I'_1(\omega)'^2_3 \qquad (G.8.21)$$

If the rigid body has kinetic energy T, then the precession amplitude is given by solving the above for A,

$$A = (2T/I'_1) - (\omega)'^2_3 \qquad \text{(G.8.22)}$$

What are the angular momentum components in Frame S' ??

$$(L)'_1 = I'_1\,(\omega)'_1 = I'_1\,A\sin(\Omega' t)$$
$$(L)'_2 = I'_1\,(\omega)'_2 = I'_1\,A\cos(\Omega' t)$$
$$(L)'_3 = I'_3\,(\omega)'_3 = I'_3\,(\omega)'_3$$

The first two components are changing in time. But it seems that since $\mathbf{N} = \dot{\mathbf{L}}$ and $\mathbf{N} = 0$ in Frame S, we should have $\mathbf{L}$ = constant. How does this fit with the moving L components above?

Just for interest, angular momentum magnitude squared for this rigid body is

$$L^2 = (L)'_i\,(L)'_i = I'_i\,(\omega)'_i\,I'_i\,(\omega)'_i = I'^2_i(\omega)'^2_i =$$

$$= I'^2_1[(\omega)'^2_1 + (\omega)'^2_2] + I'^2_3(\omega)'^2_3$$

$$= I'^2_1A^2 + I'^2_3(\omega)'^2_3 \qquad \text{(G.8.23)}$$

and is thus distributed some into the spinning direction and some into the precession directions.

Imagine the $\mathbf{L}$ vector *fixed* in Frame S and I make its components be

$$L_1 = 0$$
$$L_2 = 0$$
$$L_3 = L_3 \qquad\qquad \mathbf{L} = L_3\hat{\mathbf{z}}$$

What are the Frame S' components of this $\mathbf{L}$?

$$(L)'_1 = \mathbf{L}\bullet\hat{\mathbf{x}}' = L_3\,\hat{\mathbf{z}}\bullet\hat{\mathbf{x}}' = L_3$$

$$\hat{\mathbf{z}} = (\sin\psi\sin\theta)\hat{\mathbf{x}}' + (\cos\psi\sin\theta)\hat{\mathbf{y}}' + (\cos\theta)\,\hat{\mathbf{z}}' \qquad \text{(G.5.32)}$$

$$\hat{\mathbf{z}}\bullet\hat{\mathbf{x}}' = \sin\psi\sin\theta$$

So I conclude that

$$(L)'_x = L_3\sin\psi\sin\theta = L_3\sin\psi(t)\sin\theta = I'_1\,A\sin(\Omega' t)$$

$$(L)'_y = L_3\cos\psi(t)\sin\theta = I'_1\,A\cos(\Omega' t)$$

$(L)'_z = L_3\cos\theta = I'_3 (\omega)'_3$

These equations seem inconsistent. The only solution is to have $\psi(t) = \Omega' t$ but I thought we had $\psi(t)$ being rotation about the figure axis which is more like $\psi(t) = (\omega)'_3 t$ which is a large number.

Let's try brute force to compute the Frame S components of $\boldsymbol{\omega}$. What I have in Frame S' is this

$$(\omega)'_1 = A\sin(\Omega' t) = \dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$(\omega)'_2 = (\dot{\omega})'_1 / \Omega' = A\cos(\Omega' t) = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$(\omega)'_3 = K = \dot{\phi}\cos\theta + \dot{\psi}$$

I know that

$$(\boldsymbol{\omega})' = R\boldsymbol{\omega} \quad \text{so} \quad \boldsymbol{\omega} = R^{-1}(\boldsymbol{\omega})'$$

$$\omega_i = (R^{-1})_{ij}\ (\omega)'_j$$

But what is R? Should I use the Euler angles R to relate body frame to space frame? Let's try that. Then

$$R = R_z(-\psi)R_x(-\theta)R_z(-\varphi)$$

$$R^{-1} = R_z(\varphi)R_x(\theta)R_z(\psi)$$

Here it is

```
evalm(Rz(phi) &* Rx(theta) &* Rz(psi));
```

$$\begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & -\sin(\psi)\cos(\phi) - \cos(\psi)\cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) \\ \cos(\psi)\sin(\phi) + \sin(\psi)\cos(\theta)\cos(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & -\sin(\theta)\cos(\phi) \\ \sin(\psi)\sin(\theta) & \cos(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Well, I can just use (G.5.30) I think, so I get

$\omega_1 = (\cos\varphi\cos\psi - \cos\theta\sin\varphi\sin\psi)\ A\sin(\Omega' t) + (-\cos\varphi\sin\psi - \cos\theta\sin\varphi\cos\psi)\ A\cos(\Omega' t) + \sin\theta\sin\varphi\ (\omega)'_3$
$\omega_2 = (\sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi)\ A\sin(\Omega' t) + (-\sin\varphi\sin\psi + \cos\theta\cos\varphi\cos\psi)\ A\cos(\Omega' t) - \sin\theta\cos\varphi\ (\omega)'_3$
$\omega_3 = \sin\theta\sin\psi\ A\sin(\Omega' t) + \sin\theta\cos\psi\ A\cos(\Omega' t) + \cos\theta\ (\omega)'_3.$

How would I show this is conical motion? It just seems a total mess. At is at least moving. If this is supposed to match the picture of Marion on page 392 it seems that if L is in the $\hat{\mathbf{z}}$ direction I would expect to see that ω_3 = constant. But that seems very unlikely looking at the last line above. Unless

$$\psi(t) = \Omega' t$$

But how would I justify that? Well go back and add new information:

$$(\omega)'_1 = A\sin(\Omega' t) = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$(\omega)'_2 = (\dot{\omega})'_1 / \Omega' = A\cos(\Omega' t) = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$(\omega)'_3 = K = \dot{\varphi}\cos\theta + \dot{\psi}$$

So I really have these three equations never yet looked at

$$A\sin(\Omega' t) = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$A\cos(\Omega' t) = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$K = \dot{\varphi}\cos\theta + \dot{\psi}$$

OK, the first two imply that

$$A^2 = [\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]^2 + [\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]^2$$

Cross terms cancel, leaving just the 4 main terms. Then

$$A^2 = \dot{\varphi}^2\sin^2\theta + \dot{\theta}^2$$

So now I have these two equations

$$A^2 = \dot{\varphi}^2\sin^2\theta + \dot{\theta}^2$$
$$K = \dot{\varphi}\cos\theta + \dot{\psi}$$

Not going anywhere fast. Go back to the first two

$$A\sin(\Omega' t) = \dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$A\cos(\Omega' t) = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$

so

$$\sin\psi A\sin(\Omega' t) = \dot{\varphi}\sin\theta\sin^2\psi + \dot{\theta}\cos\psi\sin\psi$$
$$\cos\psi A\cos(\Omega' t) = \dot{\varphi}\sin\theta\cos^2\psi - \dot{\theta}\sin\psi\cos\psi$$

Add these to get

$$\sin\psi A\sin(\Omega' t) + \cos\psi A\cos(\Omega' t) = \dot{\varphi}\sin\theta$$

Do it differently now to cancel the other term

$$\cos\psi A \sin(\Omega' t) = \dot{\varphi} \sin\theta \sin\psi \cos\psi + \dot{\theta} \cos^2\psi$$

$$-\sin\psi A \cos(\Omega' t) = -\dot{\varphi} \sin\theta \sin\psi \cos\psi + \dot{\theta} \sin^2\psi$$

Then

$$\cos\psi A \sin(\Omega' t) - \sin\psi A \cos(\Omega' t) = \dot{\theta}$$

So we have at least two new and interesting equations

$$\sin\psi A \sin(\Omega' t) + \cos\psi A \cos(\Omega' t) = \dot{\varphi} \sin\theta$$

$$\cos\psi A \sin(\Omega' t) - \sin\psi A \cos(\Omega' t) = \dot{\theta}$$

Write as

$$A\cos(\psi - \Omega' t) = \dot{\varphi} \sin\theta$$

$$A\sin(\Omega' t - \psi) = \dot{\theta}$$

So I now have four interesting equations

$$A\cos(\psi - \Omega' t) = \dot{\varphi} \sin\theta$$

$$A\sin(\psi - \Omega' t) = -\dot{\theta}$$

$$A^2 = \dot{\varphi}^2 \sin^2\theta + \dot{\theta}^2$$

$$K = \dot{\varphi} \cos\theta + \dot{\psi}$$

There are four unknowns here: $\dot{\varphi}$, θ, $\dot{\theta}$, $\dot{\psi}$ so maybe I can somehow solve. I want to show $\dot{\theta} = 0$ I think, but not clear how to do that. All four are not independent since third is sum of squares of 1 and 2.

Ansatz: Just *try* to find a solution with $\dot{\theta} = 0$. The equations then say

$$A\cos(\psi - \Omega' t) = \dot{\varphi} \sin\theta$$

$$A\sin(\psi - \Omega' t) = 0$$

$$A^2 = \dot{\varphi}^2 \sin^2\theta$$

$$K = \dot{\varphi} \cos\theta + \dot{\psi}$$

The 2nd seems to say that $\psi = \Omega' t$. The other info left is only this

$$A = \dot{\varphi} \sin\theta$$

$$K = \dot{\varphi} \cos\theta + \Omega'$$

Recall that

$$A = (2T/I'_1) - K^2$$

not very useful. Instead process

$$A\cos\theta = \dot{\phi}\sin\theta\cos\theta$$
$$K\sin\theta = \dot{\phi}\cos\theta\sin\theta + \Omega'$$

$$A\cos\theta - K\sin\theta = -\Omega' = -K(I'_1-I'_3)/I'_1$$

$$(A/K)\cos\theta - \sin\theta = -(I'_1-I'_3)/I'_1$$

This must be how θ gets determined? Maple at least gives solutions although ugly.

Let's now go back to the brute force result:

$$\omega_1 = (\cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi)\ A\sin(\Omega't) + (-\cos\varphi\sin\psi-\cos\theta\sin\varphi\cos\psi)\ A\cos(\Omega't) + \sin\theta\sin\varphi\ (\omega)'_3$$
$$\omega_2 = (\sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi)\ A\sin(\Omega't) + (-\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi)\ A\cos(\Omega't) - \sin\theta\cos\varphi\ (\omega)'_3$$
$$\omega_3 = \sin\theta\sin\psi\ A\sin(\Omega't) + \sin\theta\cos\psi\ A\cos(\Omega't) + \cos\theta\ (\omega)'_3.$$

But now set Ω't = ψ to get some simplification maybe,

$$\omega_1 = (\cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi)\ A\sin\psi + (-\cos\varphi\sin\psi-\cos\theta\sin\varphi\cos\psi)\ A\cos\psi + \sin\theta\sin\varphi\ K$$
$$\omega_2 = (\sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi)\ A\sin\psi + (-\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi)\ A\cos\psi - \sin\theta\cos\varphi K$$
$$\omega_3 = \sin\theta\sin\psi\ A\sin\psi + \sin\theta\cos\psi\ A\cos\psi + \cos\theta K.$$

Simplify them one at a time:

$$\omega_1 = A\{\cos\varphi\cos\psi\sin\psi-\cos\theta\sin\varphi\sin^2\psi - \cos\varphi\sin\psi\cos\psi-\cos\theta\sin\varphi\cos^2\psi\} + \sin\theta\sin\varphi\ K$$

$$= A\{-\cos\theta\sin\varphi\sin^2\psi - \cos\theta\sin\varphi\cos^2\psi\} + \sin\theta\sin\varphi\ K$$

$$= A\cos\theta\{-\sin\varphi\sin^2\psi - \sin\varphi\cos^2\psi\} + \sin\theta\sin\varphi\ K$$

$$= -A\cos\theta\sin\varphi + \sin\theta\sin\varphi\ K$$

$$= (-A\cos\theta + K\sin\theta)\sin\varphi \qquad \text{ok}$$

$$\omega_2 = (\sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi)\ A\sin\psi + (-\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi)\ A\cos\psi - \sin\theta\cos\varphi K$$

$$= A\{\sin\varphi\sin\psi\cos\psi + \cos\theta\cos\varphi\sin^2\psi - \sin\varphi\sin\psi\cos\psi+\cos\theta\cos\varphi\cos^2\psi\} - \sin\theta\cos\varphi K$$

$$= A\{\cos\theta\cos\varphi\sin^2\psi+\cos\theta\cos\varphi\cos^2\psi\} - \sin\theta\cos\varphi K$$

$= A\{\cos\theta\cos\varphi\} - \sin\theta\cos\varphi K$

$= A\cos\theta\cos\varphi - \sin\theta\cos\varphi K$

$= (A\cos\theta - K\sin\theta)\cos\varphi$

$\omega_3 = \sin\theta\sin\psi\ A\sin\psi + \sin\theta\cos\psi\ A\cos\psi + \cos\theta K$

$= \sin\theta\sin^2\psi\ A + \sin\theta\cos^2\psi\ A + K\cos\theta$

$= A\sin\theta + K\cos\theta$

Here then are my "brute force" results where I inserted $\psi(t) = \Omega' t$:

$\omega_1 = (-A\cos\theta + K\sin\theta)\sin\varphi$

$\omega_2 = (A\cos\theta - K\sin\theta)\cos\varphi$

$\omega_3 = A\sin\theta + K\cos\theta$

Now recall from above that

$A = \dot{\varphi}\sin\theta \Rightarrow \quad \dot{\varphi} = A/\sin\theta \quad \Rightarrow \quad \varphi = (A/\sin\theta)\, t$

The results are now

$\omega_1 = -(A'\cos\theta - K\sin\theta)\sin[(A'/\sin\theta)\, t]$

$\omega_2 = (A'\cos\theta - K\sin\theta)\cos[(A'/\sin\theta)\, t]$

$\omega_3 = A'\sin\theta + K\cos\theta$

This does show precession! The amplitude and frequency seem strange.

$\omega_1 = -A\sin[\Omega t]$ $\qquad A = A'\cos\theta - K\sin\theta$

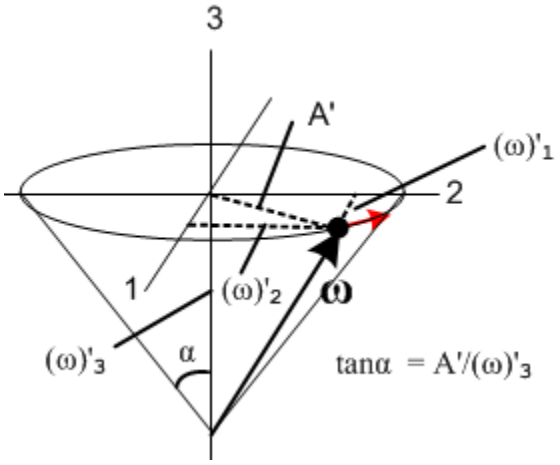
$\omega_2 = A\cos[\Omega t]$ $\qquad \Omega = (A'/\sin\theta)$

$\omega_3 = A'\sin\theta + K\cos\theta$

I think it goes backwards?

How can I verify this result somewhere? I think it has to be a textbook somewhere.

I have found a few things. going back to my picture



Note the angle α. It seems clear that

$\tan\alpha = A'/K$

$\sin\alpha = A'/\omega$

Then my result above can be written

$\dot{\phi} = A'/\sin\theta = \omega\sin\alpha/\sin\theta \qquad \varphi = (\omega\sin\alpha/\sin\theta)t$

and I do see this in a paper I am looking at

$$\dot{\phi} = \frac{\sin\alpha}{\sin\theta}\omega, \qquad 4.8.19$$

so there at least is some kind of confirmation of what I am doing.

But he also makes this claim:

$$\tan\theta = \frac{I_1}{I_3}\tan\alpha. \qquad 4.8.18$$

I can write my body frame stuff as

$$\begin{aligned}(\omega)'_1 &= \omega\sin\alpha\sin(\Omega't)\\ (\omega)'_2 &= \omega\sin\alpha\cos(\Omega't)\\ (\omega)'_3 &= K\end{aligned} \qquad (G.8.20)$$

How about this idea

$\omega^2 = (\omega)'_i{}^2 = \omega^2 \sin^2\alpha + K^2$

which says

$\omega^2(1-\sin^2\alpha) = K^2$

$\omega^2 \cos^2\alpha = K$

Then nice result is

$K = \omega \cos\alpha$

So now I have

$A' = \omega\sin\alpha$
$K = \omega\cos\alpha$

$A'/K' = \tan\alpha$ which I already knew.

$A'^2 + K^2 = \omega^2$ obvious from picture, but still new to me.

Now my author is claiming that

$\omega\sin\alpha = \omega_{y'}$ A'

$\omega\cos\alpha = \omega_{z'}$ K

He claims also this

$$L_{x'} = 0, \qquad L_{y'} = L\sin\theta, \qquad L_{z'} = L\cos\theta \ . \qquad 4.8.12$$

where is that coming from? These are "intermediate axes" for Euler I think. The ξ' and η' guys.

$L_{\xi'} = 0$ $L_{\eta'} = L\sin\theta$ $L_{z'} = L\cos\theta$

Looking at my EA picture I think the above are correct. I agree that

$L_{\eta'} / L_{z'} = \tan\theta$

But he is claiming next that

$L_{\eta'} = I'_1 \omega_{\eta'}$ $L_{z'} = I'_3 \omega_{z'}$

He is somehow using those intermediate angles. The right one here seems OK.

I get some hints here, but just not very systematic!

Go back to

$\omega_1 = (-A'\cos\theta + K\sin\theta)\sin\varphi$

$\omega_2 = (A'\cos\theta - K\sin\theta)\cos\varphi$

$\omega_3 = A'\sin\theta + K\cos\theta$

and use

$A' = \omega\sin\alpha$
$K = \omega\cos\alpha$

Then these say

$\omega_1 = (-\omega\sin\alpha\cos\theta + \omega\cos\alpha\sin\theta)\sin\varphi$
$\omega_2 = (\omega\sin\alpha\cos\theta - \omega\cos\alpha\sin\theta)\cos\varphi$
$\omega_3 = \omega\sin\alpha\sin\theta + \omega\cos\alpha\cos\theta$

or

$\omega_1 = -\omega(\sin\alpha\cos\theta + \cos\alpha\sin\theta)\sin\varphi$
$\omega_2 = \omega(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\cos\varphi$
$\omega_3 = \omega(\sin\alpha\sin\theta + \cos\alpha\cos\theta)$

Now

$\sin\alpha\cos\theta - \cos\alpha\sin\theta = \sin(\alpha-\theta)$
$\sin\alpha\sin\theta + \cos\alpha\cos\theta = \cos(\alpha-\theta)$

Then have

$\omega_1 = -\omega\sin(\alpha-\theta)\sin\varphi$ $\qquad\qquad$ $\varphi = (\omega\sin\alpha/\sin\theta)t$
$\omega_2 = \omega\sin(\alpha-\theta)\cos\varphi$
$\omega_3 = \omega\cos(\alpha-\theta)$

$\mathbf{L} = L\mathbf{e}_3$

Getting pretty simple now! It seems like the Frame S cone angle is not θ, but is α-θ.

Here is a picture the guy draws,

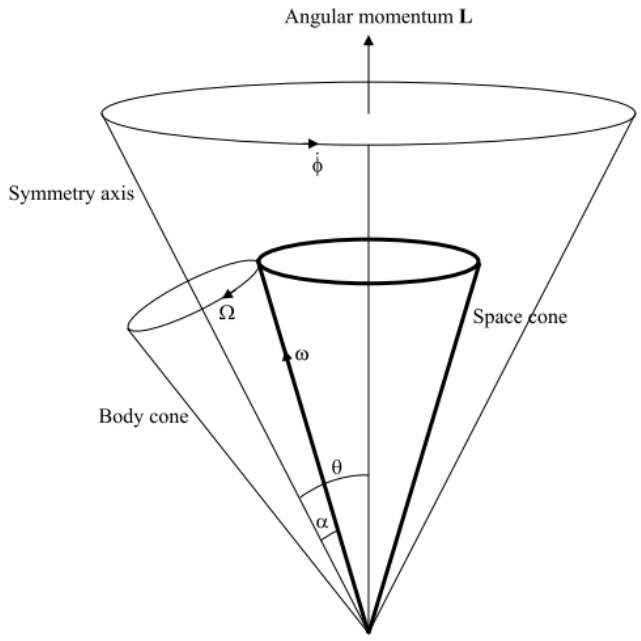
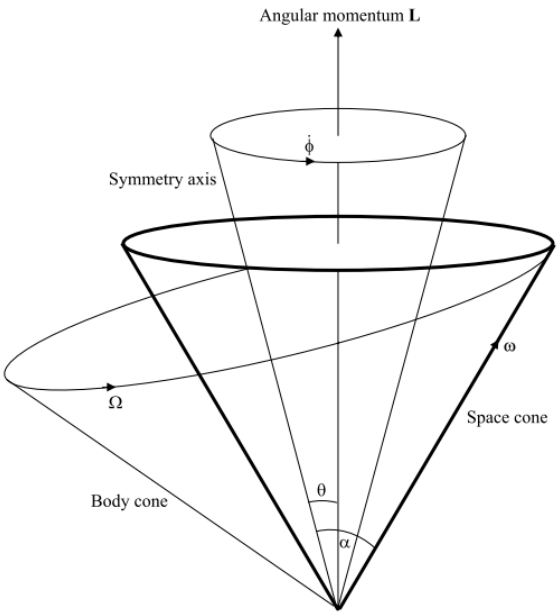


FIGURE IV.16

You see that indeed the space cone has half angle $\theta - \alpha$, so confirmation. Not sure what the Big Cone is in this picture.



Again the space cone has α-θ as its half angle. What is that other cone??

New idea 3.15.17.

What happens if I back map **L** from Frame S' to Frame S and require it be z-aligned in Frame S? Maybe better to use the forward map which is

$$x' = (\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi)\, x + (\cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi)\, y + \sin\psi\sin\theta\, z$$
$$y' = (-\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi)\, x + (-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi)\, y + \cos\psi\sin\theta\, z$$
$$z' = \sin\theta\sin\varphi\, x - \sin\theta\cos\varphi\, y + \cos\theta\, z \quad . \qquad \text{(G.5.26)}$$

Write as

$$(L')_1 = (\cos\psi\cos\varphi - \sin\psi\cos\theta\sin\varphi)\, L_1 + (\cos\psi\sin\varphi + \sin\psi\cos\theta\cos\varphi)\, L_2 + \sin\psi\sin\theta\, L_3$$
$$(L')_2 = (-\sin\psi\cos\varphi - \cos\psi\cos\theta\sin\varphi)\, L_1 + (-\sin\psi\sin\varphi + \cos\psi\cos\theta\cos\varphi)\, L_2 + \cos\psi\sin\theta\, L_3$$
$$(L')_3 = \sin\theta\sin\varphi\, L_1 - \sin\theta\cos\varphi\, L_2 + \cos\theta\, L_3 \quad . \qquad \text{(G.5.26)}$$

Then require that $L_1 = L_2 = 0$ in Frame S to get

$$(L')_1 = \sin\psi\sin\theta\, L_3$$
$$(L')_2 = \cos\psi\sin\theta\, L_3$$
$$(L')_3 = \cos\theta\, L_3. \qquad (*)$$

But then we know that

$$(L')_1 = I'_1(\omega)'_1 = I'_1\, \omega\sin\alpha\, \sin(\Omega' t)$$
$$(L')_2 = I'_1(\omega)'_2 = I'_1\, \omega\sin\alpha\, \cos(\Omega' t)$$

$(L')_3 = I'_1(\omega)'_3 = I'_3\,\omega\cos\alpha$ (**)

Compare this to

$(\omega)'_1 = \omega\sin\alpha\sin(\Omega't)$
$(\omega)'_2 = \omega\sin\alpha\cos(\Omega't)$. (G.8.20)
$(\omega)'_3 = \omega\cos\alpha$

So viewed in Frame S', the vectors **L** and **ω** are different, but they both precess in Frame S'.

Meanwhile, from (*) and (**) we have this possibly new information

$I'_1\,\omega\sin\alpha\sin(\Omega't) = \sin\psi\sin\theta\,L_3 = \sin(\Omega't)\sin\theta\,L_3$
$I'_1\,\omega\sin\alpha\cos(\Omega't) = \cos\psi\sin\theta\,L_3 = \cos(\Omega't)\sin\theta\,L_3$
$I'_3\,\omega\cos\alpha = \cos\theta\,L_3$

or // here I assume that $\psi = \Omega't$

$I'_1\,\omega\sin\alpha = \sin\theta\,L_3$
$I'_1\,\omega\sin\alpha = \sin\theta\,L_3$
$I'_3\,\omega\cos\alpha = \cos\theta\,L_3$

Divide the last two equations to get

$(I'_1/I'_3)\tan\alpha = \tan\theta$

which compare to the PDF claim

$$\tan\theta = \frac{I_1}{I_3}\tan\alpha. \qquad 4.8.18$$

So I have finally obtained this key result. It relates the angles θ and α for the first time. I think that θ is the cone half angle for the Frame S precession! This also says θ = constant, something I was looking for! At this point this is what I know about Frame S viewing

$\omega_1 = -\omega\sin(\alpha-\theta)\sin\varphi$ $\varphi = (\omega\sin\alpha/\sin\theta)t$
$\omega_2 = \omega\sin(\alpha-\theta)\cos\varphi$
$\omega_3 = \omega\cos(\alpha-\theta)$
$\mathbf{L} = L\mathbf{e}_3$

I also know that L is same in both frames as a mag, and I already computed,

$L^2 = I'^2_1 A'^2 + I'^2_3(\omega)'^2_3 = (I'_1\omega\sin\alpha)^2 + (I'_3\omega\cos\alpha)^2$ = a nice expression

$L^2 = \omega^2 [(I'_1 \sin\alpha)^2 + (I'_3 \cos\alpha)^2]$

so L and ω are locked to each other in this simple manner.

Summary of what I now know:

1. α is the cone angle for ω motion viewed in Frame S'.

2. In Frame S' the motion of the **ω** vector is this (cone angle α)

$(\omega)'_1 = \omega\sin\alpha \sin(\Omega' t)$
$(\omega)'_2 = \omega\sin\alpha \cos(\Omega' t)$
$(\omega)'_3 = \omega\cos\alpha$

Given the above, we would compute the cone angle this way

tan(cone half angle) = $(\omega)'_1$ coefficient / $(\omega)'_3$ = $\omega\sin\alpha / \omega\cos\alpha = \tan\alpha$

and therefore : cone half angle = α

2. In Frame S' the motion of the **L** vector is this,

$(L')_1 = I'_1 \omega\sin\alpha \sin(\Omega' t)$
$(L')_2 = I'_1 \omega\sin\alpha \cos(\Omega' t)$
$(L')_3 = I'_3 \omega \cos\alpha$

tan(cone half angle) = $(L')_1$ coefficient / $(L')_3$ = $I'_1 \omega\sin\alpha / I'_3 \omega \cos\alpha = (I'_1/I'_3) \tan\alpha$

3. I have shown also that $(I'_1/I'_3) \tan\alpha = \tan\theta$.

Combining this with the previous shows that the **L** precession half angle is just θ !!

I think I then know about all three cones in the guy's pictures.

Update 3.15.17

I have tried to write some of this up in line in "draft of G.8" but it is a bit messy. I solve for the Euler angle and get what appears to be conical motion of the rotor. But what are my conclusions about the ω and the **L** vector in Frame S? I know that **L** is fixed in the z direction, yes. I know that the rotor figure axis is connected to the Euler angles. But what is the **ω** vector doing? Maybe I just use these results,

$(\omega)_x = \dot{\psi} \sin\theta\sin\varphi + \dot{\theta} \cos\varphi$
$(\omega)_y = -\dot{\psi} \sin\theta\cos\varphi + \dot{\theta} \sin\varphi$
$(\omega)_z = \dot{\psi} \cos\theta + \dot{\varphi}$. // Frame S ok (G.6.4)

which I can rewrite as [using

$(\omega)_x = \Omega' \sin\theta \sin[(L/I'_1)t]$
$(\omega)_y = -\Omega' \sin\theta \cos[(L/I'_1)t]$
$(\omega)_z = \Omega' \cos\theta + (L/I'_1)$.

since

$\theta(t) = \tan^{-1}([I'_1/I'_3]\tan\alpha)$ // = constant
$\psi(t) = \Omega' t$ // $\Omega' = \omega\cos\alpha\,(I'_1-I'_3)/I'_1$
$\varphi(t) = (L/I'_1)t$

Does the mag of ω work out right or not? It looks a bit fishy,

$\omega^2 = \Omega'^2\sin^2\theta + (\Omega'\cos\theta + (L/I'_1))^2$??

$\omega^2 = \Omega'^2\sin^2\theta + \Omega'^2\cos^2\theta + (L/I'_1)^2 + 2(L/I'_1)\Omega'\cos\theta$??

$\omega^2 = \Omega'^2 + (L/I'_1)^2 + 2(L/I'_1)\Omega'\cos\theta$??

I know that

$\cos\theta\, L = I'_3\omega\cos\alpha$

so I can eliminate that to get

$\omega^2 = \Omega'^2 + (L/I'_1)^2 + 2(1/I'_1)\Omega' I'_3\omega\cos\alpha$??

But I know that

$\Omega' = \omega\cos\alpha\,(I'_1-I'_3)/I'_1$

So then have

$\omega^2 = \omega^2\cos^2\alpha\,(I'_1-I'_3)^2/I'^2_1 + (L/I'_1)^2 + 2(I'_3/I'_1)\{\omega\cos\alpha\,(I'_1-I'_3)/I'_1\}\omega\cos\alpha$??

$\omega^2 = \omega^2\cos^2\alpha\,(I'_1-I'_3)^2/I'^2_1 + (L/I'_1)^2 + 2(I'_3/I'_1)\omega^2\cos^2\alpha\,(I'_1-I'_3)/I'_1$??

$I'^2_1\,\omega^2 = \omega^2\cos^2\alpha\,(I'_1-I'_3)^2 + L^2 + 2I'_3\omega^2\cos^2\alpha\,(I'_1-I'_3)$??

I also know that

$L^2 = \omega^2[(I'_1\sin\alpha)^2 + (I'_3\cos\alpha)^2] = \omega^2 I'^2_1\sin^2\alpha + \omega^2 I'^2_3\cos^2\alpha$

so put that in as well to get

$I'_1{}^2\ \omega^2 = \omega^2\cos^2\alpha\,(I'_1-I'_3)^2 + \omega^2\, I'_1{}^2\sin^2\alpha + \omega^2\, I'_3{}^2\cos^2\alpha + 2\, I'_3\omega^2\cos^2\alpha\,(I'_1-I'_3)$??

$I'_1{}^2 = \cos^2\alpha\,(I'_1-I'_3)^2 + I'_1{}^2\sin^2\alpha + I'_3{}^2\cos^2\alpha + 2\, I'_3\cos^2\alpha\,(I'_1-I'_3)$??

$I'_1{}^2 = \cos^2\alpha\,(I'_1-I'_3)^2 + I'_1{}^2 - I'_1{}^2\cos^2\alpha + I'_3{}^2\cos^2\alpha + 2\, I'_3\cos^2\alpha\,(I'_1-I'_3)$??

$0 = \cos^2\alpha\,(I'_1-I'_3)^2 - I'_1{}^2\cos^2\alpha + I'_3{}^2\cos^2\alpha + 2\, I'_3\cos^2\alpha\,(I'_1-I'_3)$??

$0 = (I'_1-I'_3)^2 - I'_1{}^2 + I'_3{}^2 + 2\, I'_3(I'_1-I'_3)$??

$0 = I'_1{}^2 + I'_3{}^2 - 2\, I'_1\, I'_3 - I'_1{}^2 + I'_3{}^2 + 2\, I'_3 I'_1 - 2\, I'_3{}^2$??

$0 = I'_3{}^2 - 2\, I'_1\, I'_3 + I'_3{}^2 + 2\, I'_3 I'_1 - 2\, I'_3{}^2$??

$0 = -2\, I'_1\, I'_3 + 2\, I'_3 I'_1$??

$0 = 0$ YES!! ??

Now go back to

$(\omega)_x = \Omega' \sin\theta\sin[(L/I'_1)t]$ $\Omega' = \omega\cos\alpha\,(I'_1-I'_3)/I'_1$

$(\omega)_y = -\,\Omega' \sin\theta\cos[(L/I'_1)t]$

$(\omega)_z = \Omega' \cos\theta + (L/I'_1)$.

I don't like this form much. Go back to

$x = (\cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi)\, x' + (-\cos\varphi\sin\psi-\cos\theta\sin\varphi\cos\psi)\, y' + \sin\theta\sin\varphi\, z'$

$y = (\sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi)\, x' + (-\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi)\, y' - \sin\theta\cos\varphi\, z'$

$z = \sin\theta\sin\psi\, x' + \sin\theta\cos\psi\, y' + \cos\theta\, z'$. (G.5.30)

Apply this to the ω vector to get

$\omega_1 = (\cos\varphi\cos\psi-\cos\theta\sin\varphi\sin\psi)\,(\omega)'_1 + (-\cos\varphi\sin\psi-\cos\theta\sin\varphi\cos\psi)\,(\omega)'_2 + \sin\theta\sin\varphi\,(\omega)'_3$

$\omega_2 = (\sin\varphi\cos\psi + \cos\theta\cos\varphi\sin\psi)\,(\omega)'_1 + (-\sin\varphi\sin\psi+\cos\theta\cos\varphi\cos\psi)\,(\omega)'_2 - \sin\theta\cos\varphi\,(\omega)'_3$

$\omega_3 = \sin\theta\sin\psi\,(\omega)'_1 + \sin\theta\cos\psi\,(\omega)'_2 + \cos\theta\,(\omega)'_3$.

Now insert our Frame S' solutions which were

$(\omega)'_1 = \omega\sin\alpha\,\sin\psi$

$(\omega)'_2 = \omega\sin\alpha\,\cos\psi$

$(\omega)'_3 = \omega\cos\alpha$ α = cone half-angle

****

Our Frame S' solution for $(\boldsymbol{\omega})'$ was

$$(\boldsymbol{\omega})' = \begin{pmatrix} \omega\sin\alpha\,\sin\psi \\ \omega\sin\alpha\,\cos\psi \\ \omega\cos\alpha \end{pmatrix}$$

To obtain the Frame S components of $\boldsymbol{\omega}$ we have Maple compute $\boldsymbol{\omega} = R^{-1}(\boldsymbol{\omega})'$:

```
RINV := evalm(Rz(phi) &* Rx(theta) &* Rz(psi));
```

$$RINV := \begin{bmatrix} \cos(\phi)\cos(\psi) - \cos(\theta)\sin(\phi)\sin(\psi) & -\cos(\phi)\sin(\psi) - \cos(\theta)\sin(\phi)\cos(\psi) & \sin(\theta)\sin(\phi) \\ \sin(\phi)\cos(\psi) + \cos(\theta)\cos(\phi)\sin(\psi) & -\sin(\phi)\sin(\psi) + \cos(\theta)\cos(\phi)\cos(\psi) & -\sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\psi) & \sin(\theta)\cos(\psi) & \cos(\theta) \end{bmatrix}$$

```
wp1 := omega*sin(alpha)*sin(psi):
wp2 := omega*sin(alpha)*cos(psi):
wp3 := omega*cos(alpha):
wp := matrix(3,1,[wp1,wp2,wp3]);
```

$$wp := \begin{bmatrix} \omega\sin(\alpha)\sin(\psi) \\ \omega\sin(\alpha)\cos(\psi) \\ \omega\cos(\alpha) \end{bmatrix}$$

```
W := evalm(RINV &* wp): W := simplify(W);
```

$$W := \begin{bmatrix} -\omega\sin(\alpha)\cos(\theta)\sin(\phi) + \sin(\theta)\sin(\phi)\,\omega\cos(\alpha) \\ \omega\sin(\alpha)\cos(\theta)\cos(\phi) - \sin(\theta)\cos(\phi)\,\omega\cos(\alpha) \\ \sin(\theta)\,\omega\sin(\alpha) + \cos(\theta)\,\omega\cos(\alpha) \end{bmatrix}$$

where W = $\boldsymbol{\omega}$. The resulting vector can be simplified by combining the trig terms:

```
W[1,1]/sin(phi):simplify(%):combine(%);
```

$$\omega\sin(-\alpha+\theta)$$

```
W[2,1]/cos(phi):simplify(%):combine(%);
```

$$-\omega\sin(-\alpha+\theta)$$

```
W[3,1]:simplify(%):combine(%);
```

$$\omega\cos(-\alpha+\theta)$$

The final result is then

$\omega_1 = \omega\,\sin\varphi\,\sin(\theta-\alpha)$
$\omega_2 = -\,\omega\cos\varphi\,\sin(\theta-\alpha)$
$\omega_3 = \omega\cos(\theta-\alpha)$

Using $\varphi = (L/I'_1)t$ this becomes

$\omega_1 = \omega\,\sin(\theta-\alpha)\sin[(L/I'_1)t\,]$
$\omega_2 = -\,\omega\,\sin(\theta-\alpha)\cos[(L/I'_1)t\,]$
$\omega_3 = \omega\,\cos(\theta-\alpha)$

This set of equations describes the precession in Frame S components of the vector **ω** at rate $\varphi = (L/I'_1)t$ with a cone half-angle of $\theta-\alpha$.

Verify the minus sign:

$\omega \sin\alpha \cos\theta\cos\varphi - \sin\theta\cos\varphi\omega\cos\alpha$

$= \omega\cos\varphi (\sin\alpha \cos\theta - \cos\alpha\sin\theta) =$

$= \omega\cos\varphi \sin(\alpha-\theta)$

$= - \omega\cos\varphi \sin(\theta-\alpha)$

First line is

$\sin\theta\cos\alpha - \sin\alpha\cos\theta = \sin(\theta-\alpha)$

signs are correct! Ω' figure axis

****

$\sin\theta\, L = I'_1\omega\sin\alpha$?

$\sin^2\theta\, \omega^2 [(I'_1\sin\alpha)^2 +(I'_3\cos\alpha)^2] = I'^2_1\omega^2\sin^2\alpha$?

$\sin^2\theta\, [(I'_1\sin\alpha)^2 +(I'_3\cos\alpha)^2] = (I'_1\sin\alpha)^2$?

$\sin^2\theta\, [1 +([I'_3/I'_1]\cot\alpha)^2] = 1$?

$[1 +([I'_3/I'_1]\cot\alpha)^2] = \csc^2\theta = 1 + \cot^2\theta$?

$\equiv \cot\theta = [I'_2/I'_1]\cot\alpha$ yes! same as ***

****

$L^2 = \omega^2 (I'^2_1\sin^2\alpha + I'^2_3\cos^2\alpha)$

$2T = \omega^2 (I'_1\sin^2\alpha + I'_3\cos^2\alpha)$

$$2TI'_1 = \omega^2 (I'_1{}^2\sin^2\alpha + I'_3 I'_1\cos^2\alpha)$$

$$2TI'_1 - L^2 = \omega^2 (I'_1{}^2\sin^2\alpha + I'_3 I'_1\cos^2\alpha) - \omega^2 (I'_1{}^2\sin^2\alpha + I'_3{}^2\cos^2\alpha)$$

$$= \omega^2 [I'_3 I'_1\cos^2\alpha - I'_3{}^2\cos^2\alpha]$$

$$= \omega^2\cos^2\alpha\, I'_3(I'_1-I'_3)$$

which says

$$\omega^2\cos^2\alpha = (2TI'_1 - L^2) / [I'_3(I'_1-I'_3)]$$

Do it again this way

$$L^2 = \omega^2 (I'_1{}^2\sin^2\alpha + I'_3{}^2\cos^2\alpha)$$
$$2TI'_3 = \omega^2 (I'_1 I'_3\sin^2\alpha + I'_3{}^2\cos^2\alpha)$$

$$2TI'_3 - L^2 = \omega^2 (I'_1 I'_3\sin^2\alpha - I'_1{}^2\sin^2\alpha) = \omega^2 I'_1 (I'_3 - I'_1)\sin^2\alpha$$

which says

$$\omega^2\sin^2\alpha = (2TI'_3 - L^2) / I'_1(I'_3 - I'_1)$$

So at this point we have

$$\omega^2\cos^2\alpha = (2TI'_1 - L^2) / [I'_3(I'_1-I'_3)]$$
$$\omega^2\sin^2\alpha = (2TI'_3 - L^2) / I'_1(I'_3 - I'_1)$$

$$\omega^2\cos^2\alpha = \frac{|2TI'_1 - L^2|}{I'_3|I'_1-I'_3|} \qquad \omega^2\sin^2\alpha = \frac{|2TI'_3 - L^2|}{I'_1|I'_1 - I'_3|}$$

Divide to get

$$\tan^2\alpha = \frac{|2TI'_3 - L^2|}{I'_1|I'_1 - I'_3|} \Big/ \frac{|2TI'_1 - L^2|}{I'_3|I'_1-I'_3|} = (I'_3/I'_1)\,\frac{|2TI'_3 - L^2|}{|2TI'_1 - L^2|}$$

$$\omega^2 = \frac{|2TI'_1 - L^2|}{I'_3|I'_1-I'_3|} + \frac{|2TI'_3 - L^2|}{I'_1|I'_1 - I'_3|}$$

As verification, let $I'_1 = a$ and $I'_3 = b$ so that

```
A := (2*T*a - L^2)/(b*(a-b));
```

$$A := \frac{2Ta - L^2}{b(a-b)}$$

```
B := (2*T*b - L^2)/(a*(b-a));
```

$$B := \frac{2Tb - L^2}{a(b-a)}$$

```
A+B;
```

$$\frac{2Ta - L^2}{b(a-b)} + \frac{2Tb - L^2}{a(b-a)}$$

```
simplify(%);
```

$$\frac{2Ta + 2Tb - L^2}{ab}$$

I then get this result from adding the two equations

$\omega^2 = [2T(a+b)-L^2]/ab$

But I started with

$L^2 = \omega^2 (a^2\sin^2\alpha + b^2\cos^2\alpha)$

$2T = \omega^2 (a\sin^2\alpha + b\cos^2\alpha)$

I would then like to show that

$\omega^2 = [\omega^2 (a\sin^2\alpha + b\cos^2\alpha)(a+b) - \omega^2 (a^2\sin^2\alpha + b^2\cos^2\alpha)] /ab$

or

$1 = [(a\sin^2\alpha + b\cos^2\alpha)(a+b) - (a^2\sin^2\alpha + b^2\cos^2\alpha)] /ab$

or

$(a\sin^2\alpha + b\cos^2\alpha)(a+b) - (a^2\sin^2\alpha + b^2\cos^2\alpha) = ab$?

$\sin^2\alpha(a^2+ab - a^2) + \cos^2\alpha(ab+b^2 - b^2) = ab$?

$\sin^2\alpha\, ab + \cos^2\alpha\, ab = ab$ OK

So my solution is

$\omega^2\sin^2\alpha = (2TI'_3 - L^2) / [I'_1(I'_3 - I'_1)]$

$\omega^2\cos^2\alpha = (2TI'_1 - L^2) / [I'_3(I'_1-I'_3)]$

$\tan^2\alpha = (2TI'_3 - L^2) / I'_1(I'_3 - I'_1) \;//\; (2TI'_1 - L^2) / [I'_3(I'_1-I'_3)]$

$= - (2TI'_3 - L^2) / I'_1 \;//\; (2TI'_1 - L^2) / I'_3$

$= - (I'_3/I'_1)\ (2TI'_3 - L^2) / (2TI'_1 - L^2)$