Inertial moments of an oblate spheroid (approx) PhL 3.19.17

Texas reference:

http://farside.ph.utexas.edu/teaching/336k/Newtonhtml/node109.html

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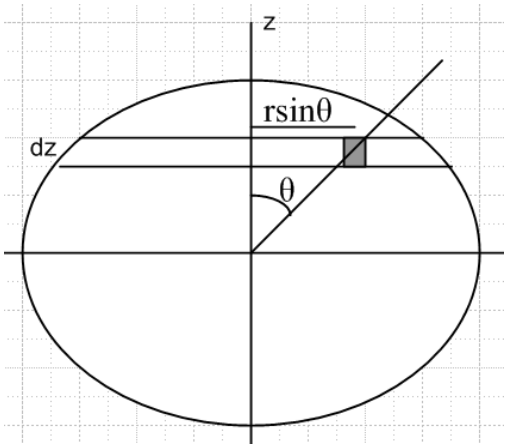
Here is the picture



Use spherical coordinates (r,θ,φ), Assume unit mass density ρ. Then

$$dm = dV = \rho r^2 dr d\varphi \sin\theta d\theta.$$

Note that

$$z = r\cos\theta$$

The moment of inertia of dm is given by

$$dI = dm(r\sin\theta)^2 = [\rho r^2 dr d\varphi \sin\theta d\theta](r\sin\theta)^2$$

The boundary of the oblate spheroid is assumed to be this, where $\varepsilon << 1$

$$R(\theta) = a[1 - (2/3)\varepsilon P_2(\cos\theta)]$$

Here then is the moment of inertial about the vertical axis

$$I_3 = \rho \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin^3\theta \int_0^{R(\theta)} r^4$$

$$= 2\pi\rho \int_0^{\pi} d\theta \sin^3\theta \int_0^{R(\theta)} r^4$$

$$= (2\pi\rho/5) \int_0^{\pi} d\theta \sin^3\theta \; R^5(\theta)$$

Now

$$R(\theta) = a[1 - (2/3)\varepsilon P_2(\cos\theta)]$$

$$R^5(\theta) \approx a^5 \; [1 - (10/3)\varepsilon P_2(\cos\theta)]$$

Then

$$I_3 \approx (2\pi\rho/5)\, a^5 \int_0^{\pi} d\theta \sin^3\theta \, [1 - (10/3)\varepsilon P_2(\cos\theta)]$$

The first integral is this

```
Int(sin(theta)^3,theta = 0..Pi);
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$$\int_0^{\pi} \sin(\theta)^3 \, d\theta$$

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value(%);
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$$\frac{4}{3}$$

We know that $P_2(\cos\theta) = (1/2)(3\cos^2\theta - 1)$

Then we want

$$\int_0^{\pi} d\theta \sin^3\theta \, P_2 = (1/2) \int_0^{\pi} d\theta \sin^3\theta \, (3\cos^2\theta - 1)$$

Maple says

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P2 := (1/2)*(3*cos(theta)^1-1);
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$$P2 := \frac{3}{2}\cos(\theta) - \frac{1}{2}$$

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Int(sin(theta)^3*P2,theta = 0..Pi);
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$$\int_0^{\pi} \sin(\theta)^3 \left(\frac{3}{2}\cos(\theta) - \frac{1}{2}\right) d\theta$$

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value(%);
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$$\frac{-2}{3}$$

Therefore

$I_3 \approx (2\pi\rho/5)\, a^5 \int_0^{\pi} d\theta \sin^3\theta\, [1 - (10/3)\varepsilon\, P_2(\cos\theta)]$

$= (2\pi\rho/5)\, a^5\, [\, (4/3) - (10/3)\varepsilon\, (-2/3)\,]$

$= (2\pi\rho/5)\, a^5\, [\, (4/3) + (20/9)\varepsilon\,]$

$= (2\pi\rho/5)\, (4/3)\, a^5\, [\, 1 + (5/3)\varepsilon\,]$

$= \rho\, (8\pi/15)\, a^5\, [\, 1 + (5/3)\varepsilon\,]$

Now approximate that

$M = (4/3)\pi a^3 \rho \qquad \Rightarrow \qquad \rho = (3/4)\, M/(\pi a^3)$

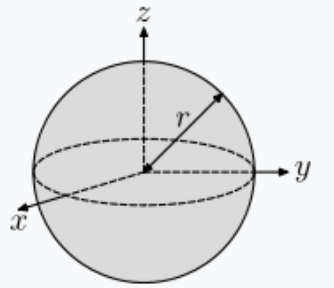
Then have

$I_3 \approx (3/4)\, M/(\pi a^3) * (8\pi/15)\, a^5\, [\, 1 + (5/3)\varepsilon\,]$

$= (3/4)\, M\, (8/15)\, a^2\, [\, 1 + (5/3)\varepsilon\,]$

$= (2/5)\, M\, a^2\, [\, 1 + (5/3)\varepsilon\,]$

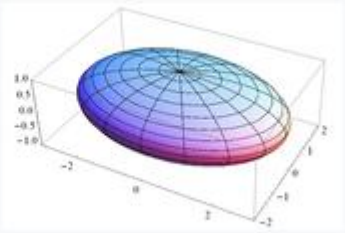
I now go check the list of inertias on wiki,



$$I = \frac{2mr^2}{5}$$ [1]

So that is good and we the correction for oblateness to first order in ε.

But now I back up because I see this result in that list

Ellipsoid (solid) of semiaxes *a*, *b*, and *c* with mass *m*		$I_a = \frac{m}{5}\left(b^2 + c^2\right)$ $I_b = \frac{m}{5}\left(a^2 + c^2\right)$ $I_c = \frac{m}{5}\left(a^2 + b^2\right)$

If we have spheroid, set A = B and you then get

$I_3 = (m/5)\,(A^2+C^2)$

$I_1 = (m/5)\,(2A^2)$

Thus we have

$I_3 - I_1 = (m/5)\,(A^2+C^2) - (m/5)\,(2A^2)$

$= (m/5)\,(C^2-A^2)$

In our model we have

$R(\theta) = a[1 - (2/3)\varepsilon\, P_2(\cos\theta)]$

Therefore

$C = a[1 - (2/3)\varepsilon\, P_2(0)] = a[1 - (2/3)\varepsilon\,(-1/2)] = a\,[1 + \varepsilon/3]$

$A = a[1 - (2/3)\varepsilon\, P_2(1)]\ \ a[1 - (2/3)\varepsilon\, 1] = a[1 - 2\varepsilon/3]$

$C+A \approx 2a$

$C-A = \varepsilon a$

since

$P_2(\cos\theta) = (1/2)(3\cos^2\theta - 1)$.

Then

$I_3 - I_1 = (m/5)(C+A)(C-A) \approx (m/5)(2a)(\varepsilon a) = (2/5)m\varepsilon a^2$

THAT took a long time for me to get!!

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On scratch I have showed that the torque on an oblate Earth is given by

$$\mathbf{N} = 3(2/5)\, GMa^2\varepsilon/r^3 \cos\theta\sin\theta \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix}$$

$$= 3\{(2/5)M\varepsilon a^2\}\, G/r^3 \cos\theta\sin\theta \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix}$$

$$= 3\{I_3 - I_1\}\, G/r^3 \cos\theta\sin\theta \begin{pmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{pmatrix}$$

Compare this to Williams result

$$\underline{T} = \frac{3Gm(C-A)\sin\delta\cos\delta}{r^3} \begin{pmatrix} \sin\alpha \\ -\cos\alpha \\ 0 \end{pmatrix},$$

I think I am on to it finally. Note that

$\cos\theta\sin\theta = \cos(\pi/2-\delta)\sin(\pi/2-\delta) = \sin\delta\cos\delta$

So I have lost a mass somewhere.

********** **Here is the calculation of the torque on the Earth** **********

STEP 1

Away from sources of charge, the electrostatic potential must satisfy the Laplace equation. The same is true for the gravitational potential away from mass sources. In general any such potential can be expressed as a sum over the "atomic forms" (harmonics) of a given coordinate system. For spherical coordinates the atomic forms can be taken as

$$[r^n\ ,\ r^{-n-1}]\ [\ P_n^m(z),\ Q_n^m(z)\ ]\ [e^{im\varphi},\ e^{-im\varphi}] \qquad z = \cos\theta$$

where n and m are non-negative integers. These are appropriate for regions of space which have the full range of angle φ. In such a region, for the potential to be single valued in azimuth one must have m = integer so $\exp(im(\varphi+2\pi)) = \exp(im\varphi)$. This in turn forces n to be an integer, and $m \leq n$. If $\theta = 0$ is part of the range of interest, one cannot have $Q_n^m(z)$ terms since they blow up at $z = 0$. In this case one has

$$[r^n\ ,\ r^{-n-1}]\ [\ P_n^m(z)]\ [e^{im\varphi},\ e^{-im\varphi}] \qquad z = \cos\theta$$

which gives the familiar spherical harmonics expansion of a potential. If the potential is azimuthally symmetric, one must have m = 0 reducing the atomic form set to

$$[r^n\ ,\ r^{-n-1}]\ P_n(z)$$

If a region of interest extends to $r = \infty$, only the r^{-n-1} powers are allowed. We then arrive at this expansion for the gravitational potential outside an azimuthally symmetric mass distribution.

$$V(r,\theta) = \Sigma_{n=0}^{\infty}\ V_n\ r^{-m-1}\ P_n(\cos\theta)$$

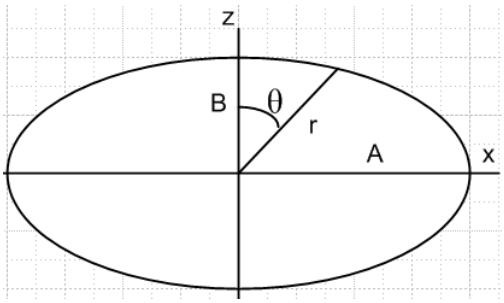
If the potential is also symmetric under z reflections, only even n contribute in the sum, and this is the situation for the potential of an oblate spheroid with z as the symmetry axis. If the spheroid is only very slightly oblate, we keep just the first two terms to get

$$V(r,\theta) = V_0/r\ + V_2\ r^{-3}\ P_2(\cos\theta)$$

STEP 2

Consider an ellipse of the form $x^2/A^2 + z^2/B^2 = 1$. If we measure polar angle θ down from the z axis, then $x = r\sin\theta$ and $z = r\cos\theta$ and the ellipse equation becomes

$$r^2 = \frac{A^2B^2}{B^2\sin^2\theta + A^2\cos^2\theta}$$



For an "oblate" (wide) ellipsoid we have A > B. If the ellipse is only slightly oblate, write A = B+δ

In this case we get

$$r^2 = \frac{A^2B^2}{B^2\sin^2\theta+(B+\delta)^2\cos^2\theta} \approx \frac{A^2B^2}{B^2 + 2B\delta\cos^2\theta} \approx \frac{A^2}{1+ 2(\delta/B)\cos^2\theta}$$

$$\approx A^2 [\ 1-2(\delta/B)\cos^2\theta\ ] \approx A^2 [\ 1-2(\delta/A)\cos^2\theta\ ]$$

Then

$$r \approx A [\ 1-(\delta/A)\cos^2\theta\ ]$$

The eccentricity of the ellipse is

$$\varepsilon^2 = (A^2-B^2)/A^2 = (A+B)(A-B)/A^2 \approx 2A\delta/A^2 = 2\delta/A$$

Then have

$$r = A [\ 1- (1/2)(2\delta/A)\cos^2\theta\ ] = [\ 1- (1/2)\varepsilon^2\cos^2\theta\ ]$$

STOP. My Texas document claims

$$r = a_\theta(\theta) = a\left[1 - \frac{2}{3}\,\epsilon\, P_2(\cos\theta)\right]$$

and you see the ε is not squared, and is called ellipticity which other sides say is the usual eccentricity

Now write

$$P_2 = (1/2)\,(3\cos^2\theta - 1) \quad \equiv \quad 2P_2 = 3\cos^2\theta - 1 \quad \Rightarrow\ 3\cos^2\theta = 2P_2 + 1$$

so $\cos^2\theta = (1/3)(2P_2 + 1)$

Then have

$r = [\ 1 - (1/2)\varepsilon^2\cos^2\theta\ ] = [\ 1 - (1/6)\varepsilon^2(2P_2 + 1)\ ]$

How am I ever doing to get this result:

$$r = a_\theta(\theta) = a\left[1 - \frac{2}{3}\,\epsilon\, P_2(\cos\theta)\right]$$

Try to do it:

$r = 1 - (1/3)\varepsilon^2 P_2 - (1/6)\varepsilon^2$

$= (1 - (1/6)\varepsilon^2) - (1/3)\varepsilon^2 P_2$

$= (1 - (1/6)\varepsilon^2)\ [\ 1 - (1/3)\varepsilon^2 P_2 / (1 - (1/6)\varepsilon^2)\ ]$

$= d\ [\ 1 - (1/3)\varepsilon^2 P_2 / d\ ]$ $\qquad d = (1 - (1/6)\varepsilon^2)$

$\approx\ d\ [\ 1 - (1/3)\varepsilon^2 P_2\ ]$

So the *fraction* is wrong and the *power* of ε is wrong!

What is the "mean radius" of this oblate spheroid? Go back to

$r = [\ 1 - (1/2)\varepsilon^2\cos^2\theta\ ]$

Then

$\langle r \rangle = [\ 1 - (1/2)\varepsilon^2\langle\cos^2\theta\rangle\ ]$
$= [\ 1 - (1/2)\varepsilon^2(1/3)]$ if I did it right

$= [\ 1 - (1/6)\varepsilon^2]$ if I did it right

$= a$ of Texas author

But this is exactly my d from above!

Conclusion: ether the Texas author has goofed up, or I have goofed up royally.

I now need another source on the Texas claim which is the STARTING point of his presentation:

Let us now calculate the gravitational potential generated outside a *spheroid* of uniform mass density γ and mean radius a. A spheroid is the solid body produced by rotating an ellipse about a major or minor axis. Let the axis of rotation coincide with the z-axis, and let the outer boundary of the spheroid satisfy

$$r = a_\theta(\theta) = a\left[1 - \frac{2}{3}\epsilon P_2(\cos\theta)\right], \qquad (901)$$

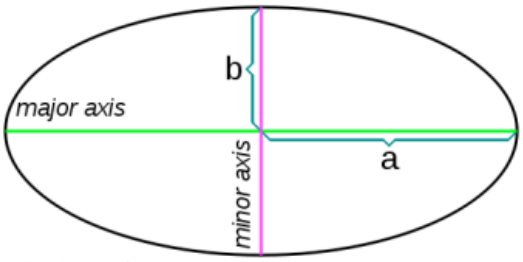
I found a doc which says this:

Figure of the Earth

Earth's actual surface is an equipotential surface (sea level), a surface for which $U_G + U_c$ = constant. The figure of the Earth a smooth surface that approximates this shape and upon which more complicated topography can be represented. The earth approximates an oblate spheroid, which means it is elliptically-shaped with a longer equatorial radius than a polar radius.

The flattening (or oblateness) is the ratio of the difference in radii to the equatorial radius:

$$f = \frac{a-b}{a}$$



For earth, f=0.00335287, or 1/298.252, and the difference in the polar and equatorial radii is about 21 km.

This author is using the "flatness" factor f. If I go back to

$$r \approx A\,[\,1-(\delta/A)\cos^2\theta\,]$$

then

$$f = (A-B)/A = \delta/A$$

which compare to

$$\varepsilon^2 = (A^2-B^2)/A^2 = (A+B)(A-B)/A^2 \approx 2A\delta/A^2 = 2\delta/A$$

Thus

$$\varepsilon^2 = 2f$$

and then my result is

$r = d\,[\ 1 - (1/3)\varepsilon^2 P_2\ ] = d\,[\ 1 - (1/3)2fP_2\ ] = d[\ 1 - (2/3)\ f\ P_2\ ]$

so the Texas offer is using symbol f to be "flatness", not eccentricity. That would explain it!

In fact, later Texas does say this. confirming that his ε is really the flattening factor

$$\frac{a_e - a_p}{a} = \epsilon = \frac{5}{4}\frac{\Omega^2 a^3}{G M}.$$

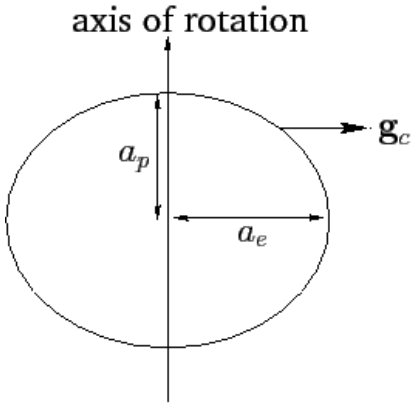


Figure 40: *Rotational flattening.*

This same new author says

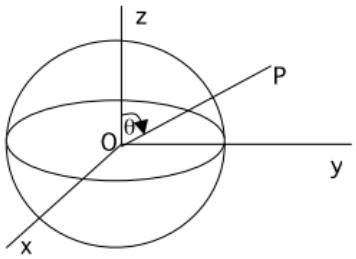
The gravitational potential of an ellipsoid is given by:

$$U_G = -G\frac{M_E}{r} - G\frac{(C-A)}{r^3}\frac{\left(3\cos^2\theta - 1\right)}{2} = -G\frac{M_E}{r} - G\frac{(C-A)}{r^3}P_2(\cos\theta)$$

where A and C are the moments of inertia about the equatorial and polar axes.

More generally:

$$U_G = -G\frac{M_E}{r}\left(1 - \sum_{n=2}^{\infty}\left(\frac{R}{r}\right)^2 J_n P_n(\cos\theta)\right)$$



Where P_n are the Legendre polynomials and the coefficients J_n are measured for Earth. The most important is the dynamical form factor: $J_2 = \frac{C-A}{M_E R^2} = 1082.6 \times 10^{-6}$

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Another approach: in curvilinear /sphericals I have

$V(r,\theta,\varphi) = \Sigma_{n=0}^{\infty} \Sigma_{m=-n}^{n} a_{nm}(r)\ P_n^m(\cos\theta)\ e^{im\varphi}$

$a_{nm}(r) = (2n+1)\ f(n,-m)\ (1/4\pi) \int d\Omega\ V(r,\theta,\varphi)\ e^{-im\varphi}\ P_n^m(z)$

If V goes not depend on φ, only m = 0 can contribute in the projection so

$a_{n0}(r) = (2n+1)\ f(n,0)\ (1/4\pi) \int d\Omega\ V(r,\theta) P_n(z)$

$$V(r,\theta) = \Sigma_{n=0}^{\infty} a_{n0}(r)\ P_n(\cos\theta)$$

In this case just write as

$$a_n(r) = (2n+1)f(n,0)\ (1/4\pi) \int d\Omega\ V(r,\theta)P_n(z)$$
$$V(r,\theta) = \Sigma_{n=0}^{\infty} a_n(r)\ P_n(\cos\theta)$$

Note that

$$P_n^m(z) = f(n,m)\ P_n^{-m}(z)$$

$$f(n,m) = \Gamma(n+m+1)/(n-m+1) \qquad f(n,0) = \Gamma(n+1)/(n+1) = 1$$

So here we are

$$v_n(r) = (2n+1)\ (1/4\pi) \int d\Omega\ V(r,\theta)P_n(z)$$
$$V(r,\theta) = \Sigma_{n=0}^{\infty} v_n(r)\ P_n(\cos\theta) \qquad \approx V_0/r + V_2\ r^{-3}\ P_2(\cos\theta)$$

Now above I said that

$$V(r,\theta) = V_0/r + V_2\ r^{-3}\ P_2(\cos\theta)$$

Now are things connected here? I guess it is this

$$v_0(r) = V_0/r \qquad v_2(r) = V_2/r^3$$

Then for example

$$v_0(r) = (1/4\pi) \int d\Omega\ V(r,\theta)P_0(z) = (1/4\pi) \int d\Omega\ V(r,\theta)$$

But this is not helping because I want a different thing. Start over:

$$V(\mathbf{r}) = \int dV'\ \rho(\mathbf{r}') / |\mathbf{r}-\mathbf{r}'| \qquad \text{Jackson scale electrostatics}$$

$$1/R = \Sigma_{n=0}^{\infty} \Sigma_{m=0}^{\infty} \varepsilon_m\ f(n,-m)\ (r_<)^n (r_>)^{-n-1}\ P_n^m(z)\ P_n^m(z')\ \cos[m(\varphi-\varphi')] \qquad (**)$$

If mass density $\rho(\mathbf{r}')$ does not depend on φ', only the m=0 terms survive so in effect

$$1/R = \Sigma_{n=0}^{\infty}\ \varepsilon_0\ f(n,0)\ (r_<)^n (r_>)^{-n-1}\ P_n(z)\ P_n(z')$$

$$= \Sigma_{n=0}^{\infty}\ (r_<)^n (r_>)^{-n-1}\ P_n(z)\ P_n(z')$$

Then put this in to get

$$V(\mathbf{r}) = \int dV'\, \rho(\mathbf{r}')\ \Sigma_{n=0}^{\infty} (r_<)^n (r_>)^{-n-1}\ P_n(z)\, P_n(z')$$

Now if **r** is outside the entire mass distribution, $r_> = \mathbf{r}$ and you have

$$V(\mathbf{r}) = \int dV'\, \rho(\mathbf{r}')\ \Sigma_{n=0}^{\infty} r'^n r^{-n-1}\ P_n(z)\, P_n(z')$$

$$= \Sigma_{n=0}^{\infty} r^{-n-1} P_n(z) \int dV'\, r'^n \rho(\mathbf{r}') P_n(z')$$

$$= \Sigma_{n=0}^{\infty} r^{-n-1} P_n(z) \{ 2\pi \int r'^{2+n} dr' \int \sin\theta' d\theta'\, \rho(r',\cos\theta') P_n(z') \}$$

$$= \Sigma_{n=0}^{\infty} r^{-n-1} P_n(z) \{ \ 2\pi \int r'^{2+n} dr' \int dz'\, \rho(r',z') P_n(z') \}$$

Add factor -G on the right to get

$$V(\mathbf{r}) = \Sigma_{n=0}^{\infty} r^{-n-1} P_n(z) \{ \ -2\pi G \int r'^{2+n} dr' \int dz'\, \rho(r',z') P_n(z') \}$$

$$= \Sigma_{n=0}^{\infty} r^{-n-1} P_n(z)\, V_n$$

where

$$V_n = -2\pi G \int r'^{2+n} dr' \int dz'\, \rho(r',z') P_n(z')$$

So I am now agreeing with Texas who says

$$\Phi(r,\theta) = \sum_{n=0,\infty} J_n \frac{P_n(\cos\theta)}{r^{n+1}},$$

$$J_n = -2\pi G \int\int r^{2+n} \rho(r,\theta)\, P_n(\cos\theta)\, \sin\theta\, dr\, d\theta.$$

So I could do the rest of his page and conclude as he does that

$$\Phi(r,\theta) = -\frac{G\,M}{r} + \frac{2}{5}\frac{G\,M\,a^2}{r^3}\,\epsilon\,P_2(\cos\theta) + \mathcal{O}(\epsilon^2).$$

So I think I can derive this given his oblate form

$$r = a_\theta(\theta) = a\left[1 - \frac{2}{3}\,\epsilon\, P_2(\cos\theta)\right]$$

where I have shown that ε is really the flatness.

I would next like to show now the projection integrals relate to moments!

$V_n = -2\pi G \int r'^{2+n} dr' \int dz'\, \rho(r',z')\, P_n(z')$

$V_0 = -2\pi G \int r'^2 dr' \int dz'\, \rho(r',z') = -GM$ M = total mass

$V_2 = -2\pi G \int r'^4 dr' \int dz'\, \rho(r',z')\, P_2(z') = ???$

Remember that z' = cosθ, it is not a Cartesian z'. I think the inertial radius is R = r' sinθ'

$I_{33} \equiv \int dxdydz\, \rho(\mathbf{x})[\, r^2 - z^2]$

$I_{11} \equiv \int dxdydz\, \rho(\mathbf{x})[\, r^2 - x^2]$

$I_{33} - I_{11} = \int dxdydz\, \rho(\mathbf{x})\, [x^2 - z^2]$

In sphericals maybe have x = rsinθcosφ and z = rcosθ. Then

$I_{33} - I_{11} = \int d\varphi \int \sin\theta d\theta \int r^2 dr\, \rho(\mathbf{x})\, [\, r^2\sin^2\theta\cos^2\varphi - r^2\cos^2\theta\,]$

Now go from the Cartesian z to the spherical z and write this as

$I_{33} - I_{11} = \int d\varphi \int dz \int r^4 dr\, \rho(\mathbf{x})\, [\, (1-z^2)\cos^2\varphi - z^2\,]$

Then use $\int d\varphi \cos^2\varphi = \pi$ to get

$I_{33} - I_{11} = \int dz \int r^4 dr\, \rho(\mathbf{x})\, [\, \pi(1-z^2)\cos^2\varphi - 2\pi z^2\,]$

$= \pi \int dz \int r^4 dr\, \rho(\mathbf{x})\, [\, (1-z^2) - 2z^2\,]$

$$= \pi \int dz \int r^4 dr\, \rho(\mathbf{x}) \left[(1-3z^2) \right]$$

But $P_2 = (1/2)(3\cos^2\theta - 1$ so $P_2(z) = (1/2)(3z^2-1)$ so

$$(1-3z^2) = -2P_2(z)$$

and then I get

$$I_{33} - I_{11} = -2\pi \int dz \int r^4 dr\, \rho(\mathbf{x}) P_2(z)$$

Compare this to result above

$$V_2 = -2\pi G \int r'^4 dr' \int dz'\, \rho(r',z')\, P_2(z')$$

Therefore I think

$$V_0 = -GM$$

$$V_2 = G\,(I_{33} - I_{11})$$

Then I get

$$V(r,\theta) = V_0/r + V_2\, r^{-3}\, P_2(\cos\theta)$$

$$= -GM/r + G\,(I_{33} - I_{11})\, P_2(\cos\theta)/\, r^3$$

This agrees with

The gravitational potential of an ellipsoid is given by:

$$U_G = -G\frac{M_E}{r} - G\frac{(C-A)}{r^3}\frac{\left(3\cos^2\theta - 1\right)}{2} = -G\frac{M_E}{r} - G\frac{(C-A)}{r^3}P_2\left(\cos\theta\right)$$

so I guess I have obtained this thing two ways now. The C and A are inertia moments. Then all you have to do is compute the torque on this as I have already done on scratch.