G.8 Comments on Rigid Body Dynamics

In this section we assume the Special Case #4 version of Fig 1 where Frame S and Frame S' have the same origin.

Assume Frame S is inertial and Frame S' is embedded into a rotating rigid body. At some instant in time Frame S' might have an orientation relative to Frame S according to the Euler angles of Fig (G.5.5).

From (11.3.2) Newton's Second Law (angular motion) applied in inertial Frame S states,

$$\mathbf{N} = \dot{\mathbf{L}} \qquad // \ \dot{\mathbf{L}} = \partial_{\mathtt{S}}\mathbf{L}$$

where we set the torque and angular momentum reference point to the common Frames' origin. Using the G Rule (2.1) one finds,

$$\dot{\mathbf{L}} = \partial_{\mathtt{S}}\mathbf{L} = \partial_{\mathtt{S}'}\mathbf{L} + \boldsymbol{\omega} \times \mathbf{L}$$

so Newton says

$$(\partial_{\mathtt{S}'}\mathbf{L}) + \boldsymbol{\omega} \times \mathbf{L} = \mathbf{N} \ .$$

Note that $(\partial_{\mathtt{S}'}\mathbf{L})$ is not a "natural" object in the sense of Section 1.8 since **L** is a Frame S object, but the time derivative taken in Frame S'.

We wish to take the Frame S' component of the above equation. For $(\partial_{\mathtt{S}'}\mathbf{L})$ we write

$$(\partial_{\mathtt{S}'}\mathbf{L})'_{\mathtt{i}} = \partial_{\mathtt{S}'}[\ (\mathrm{L})'_{\mathtt{i}}] = \partial_{\mathtt{t}}(\mathrm{L})'_{\mathtt{i}}$$

In the first equality we have used the commutation theorem (1.11.1) that $(\partial_{\mathtt{S}'}\mathbf{a})'_{\mathtt{j}} = \partial_{\mathtt{S}'}[(\mathrm{a})'_{\mathtt{j}}]$. In the second equality we use the fact (1.10.1) that the time derivative of a tensor *component* requires no frame specification. The Frame S' components of ** are therefore

$$\partial_{\mathtt{t}}(\mathrm{L})'_{\mathtt{i}} + \varepsilon_{\mathtt{ijk}}(\omega)'_{\mathtt{j}}(\mathrm{L})'_{\mathtt{k}} = (\mathrm{N})'_{\mathtt{i}}$$

$$(\partial_{\mathtt{S}'}\mathbf{L})'_{\mathtt{i}} = \partial_{\mathtt{S}'}[\ (\mathrm{L})'_{\mathtt{i}}] = \partial_{\mathtt{t}}(\mathrm{L})'_{\mathtt{i}} \qquad \text{// using below 1.11.1}$$

Then we really should write rigid body doc

$$\partial_{\mathtt{t}}(\mathrm{L})'_{\mathtt{i}} + [\ \boldsymbol{\omega} \times \mathbf{L}]'_{\mathtt{i}} = \mathrm{N}'_{\mathtt{i}}$$

or

$$\mathrm{d}(\mathrm{L})'_{\mathtt{i}}/\mathrm{dt} + \varepsilon_{\mathtt{ijk}}(\omega)'_{\mathtt{j}}(\mathrm{L})'_{\mathtt{k}} = (\mathrm{N})'_{\mathtt{i}}$$

Meanwhile, the angular version of $\mathbf{p} = m\mathbf{v}$ is $\mathbf{L} = I\boldsymbol{\omega}$ where $\boldsymbol{\omega}$ is the angular momentum vector and I is the inertia tensor. Then

$$\dot{\mathbf{L}} = I\dot{\boldsymbol{\omega}}$$

$$\partial_t (L)'_i = I_{ij}\ \partial_t (\omega)'_j = I_{ij}\ (\dot{\omega})'_j$$

Question: Is I the Frame S inertia tensor? I think yes, since I have the Frame S $\mathbf{L}$, and $\dot{\boldsymbol{\omega}}$ is the same object in both Frames from ***. But somehow you want it to be I' which is diagonal only in Frame S'. Have to mess with that a bit. This same question is fudged in my Goldstein section 5.5 notes!

Then

$$I_{ij}\ (\dot{\omega})'_j + \varepsilon_{ijk} (\omega)'_j (L)'_k = (N)'_i$$

If Frame S' is chosen so that the inertia tensor of the rigid body is diagonal, $I = \mathrm{diag}(I_1, I_2, I_3)$, then

$$I_i\ (\dot{\omega})'_i + \varepsilon_{ijk} (\omega)'_j (L)'_k = (N)'_i$$

meanwhile again

$$\mathbf{L} = I\boldsymbol{\omega} \ \Rightarrow\ (L)'_i = I_i (\omega)'_i$$

Then it is

$$I_i\ (\dot{\omega})'_i + \varepsilon_{ijk} (\omega)'_j\ I_k\ (\omega)'_k = (N)'_i$$

$$I_1\ (\dot{\omega})'_1 + \quad (\omega)'_2\ I_3\ (\omega)'_3\ - (\omega)'_3\ I_2\ (\omega)'_2\ =\ (N)'_1$$

or

$$I_1\ (\dot{\omega})'_1 + (\omega)'_2\ (\omega)'_3 [\ I_3\ - I_2] =\ (N)'_1$$

I guess you then say

$$(N)'_1 = N_1$$

(the true torque in both frames is the same)

Then you have

$$I_1\ (\dot{\omega})'_1 - (\omega)'_2\ (\omega)'_3 [\ I_2\ - I_3] =\ N_1$$

This then explains the equations I see in Goldstein page 159.

This is pretty much all I wanted to do in this Comment. Relate things to Goldstein equations, but show the Frame S' stuff carefully in terms of notation. This does bring in some frames doc facts!

I guess I should also read through this Goldstein section first.

Update 2.27.17

I have reviewed Goldstein on this topic, and he fudges the issue of what frame L and I are "in". He just says nothing about it. So I will need to look elsewhere. It seems to me right now that in terms of a spinning symmetric top, say, that I will only be diagonal in the body frame. So we have this contradiction that in writing **L** = I**ω** Goldstein would be using the space L and the body I, which to me does not make sense. I don't think that **L** and **L'** are the same. In my Special Case #4 summary I say

$$\mathbf{L} = \mathbf{L'} + m\mathbf{r'} \times (\boldsymbol{\omega} \times \mathbf{r'}) \qquad \text{(f)}$$

for a point particle in the rigid body. So you cannot say **L = L'** and escape that way! I will need another source on this topic! I will start with the usuals:

Marion: On page 365 (12.17) he writes $\mathbf{L} = m\mathbf{r'} \times (\boldsymbol{\omega} \times \mathbf{r'})$ summed over particles in the rigid body. How does this relate to my equation above?? I think he is implying that this is all in the body system. I think I need to resolve this question. For his L, he uses v = ω x r as the velocity of each particle relative to the fixed origin and just sums L = r x p relative to the fixed origin point.

Resolution! In the body frame, the spinning top is at rest and so **L'** = 0, duh!! For each particle in the body frame, **p'** = 0 so **r'** x **p'** = 0. So I guess L really is the space **L** as I have assumed all along above.

So now consider **L** = I**ω** again in the space frame. How can you just "select" principal axes?

You cannot select Frame S to be aligned with the top axis because that axes is moving (rotating, not to mention nutating) but S has to be an inertial frame!

and indeed it is the Frame S' components that you want!! This last equation appears as

$$\frac{dL_i}{dt} + \epsilon_{ijk}\omega_j L_k = N_i.$$

and you see how they are using a very compact notation. But I agree, their ω_i really are the Frame S' components!!!