Value of (I_3-I_1) for a slightly oblate or prolate spheroid of constant density

which is valid for any perimeter equation r(θ). For the oblate perimeter of ** we can compute

$$I_3 - I_1 = -2\pi \int_0^{\pi} d\theta \sin\theta P_2(\cos\theta) \int_0^{r(\theta)} r^4 dr \rho(\mathbf{r})$$

$$= -2\pi\rho \int_0^{\pi} d\theta \sin\theta P_2(\cos\theta) (1/5) r(\theta)^5$$

Now recall

$$r(\theta) = a [1 - (2/3)\varepsilon P_2(\cos\theta)]$$

so

$$r(\theta)^5 \approx a^5 [1 - (10/3)\varepsilon P_2(\cos\theta)]$$

Then

$$I_3 - I_1 = -2\pi\rho(1/5) \int_0^{\pi} d\theta \sin\theta P_2(\cos\theta) a^5 [1 - (10/3)\varepsilon P_2(\cos\theta)]$$

$$= 2\pi\rho a^5 (1/5) (10/3)\varepsilon \int_0^{\pi} d\theta \sin\theta P_2(\cos\theta) P_2(\cos\theta)$$

$$= 2\pi\rho a^5 (1/5) (10/3)\varepsilon \, 1/ [2+1/2] =$$

$$= 2\pi\rho a^5 (1/5) (10/3)\varepsilon (2/5) =$$

$$= 2\pi\rho a^5 \varepsilon (20/75) = 2\pi\rho a^5 \varepsilon (4/15) = (8/15)\pi \rho a^5 \varepsilon$$

and then

$$J_2 = G(I_3 - I_1) = (8/15) G \pi \rho a^5 \varepsilon$$

Now to zeroth order we know that

$$\rho = M/[(4/3)\pi a^3]$$

so get

$J_2 = G(I_3-I_1) = (8/15)G\,\pi\,\rho a^5\varepsilon = \dfrac{(8/15)G\,\pi\,a^5\varepsilon\,M}{(4/3)\pi a^3} = (2/5)GM\,a^2\varepsilon$

=

But Texas says,

$$J_2 = \frac{8\pi G\gamma a^5 \epsilon}{15} = \frac{2}{5} G M a^2 \epsilon,$$

But this is the wrong sign on the second term, and wrong coefficient since right answer is

Source #1 (wrong)

$$U_G = -G\frac{M_E}{r} - G\frac{(C-A)}{r^3}\frac{(3\cos^2\theta - 1)}{2} = -G\frac{M_E}{r} - G\frac{(C-A)}{r^3}P_2(\cos\theta)$$

Source #2 (right)

$$\Phi(r,\theta) = -\frac{G M}{r} + \frac{2}{5}\frac{G M a^2}{r^3}\,\epsilon\, P_2(\cos\theta) + \mathcal{O}(\epsilon^2).$$

Source #3 (right)

$$U(P) = -\frac{GM}{r} + \frac{G}{r^3}(C - A)\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) \qquad (2.35)$$

where C is the different one, so this source agrees with Texas

I am dead in the water until this sign error is located. Time 11:30AM.

2

But apart from a minus sign, this is our desired integral J_2 shown above, so we conclude that

$J_2 = -(I_3-I_1)$

and then the gravitational potential of the oblate Earth is

$$V(\mathbf{r}) \approx -G M_E/r + G (I_3-I_1) P_2(\cos\theta) / r^3$$

The oblate spheroid perimeter function r(θ) of *** appears as the upper endpoint of the radial integration. We know that

$$\int_0^{2\pi} d\varphi \cos^2\varphi = \pi \qquad \text{and} \qquad \int_0^{r(\theta)} r^4 dr = (1/5) r(\theta)^5$$

so

$$I_3 - I_1 = \int_0^{\pi} d\theta \sin\theta (\pi \sin^2\theta - 2\pi \cos^2\theta) (1/5) r(\theta)^5$$

$$= (\pi/5) \int_0^{\pi} d\theta \sin\theta (1 - 3\cos^2\theta) r(\theta)^5$$

$$= -(2\pi/5) \int_0^{\pi} d\theta \sin\theta P_2(\cos\theta) r(\theta)^5$$

Now

$$r(\theta) = a [1 - (2/3)\varepsilon P_2(\cos\theta)]$$

so

$$r(\theta)^5 \approx a [1 - (10/3)\varepsilon P_2(\cos\theta)]$$

Then

$$I_3 - I_1 = -(2\pi a/5) \int_0^{\pi} d\theta \sin\theta P_2(\cos\theta) [1 - (10/3)\varepsilon P_2(\cos\theta)]$$

Now use $z = r\cos\theta$ to write

$$I_3 - I_1 = -(2\pi a/5) \int_{-1}^{1} dz P_2(z) [1 - (10/3)\varepsilon P_2(z)]$$

Recall from ** that

$$\int_{-1}^{1} dz P_n(z) P_{n'}(z) = \delta_{n,n'} / (n+1/2) \qquad \text{// orthogonality}$$

Since $P_2(z) = P_2(z)P_0(z)$, the first integral in vanishes leaving

$$I_3 - I_1 = (2\pi a/5)(10/3)\varepsilon \int_{-1}^{1} dz\, P_2(z)P_2(z)$$

$$= (2\pi a/5)(10/3)\varepsilon\,(2/5) = (2\pi a/5)(2/3)\varepsilon\,(2) = 8\pi a\varepsilon/15$$

Because $P_2(z) = P_2(z)P_0(z)$, we know the first integral vanishes because the P_n are orthogonal as shown in

something is of course wrong here. If $\varepsilon = 0$, I get $I_3 - I_1 \neq 0$.

and therefore

$$I_3 - I_1 = \int_0^{\pi} d\theta \sin\theta\,(\pi \sin^2\theta - 2\pi \cos^2\theta) \int_0^{a(\theta)} r^2 dr\; r^2$$

$$= \pi \int_0^{\pi} d\theta \sin\theta$$

If the potential of interest is azimuthally symmetric in φ (such as outside a spheroidal Earth), only those atomic forms with m = 0 can contribute, reducing the atomic form set to

$$[r^n, r^{-n-1}]\, P_n(z) \quad .$$

The four lines above then become

$g(z) = \Sigma_{n=0}^{\infty} g_n P_n(z)$	$z = \cos\theta$	// expansion
$g_n = (1/K_n^0) \int dz\, P_n(z)\, g(z)$	$K_n^0 = (n+1/2)^{-1} f(n,0)$	// projection
$\int_{-1}^{1} dz\, P_n(z)P_{n'}(z) = \delta_{n,n'} K_n^0$	$n,n' = 0,1,2..$	// orthogonality
$\Sigma_{n=0}^{\infty}(1/K_n^0)\, P_n(z')\, P_n(z) = \delta(z'-z)$	$f(n,0) = \Gamma(n+1)/\Gamma(n+1) = 1$	// completeness

Of particular interest is this expansion and projection of a potential V(z)

$$V(z) = \Sigma_{n=0}^{\infty} V_n P_n(z) \qquad \text{// expansion}$$

$$V_n = (n+1/2) \int_{-1}^{1} dz\, P_n(z)\, V(z) \qquad \text{// projection} \qquad (I.8.1)$$

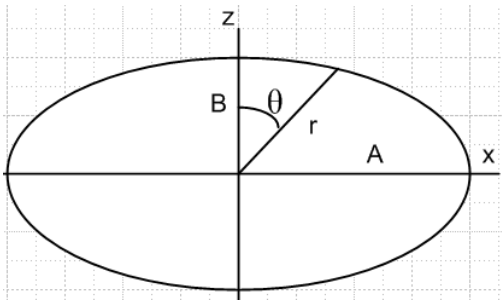
If V(z) is even in z, only even values of n appear in the expansion.

We put this Legendre transform on hold for a while to consider an oblate spheroid.

Equation of a slightly oblate ellipse and spheroid

Consider an ellipse of the form $x^2/A^2 + z^2/B^2 = 1$. If we measure polar angle θ down from the z axis, then $x = r\sin\theta$ and $z = r\cos\theta$ and the ellipse equation becomes

$$r^2 = \frac{A^2B^2}{B^2\sin^2\theta + A^2\cos^2\theta}$$



For an "oblate" (wide) ellipsoid we have A > B. If the ellipse is only slightly oblate, write

$$A = B+\delta$$

. In this case we get

$$r^2 = \frac{A^2B^2}{B^2\sin^2\theta + (B+\delta)^2\cos^2\theta} \approx \frac{A^2B^2}{B^2 + 2B\delta\cos^2\theta} \approx \frac{A^2}{1 + 2(\delta/B)\cos^2\theta}$$

$$\approx A^2\,[\,1-2(\delta/B)\cos^2\theta\,] \approx A^2\,[\,1-2(\delta/A)\cos^2\theta\,]$$

Then

$$r \approx A\,[\,1-(\delta/A)\cos^2\theta\,]$$

The "eccentricity" e of the ellipse is

$$e^2 \equiv (A^2-B^2)/A^2 = (A+B)(A-B)/A^2 \approx 2A\delta/A^2 = 2\delta/A$$

Then have

$$r = A\,[\,1-(1/2)(2\delta/A)\cos^2\theta\,] = [\,1-(1/2)e^2\cos^2\theta\,]$$

On the other hand, the "ellipticity" of an ellipse is

$$\varepsilon \equiv (A-B)/A = \delta/A \qquad // = e^2/2$$

and then we have

$$r = A\,[\,1-(1/2)(2f)\cos^2\theta\,] = A[\,1-\varepsilon\cos^2\theta\,] \ .$$

We now form a slightly oblate spheroid by rotating our slightly oblate ellipse about the vertical axis, introducing an azimuthal coordinate φ. One can then ask : what is the average value of r over this spheroid? The answer is

$$\langle r\rangle = A[\,1-\varepsilon\langle\cos^2\theta\rangle\,]$$

where, approximating the spheroid here as a sphere,

$$\langle\cos^2\theta\rangle = \frac{\int dV\cos^2\theta}{\int dV} \approx \frac{\int d\Omega\cos^2\theta}{\int d\Omega} = \frac{2\pi\int_0^\pi d\theta\,\sin\theta\cos^2\theta}{4\pi} = \frac{2\pi\,(2/3)}{4\pi} = 1/3$$

We then find that

$$\langle r\rangle = A[\,1-(1/3)\varepsilon\,] \equiv a$$

Then,

$$r = A[\,1-\varepsilon\cos^2\theta\,] = \frac{a}{1-\varepsilon/3}[\,1-\varepsilon\cos^2\theta\,]$$

$$\approx a(1+\varepsilon/3)(1-\varepsilon\cos^2\theta)$$

$$\approx a\,[\,1+\varepsilon/3-\varepsilon\cos^2\theta\,]$$

$$= a\,[\,1-(\varepsilon/3)(3\cos^2\theta-1)\,]$$

$$= a\,[\,1-(2/3)\varepsilon\,P_2(\cos\theta)\,] \qquad // \text{ since } P_2(\cos\theta) = (1/2)(3\cos^2\theta-1)$$

or

$$r(\theta) = a\,[\,1-(2/3)\varepsilon\,P_2(\cos\theta)\,] \qquad \text{(I.8.2)}$$

One can regard r(θ) as a function defining the perimeter of the oblate spheroid. If ε = 0, one of course finds that r = a for a sphere. We have assumed that ε << 1 so we are assuming slight oblateness.

The reader seeking verification will find(I.8.2) to be the very first equation on this nice web page

http://farside.ph.utexas.edu/teaching/336k/Newtonhtml/node108.html

which we have used as a guide. These are notes by Richard Fitzpatrick