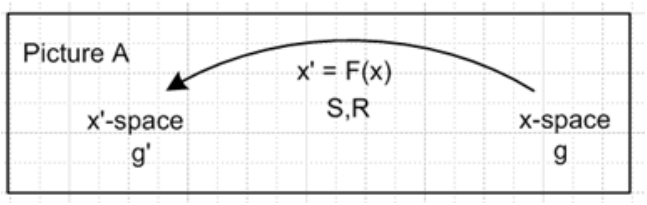
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Appendix J : Relation between this document and *Tensor Analysis and Curvilinear Coordinates*

Although we use covariant notation in this Appendix, a reader unfamiliar with that subject should not be put off , and can perhaps peruse the document mentioned for an explanation.

In our document *Tensor Analysis and Curvilinear Coordinates* there is a general transformation $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ with $d\mathbf{x}' = R d\mathbf{x}$ where $R^{i}{}_{k}(\mathbf{x}) \equiv (\partial x'^{i}/\partial x^{k})$ is the "differential" of the transformation $\mathbf{F}$, and $S = R^{-1}$ with $S^{i}{}_{k}(\mathbf{x}') \equiv (\partial x^{i}/\partial x'^{k})$. This transformation is represented by "Picture A",



(J.1)

Each space has its own covariant metric tensor: g_{ij} for x-space and g'_{ij} for x'-space.

If we let X = x-space and X' = x'-space, then $\mathbf{F}$ is a mapping $\mathbf{F}: X \rightarrow X'$ and $\mathbf{F}: \mathbf{x} \rightarrow \mathbf{x}' = \mathbf{F}(\mathbf{x})$.

A contravariant vector $\mathbf{V}$ transforms under $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ as $V'^{a} = R^{a}{}_{b} V^{b}$.

There are axis-aligned basis vectors in x-space called $\mathbf{u}_n$. Also in x-space are the tangent base vectors $\mathbf{e}_n$ which are tangent to the coordinate lines in x-space, $\mathbf{e}_n = \partial'_n \mathbf{x} = (\partial \mathbf{x}/\partial x'^{n})$. In general the $\mathbf{e}_n$ are not unit vectors, and in general they are not orthogonal to each other. In our polar coordinate example below, they happen to be orthogonal because polar coordinates is an orthogonal curvilinear coordinate system.

Notice that

$$(\mathbf{e}_n)^{i} = (\partial x^{i}/\partial x'^{n}) = S^{i}{}_{n} \tag{J.2}$$

so the tangent base vectors $\mathbf{e}_n$ are the columns of the matrix S. Note also that

$$(\mathbf{e}_n)^{i} = S^{i}{}_{n} = S^{i}{}_{m}\, \delta^{m}{}_{n} = S^{i}{}_{m}\, (\mathbf{u}_n)^{m} = (R^{-1})^{i}{}_{m} (\mathbf{u}_n)^{m} = [R^{-1}\mathbf{u}_n]^{i}$$

so

$$\mathbf{e}_n = R^{-1}\mathbf{u}_n \tag{J.3}$$

One can expand a vector $\mathbf{V}$ onto either of these basis vector sets,

$$\mathbf{V} = V^{i}\, \mathbf{u}_i = V'^{i}\, \mathbf{e}_i \,. \tag{J.4}$$

Picture A thus has a simple **Passive View** interpretation : There is a single vector **V** whose components in the $\mathbf{u}_i$ basis are V_i, and whose components in the $\mathbf{e}_i$ basis are V'_i The components are related by the equation $V'^a = R^a{}_b V^b$. In the Passive View, we can express this equation as $(V)'^a = R^a{}_b V^b$ to stress the fact that there is only one vector **V**. As a shorthand, we can then say $(\mathbf{V})' = R\mathbf{V}$.

In the **Active View** we write the vector transformation equation as $(V')^a = R^a{}_b V^b$ to suggest that R acts on the vector **V** to create a new vector **V'** whose components are $(V')^a = R^a{}_b V^b$. The original vector **V** lies in x-space, while the new vector **V'** lies in x'-space. We have R: X → X' and R: $\mathbf{V} \rightarrow \mathbf{V}' = R\mathbf{V}$.

In *Tensor* there is no "other meaning" for the vector **V'**, so we can write $(V)'^a = (V')^a = V'^a$.

With respect to the underlying transformation $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ (having differential matrix R), the transformation rules for tensors of rank 0,1,2,3 are

	Passive View	Active View	Generic View	(J.5)
0	$(s)' = s$	$(s') = s$	$s' = s$	
1	$(V)'^a = R^a{}_i V^i$	$(V')^a = R^a{}_i V^i$	$V'^a = R^a{}_i V^i$	
2	$(T)'^{ab} = R^a{}_i\, R^b{}_j T^{ij}$	$(T')^{ab} = R^a{}_i\, R^b{}_j T^{ij}$	$T'^{ab} = R^a{}_i\, R^b{}_j T^{ij}$	
3	$(T)'^{abc} = R^a{}_i R^b{}_j R^c{}_b T^{ijk}$	$(T')^{abc} = R^a{}_i R^b{}_j R^c{}_b T^{ijk}$	$T'^{abc} = R^a{}_i R^b{}_j R^c{}_b T^{ijk}$	

Example 1: The Transformation from Cartesian to Polar Coordinates

Let $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ be the non-linear transformation connecting Cartesian x,y and polar θ,r coordinates:

$$x_1 = x_2' \cos(x_1') \quad \text{or} \quad x = r\cos\theta \qquad // \ \mathbf{x} = \mathbf{F}^{-1}(\mathbf{x}') \tag{J.6}$$
$$x_2 = x_2' \sin(x_1') \quad \text{or} \quad y = r\sin\theta \ .$$

The differential matrix R and its inverse $S = R^{-1}$ are easily computed using the definitions above,

$$S^i{}_j = \begin{pmatrix} -r\sin\theta & \cos\theta \\ r\cos\theta & \sin\theta \end{pmatrix} \qquad R^i{}_j = (S^{-1})^i{}_j = \begin{pmatrix} -\sin\theta/r & \cos\theta/r \\ \cos(\theta) & \sin\theta \end{pmatrix} \ . \tag{J.7}$$

Space X is Cartesian with metric tensor $g_{ij} = \delta_{ij}$ and basis vectors

$$\mathbf{u}_1 = \hat{\mathbf{x}} \tag{J.8}$$
$$\mathbf{u}_2 = \hat{\mathbf{y}} \ .$$

The tangent base vectors can be computed using $\mathbf{e}_n = \partial\mathbf{x}/\partial x'^n$ and one finds that

$$\mathbf{e}_1 = \mathbf{e}_\theta = r\,\hat{\boldsymbol{\theta}} \qquad // \text{ note that } \mathbf{e}_1 \text{ is not a unit vector}$$

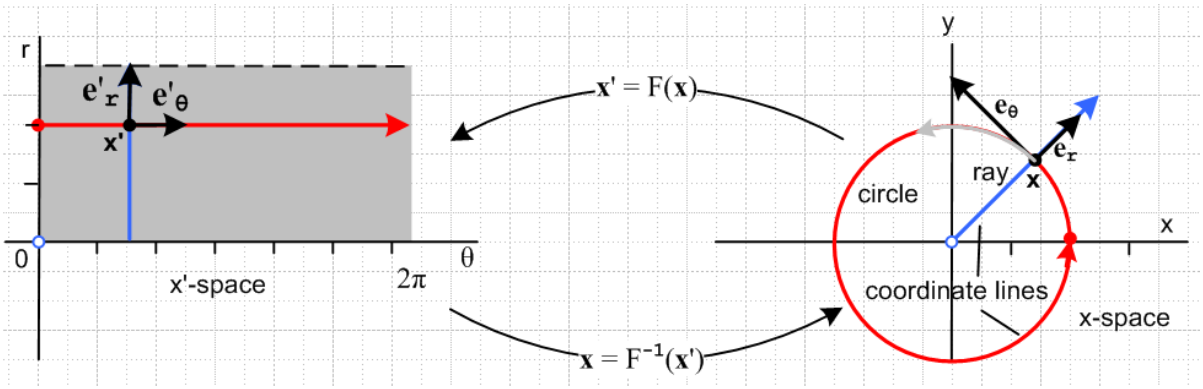
$\mathbf{e}_2 = \mathbf{e}_r = \hat{\mathbf{r}}$. // but $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal (J.9)

This is almost obvious recalling that the $\mathbf{e}_n$ are the columns of $S^i{}_j$.

The metric tensor in space X' is determined by $g'_{ij} = \mathbf{e}_i \bullet \mathbf{e}_j$ so one finds that

$$g'_{ij} = \begin{pmatrix} r^2 & 0 \\ 0 & 1 \end{pmatrix} \neq \delta_{ij} \,. \tag{J.10}$$

Picture A for this example appears this way,



(J.11)

The blue locus on the right is the "r coordinate line" obtained by varying r and holding other components constant. The tangent base vector $\mathbf{e}_r$ is tangent to the blue coordinate line at point $\mathbf{x}$. Similarly, the tangent base vector $\mathbf{e}_\theta$ is tangent to the red coordinate "line" at point $\mathbf{x}$.

The position vector $\mathbf{x} = F^{-1}(\mathbf{x}')$ does not transform as a vector so $\mathbf{x}' \neq R\mathbf{x}$. But one always has $d\mathbf{x}' = Rd\mathbf{x}$ so we know that velocity $\mathbf{v} = d\mathbf{x}/dt$ transforms as a vector. Thus, $\mathbf{v}' = R\mathbf{v}$. We can then expand a velocity vector in the two ways shown above.

$$\mathbf{v} = v^i \, \mathbf{u}_i = v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}$$

$$\mathbf{v} = v'^i \, \mathbf{e}_i = v'^1 \mathbf{e}_1 + v'^2 \mathbf{e}_2 = v'^1 (\, r \, \hat{\boldsymbol{\theta}}) + v'^2 (\hat{\mathbf{r}}) = \dot{\theta} \, r \, \hat{\boldsymbol{\theta}} + \dot{r} \, \hat{\mathbf{r}} \equiv v_\theta \hat{\boldsymbol{\theta}} + v_r \hat{\mathbf{r}} \,. \tag{J.12}$$

In the **Passive View** there is only one velocity $\mathbf{v}$, and we can determine its components in either basis. The components in the two basis are related by $\mathbf{v}' = R\mathbf{v}$ so

$v'^i = R^i{}_j v^j$ // note that R is *not* a rotation matrix

or

$$\begin{pmatrix} v'^1 \\ v'^2 \end{pmatrix} = R \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} -\sin\theta/r & \cos\theta/r \\ \cos(\theta) & \sin\theta \end{pmatrix} \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{J.13}$$

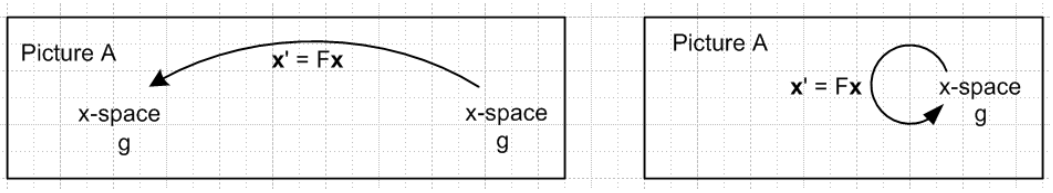
so

$$v'^1 = -(\sin\theta/r) v_x + (\cos\theta/r) v_y \quad \Rightarrow \quad r\, v'^1 = -\sin\theta v_x + \cos\theta v_y = v_\theta = r\,\dot{\theta}$$

$$v'^2 = \cos\theta\, v_x + \sin\theta\, v_y \qquad \Rightarrow \qquad v'^2 = \cos\theta\, v_x + \sin\theta\, v_y = v_r = \dot{r} \tag{J.14}$$

Example 2: Lorentz Transformations

For the special case that $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ is a *linear* transformation, one can write $\mathbf{x}' = F\mathbf{x}$ where F is a matrix, and in this case R = F. A special case of this special case occurs when spaces X and X' are the same space. We can then regard F: X → X' to be a mapping F : X → X so that space X' (our x'-space) is completely unnecessary. Picture A above then becomes



(J.15)

A further special case occurs when the elements of $F^i{}_j$ are constants and don't depend on position **x**. This is the situation for Lorentz transformations in Minkowski space (spacetime).

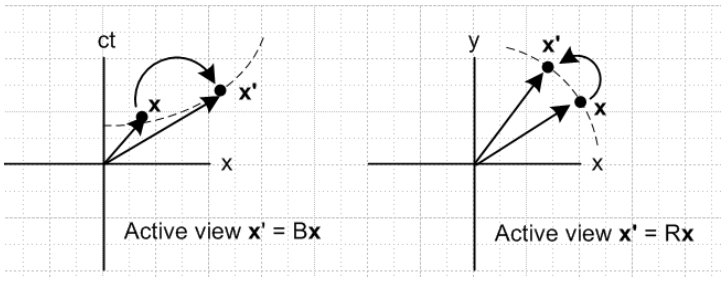
For Lorentz transformations, the space X is a four dimensional space of 4-vectors. One has

$$\mathbf{x}' = F\mathbf{x} \qquad F: \mathbf{x} \to \mathbf{x}' \qquad \begin{aligned} \mathbf{x} &= (x^1,x^2,x^3,x^4) = (x,y,z,ct) \\ \mathbf{x}' &= (x'^1,x'^2,x'^3,x'^4) = (x',y',z',ct') \end{aligned} \tag{J.16}$$

where c is the speed of light and where we show the contravariant components of **x**. The metric tensor is $g_{ij} = \text{diag}(1,1,1,-1)$ so that $\mathbf{x}\bullet\mathbf{x} = g_{ij}x^i x^j = x^2+y^2+z^2-c^2t^2$. The Lorentz transformations are restricted by the requirement that $\mathbf{x}\bullet\mathbf{x} = \mathbf{x}'\bullet\mathbf{x}'$ which makes them be rotations or boosts (velocity transformations). For example, $R_z(\theta)$ which rotates the (x,y) components into (x',y') and leaves (t,z) unchanged is a rotation, whereas $B_x(u)$ which changes (t,x) into (t',x') and leaves (y,z) unchanged is a boost.

One speaks of the coordinates x^i as being those in **Frame S** and x'^i being those for **Frame S'**, but *both these frames are in the same space X*. The idea behind $\mathbf{x}\bullet\mathbf{x} = \mathbf{x}'\bullet\mathbf{x}'$ is that all reference frames related by boosts or rotations should be equivalent, none is special (there is no "ether").

For either type of Lorentz transformation (or for any combination), we can talk about an **Active View**:

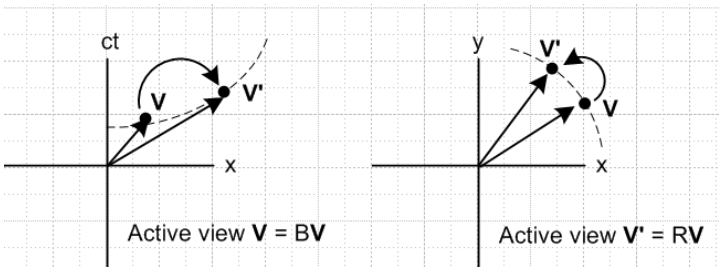


(J.17)

For either a boost or a rotation, the transformation $\mathbf{x}' = F\mathbf{x}$ starts with **x** in X and generates a new vector **x'** in the same space X. Because our pictures are both drawn in Cartesian space, it is not very easy to see

that the quantity $\mathbf{x}\bullet\mathbf{x} = \mathbf{x}'\bullet\mathbf{x}'$ for the boost picture on the left. The dotted curve represents $x^2-c^2t^2 = K < 0$, and so it is in fact true that $\mathbf{x}\bullet\mathbf{x} = x^2+y^2+z^2-c^2t^2 = x'^2+y^2+z^2-c^2t'^2 = \mathbf{x}'\bullet\mathbf{x}'$.

We can redraw the above pictures replacing the 4-vector **x** by some generic 4-vector **V** to get



(J.18)

The meaning of the axis labels is this: in either picture, the horizontal component of **V** is V_x .

A typical vector V^μ would be the momentum of a particle p^μ in which case $\mathbf{p}\bullet\mathbf{p} = p^\mu p_\mu = -m^2c^2$ where m is the particle mass, $p^\mu = (p_x, p_y, p_z, E/c)$ and $\mathbf{p}\bullet\mathbf{p} = p_x^2 + p_y^2 + p_z^2 - E^2/c^2$. If the particle is at rest, one finds that $-E^2/c^2 = -m^2c^2$ or $E = mc^2$.

In the Active View, one maps a vector **V** into a new vector **V'** whose components are $(V')^a = R^a{}_b V^b$.

The axis-aligned unit basis vectors $\mathbf{u}_i$ for space X are the expected $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$, $\hat{\mathbf{t}}$.

For any specific Lorentz transformation $\mathbf{x}' = F\mathbf{x}$ we have tangent base vectors $\mathbf{e}_n = \partial\mathbf{x}/\partial x'^n$,

$$(\mathbf{e}_n)^i = (\partial x^i/\partial x'^n) = (\partial[(F^{-1})^i{}_j x'^j]/\partial x'^n) = (F^{-1})^i{}_j(\partial x'^j/\partial x'^n) = (F^{-1})^i{}_j\,\delta_{jn} = (F^{-1})^i{}_n \,. \qquad (J.19)$$

Unlike the case for polar coordinates, these $\mathbf{e}_n$ are the same at all points in X. We can expand **V** as noted earlier

$$\mathbf{V} = V_i\,\mathbf{u}_i = V'_i\,\mathbf{e}_i \qquad (J.20)$$

and we can then take the **Passive View** that there is one 4-vector **V** and it has coordinates V_i in the $\mathbf{u}_i$ basis and $V'_i = (V)'_i$ in the $\mathbf{e}_i$ basis.

In many special relativity discussions, one writes $x = (x^0, x^1, x^2, x^3) = (t, x, y, z)$ where units are selected to make the speed of light be c = 1. The metric tensor is $g_{ij} = \text{diag}(-1,1,1,1)$ and then one has $p^\mu p_\mu = +m^2c^2 = m^2$. Component indices are Greek letters and a Lorentz transformation is written $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

Example 3: Rotations

This is a subcase of the previous case where now we ignore boosts and the only transformations are rotations. There is still of course an Active and Passive View and this is described in Section 1.3. For rotations there is no need to use covariant up and down indices since $g_{ij} = \delta_{ij}$, so we use only down indices in that Section and below.

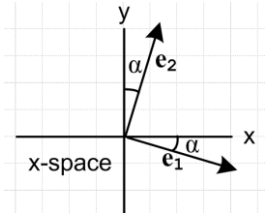
For rotations, the vector **x** is a "tensorial" vector (unlike **x** in the polar coordinates example), and we can write **x**' = R**x** where R is both the transformation matrix F and the differential matrix R. The tangent base vectors $\mathbf{u}_n$ span Frame S in space X, while the tangent base vectors $\mathbf{e}_n$ span Frame S' in the same space. Since R is a rotation, both the $\mathbf{u}_n$ and the $\mathbf{e}_n$ are orthonormal basis vectors.

Recall that the $\mathbf{e}_n$ are the columns of S. Since $S = R^{-1} = R^T$ for a rotation, this means the $\mathbf{e}_n$ are also the *rows* of R. Consider the specific rotation $R_z(\alpha)$ of (A.1) and we look only at the x and y axis. The R matrix is

$$R = R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} . \qquad \text{(J.21)}$$

Thus, the tangent base vectors are

$$\begin{aligned} \mathbf{e}_1 &= \cos\alpha\, \hat{\mathbf{x}} - \sin\alpha\, \hat{\mathbf{y}} \\ \mathbf{e}_2 &= \sin\alpha\, \hat{\mathbf{x}} + \cos\alpha\, \hat{\mathbf{y}} \end{aligned} \qquad \text{(J.22)}$$



(J.23)

Notice that these basis vectors are versions of $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ that are "back-rotated" by α.

With respect to the underlying rotation **x**' = R**x** having differential matrix R, the transformation rules for tensors of rank 0,1,2,3 are

	Passive View	Active View	Generic View	(J.24)
0	$(s)' = s$	$(s') = s$	$s' = s$	
1	$(V)'_a = R_{ai}V_i$	$(V')_a = R_{ai}V_i$	$V'_a = R_{ai}V_i$	
2	$(T)'_{ab} = R_{ai}\, R_{bj}T_{ij}$	$(T')_{ab} = R_{ai}\, R_{bj}T_{ij}$	$T'_{ab} = R_{ai}\, R_{bj}T_{ij}$	
3	$(T)'_{abc} = R_{ai}R_{bj}R_{cb}T_{ijk}$	$(T')_{abc} = R_{ai}R_{bj}R_{cb}T_{ijk}$	$T'_{abc} = R_{ai}R_{bj}R_{cb}T_{ijk}$	

As noted in Section 1.3, because the prime symbol has another meaning in our document's treatment of rotations, we use the Passive View to avoid overloading the meaning of objects like V'_a and T'_{ab}.

It happens that, for the rank 1 and rank 2 cases (only), matrix notation may be used to make the statements more compact. This is obvious for the vector case. For the rank-2 tensor T one has,

$$T'_{ab} = R_{ai}\, R_{bj} T_{ij} = R_{ai} T_{ij} R_{bj} = R_{ai} T_{ij} R^{T}_{jb} = (RTR^{T})_{ab} = (RTR^{-1})_{ab} \quad . \tag{J.25}$$

Thus we can rewrite rows 1 and 2 of the above table as

	Passive View	Active View	Generic View	
1	$(\mathbf{V})' = R\mathbf{V}$	$(\mathbf{V}') = R\mathbf{V}$	$\mathbf{V}' = R\mathbf{V}$	
2	$(T)' = RTR^{-1}$	$(T') = RTR^{-1}$	$T' = RTR^{-1}$	(J.26)

Key Fact: The connection between this Appendix and our main document is as follows:

	Basis Vectors This Appendix	Basis Vectors Main document	Relation	
Frame S	$\mathbf{u}_n$	$\mathbf{e}_n$	$\mathbf{e}_n = R^{-1}\mathbf{u}_n$	
Frame S'	$\mathbf{e}_n$	$\mathbf{e}'_n$	$\mathbf{e}'_n = R^{-1}\mathbf{e}_n$	(J.27)

It is unfortunate that the symbol $\mathbf{e}_n$ has a different meaning in the two systems.

Footnote: *Tensor* has other basis vectors it refers to as $\mathbf{e}'_n$ and $\mathbf{u}'_n$ which are axis-aligned and tangent base vectors for x'-space. Since these are associated with X' = x'-space. and since this space X' is not used in the rotation analysis, these basis vectors can be ignored.