Challenge on angular momentum PhL 2.15.17

The L section of frames is very complicated and has no external verification. But I was able to make it work in the dumbbell satellite situation. As I now resume Goldstein rotational dynamics, I realize that there is a basic equation which I never used and which should be investigated. It is this:

$$(d\mathbf{L}/dt)_S = (d\mathbf{L}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{L} \qquad (1)$$

This simple equation yields a simple analysis of things like tops. It appears on Goldstein page 158. In frames doc I deal a bit with L before I have stated the G rule which the above is for L.

I call this doc a "challenge" because I need to constantly challenge the validity and clarity of frames doc in order to get it right. I have no one else who is available to do this task. I don't mind it at all, because each such challenge increases my understanding of the rotating frame topic and erases my misconceptions.

Notation for equation (1)

When dealing with *other* vectors, for each concept there are two different vectors. For example, (12.1.2) gives us

$$\mathbf{r} = \mathbf{b} + \mathbf{r}' \qquad \mathbf{r}, \mathbf{r}'$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad \mathbf{v}, \mathbf{v}'$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \qquad \mathbf{a}, \mathbf{a}'$$

Here the primed variables are always "natural" ones such as

$$\mathbf{v}' = (d\mathbf{r}'/dt)_{S'} \qquad \mathbf{a}' = (d\mathbf{v}'/dt)_{S'}$$

Not only is there $()_{S'}$ surrounding, but the inside vector is primed as well. So equation (1) above can be written

$$\dot{\mathbf{L}} = (d\mathbf{L}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{L} \qquad (2)$$

where $(d\mathbf{L}/dt)_{S'}$ is NOT a natural vector. I have encountered such "mixed" vectors before, such as

$$\mathbf{v}_{S'} = \mathbf{v}' + \dot{\mathbf{b}}_S - \boldsymbol{\omega} \times \mathbf{b} = \mathbf{v}' + \dot{\mathbf{b}}_{S'} \qquad \textit{(6.3b,a)}$$

$$\mathbf{v}'_S = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \qquad \textit{(6.8a)} \qquad (6.9)$$

Here $\mathbf{v}_{S'}$ and $\mathbf{v}'_S$ are both mixed vectors. I guess I could write (2) above as

$$\dot{\mathbf{L}} = \dot{\mathbf{L}}_{S'} + \boldsymbol{\omega} \times \mathbf{L}$$

and that is my standard notation for a mixed vector $\dot{\mathbf{L}}_{S'}$.

My frames doc has no "section" which relates different forms of $\dot{\mathbf{L}}$ the way it does with **v** and **a** [wrong]. In such a section, presumably the object $\dot{\mathbf{L}}_{S'}$ would appear. I guess one such equation would be

$$\dot{\mathbf{L}}_{S'} = \dot{\mathbf{L}} - \boldsymbol{\omega} \times \mathbf{L}$$

Question: Generally when doing dynamics, I don't use mixed variables. But Goldstein has this mixed $\dot{\mathbf{L}}_{S'}$ sitting there on page 158 [correct]. Now it is true that $\dot{\mathbf{L}} = \mathbf{N}$ so I can say

$$\dot{\mathbf{L}}_{S'} = \mathbf{N} - \boldsymbol{\omega} \times \mathbf{L} \qquad \text{which is G (5-35).}$$

or

$$(d\mathbf{L}/dt)_{S'} = \mathbf{N} - \boldsymbol{\omega} \times \mathbf{L}$$

I have always said that the rotating observer in Frame S' can see both **L** and **L'** and can work with either one. He can compute $(d\mathbf{L}/dt)_{S'}$ for example. So I guess it is OK to have a mixed vector in a solution. [yes]

Let's pause to ask how **L** and **L'** are related, and also to consider the "reference point".

In this section I am encountering MANY notational issues, so this is a good place to try and resolve them. I will summarize the conclusions later.

G does not deal with reference points and I think he avoids that because he co-sites the two origins (why not do that simplifying action!) which causes **b** = **c** = **c'** = 0. For a spinning top, perhaps the contact point is the common origin, it is the rigid body "fixed point", I am not sure of that. Even in my general case, **c** and **c'** represent "the same point" in space as my figure (1.9.1) shows.

So how are **L** and **L'** related ? From (1..9.3) I can say

$$\mathbf{L}^{(c)}{}_{S} \equiv (\mathbf{r}-\mathbf{c}) \times m\mathbf{v}_{S} = (\mathbf{r}-\mathbf{c}) \times m\mathbf{v} = (\mathbf{r}-\mathbf{c}) \times \mathbf{p} \equiv \mathbf{L}^{(c)} = \mathbf{L}$$

$$\mathbf{L}'^{(c')}{}_{S'} \equiv (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}'_{S'} = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}' = (\mathbf{r}'-\mathbf{c}') \times \mathbf{p}' \equiv \mathbf{L}'^{(c')} . = \mathbf{L}' \qquad (1.9.3)$$

I don't attempt to show how they are related at this point in Section 1.9. But eventually I have

$$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [\,(\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_{S}] \qquad (11.2.14)$$

If I co-locate both reference points at a common origin this becomes

$$\mathbf{L}^{(0)} = \mathbf{L}'^{(0')} + m\,\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad (11.2.14)$$

or

$\mathbf{L} = \mathbf{L}' + m\,\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}')$ **

So yes, these really are different vectors.

Now before continuing, suppose m is a particle at rest *in a rigid body* in Frame S'. In fact all particles of the rigid body are at rest, so $\mathbf{v}' = 0$ and this means $\mathbf{p}' = 0$ and then $\mathbf{L}' = 0$. In this case the above becomes

$\mathbf{L} = m\,\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}')$

Question: How do you write $(d\mathbf{L}/dt)_{S'} = \mathbf{N} - \boldsymbol{\omega} \times \mathbf{L}$ in components?

Write first as

$\dot{\mathbf{L}}_{S'} = \mathbf{N} - \boldsymbol{\omega} \times \mathbf{L}$

Then in Frame S' I would write

$(\dot{\mathbf{L}}_{S'})'_i = N'_i - \varepsilon_{ijk}(\omega)'_j(L)'_k$

Now the true torque I think is the same in both frames, just as with true force **F**. For example, the force on a particle would be m**g** in either frame, but in Frame S you have $m(g)_i$ and in Frame S' have $m(g)'_i$ but as a vector equation you say $\mathbf{F} = \mathbf{F}'$. So N'_i in the above equation is the true torque whatever it might be.

But now we come to the inertial business. I will now go brush up on that section of G. Chapter 5.1. So holding here for the moment. // Got it, very simple. On page 144 G shows that with everything in the body or Frame S' frame, we can write in my Frame S' notation. (I just wrote this above so repeating)

$\mathbf{L} = m\,\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}')$

where the particle is at rest in Frame S' (as are all points in the body). This agrees with ** above because in the body, $\mathbf{L}' = 0$ since $\mathbf{L}' = \mathbf{r}' \times \mathbf{p}'$ and $\mathbf{p}' = 0$ since $\mathbf{v}' = 0$ etc etc. We then write out the cross product

$\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}') = (\mathbf{r}'\bullet\mathbf{r}')\boldsymbol{\omega} - (\mathbf{r}'\bullet\boldsymbol{\omega})\mathbf{r}' = r'^2\boldsymbol{\omega} - (\mathbf{r}'\bullet\boldsymbol{\omega})\mathbf{r}'$

so

$\mathbf{L} = m\,[\, r'^2\boldsymbol{\omega} - (\mathbf{r}'\bullet\boldsymbol{\omega})\mathbf{r}']$

Now we take the Frame S' components to get

$(\mathbf{L})'_i = m\,[\, r'^2(\omega)'_i - (\mathbf{r}'\bullet\boldsymbol{\omega})(r')_i] = m\,[\, r'^2(\omega)'_i - \Sigma_j(r')_j(\omega)'_j(r')_i]$

$= m\,[\Sigma_j\, r'^2(\omega)'_j\delta_{ij} - \Sigma_j(r')_j(\omega)'_j(r')_i] = \Sigma_j\,[m\, r'^2\delta_{ij} - m(r')_i(r')_j](\omega)'_j$

$= \Sigma_j I'_{ij} (\omega)'_j \qquad I'_{ij} \equiv m r'^2 \delta_{ij} - m(r')_i(r')_j$

and thus appears the "inertia tensor I' ". For example

$(\mathbf{L})'_1 = \Sigma_j I'_{1j} (\omega)'_j = I'_{11} (\omega)'_1 + I'_{12} (\omega)'_2 + I'_{13} (\omega)'_3$

$= m(r'^2 - m(r')_1^2) (\omega)'_1 + (- m(r')_1(r')_2)(\omega)'_2 + (- m(r')_1(r')_3)(\omega)'_3$

$= m(r'^2 - mx'^2)(\omega)'_x + (- mx'y')(\omega)'_y + (- mx'z')(\omega)'_z$

and this is Gold p 144 (5-45) if you sum over all the particles in the rigid body.

So then here is what we have

$(\mathbf{L})'_i = \Sigma_j I'_{ij} (\omega)'_j$

$(\dot{\mathbf{L}}_{S'})'_i = N'_i - \varepsilon_{ijk}(\omega)'_j(L)'_k$

Note that $(\dot{\mathbf{L}}_{S'})'_i = (\partial_{S'}\mathbf{L})'_i = [(\partial\mathbf{L}/\partial t)_{S'}]'_i$ so restate the above as

$(L)'_i = \Sigma_j I'_{ij} (\omega)'_j$

$[(\partial\mathbf{L}/\partial t)_{S'}]'_i = N'_i - \varepsilon_{ijk}(\omega)'_j(L)'_k$

Notice that $(\partial\mathbf{L}/\partial t)_{S'}$ is indeed a mixed object! **L** is the Frame S vector, but we take time derivative in Frame S'. In vector notation the above read

$\mathbf{L} = \mathbf{I}' \boldsymbol{\omega}$

$(\partial\mathbf{L}/\partial t)_{S'} = \mathbf{N} - \boldsymbol{\omega} \times \mathbf{L} \qquad \mathbf{N} = \mathbf{N}' \qquad ****$

Question: Why do I refer to I' instead of I?

Well, its definition was this

$I'_{ij} \equiv m r'^2 \delta_{ij} - m(r')_i(r')_j$ Note that this thing is symmetric in ij !

and this seems to be an object in Frame S' . I claim this really is a tensor in S'-space not S-space. So I continue to use I' . Later this is vindicated when I say $(\mathbf{I}' \boldsymbol{\omega})'_k = I'_{kn} (\omega)'_n$. This is another topic I should write up. You can only do this is I' is a tensor in Frame S'.

Question: Is there a vector **ω'** and if so, what is it?

In my Fig 1 or other figures you see **ω** and **b** as vectors. The drawing shows **r**, **r'** and **v**,**v'** because these are things related to the two Frames. But **b** and **ω** are vectors in Fig 1 which don't relate to either Frame.

In some sense we could argue **b**' = -**b** and **ω**' = -**ω** if you make Frame S' fixed to the paper, as I have shown in a few places. I might want to say something about that. These two vectors are very different from the other vectors **r**,**v**,**a**,**L** .

Now write **** in components

$$[(\partial \mathbf{L}/\partial t)_{S'}]'_i = N_i - \varepsilon_{ijk}\ (\omega)'_j\ (L)'_k$$

for example

$$[(\partial \mathbf{L}/\partial t)_{S'}]'_1 = N_1 - \varepsilon_{1jk}\ (\omega)'_j\ (L)'_k$$

$$= N_1 - (\omega)'_2\ (L)'_3 + (\omega)'_3\ (L)'_2$$

or

$$[(\partial \mathbf{L}/\partial t)_{S'}]'_x = N_x - (\omega)'_y\ (L)'_z + (\omega)'_z\ (L)'_z$$

or

$$[(\partial \mathbf{L}/\partial t)_{S'}]'_x + (\omega)'_y\ (L)'_z - (\omega)'_z\ (L)'_z = N_x \qquad \text{matches G (5-35)}$$

G's notation is MUCH more compact. In our swap notation the above would read

$$[(\partial \mathbf{L}'/\partial t)_S]_x + (\omega)_y\ (L')_z - (\omega)_z\ (L')_z = N_x$$

$$[(\partial \mathbf{L}'/\partial t)_S]_x + \omega_y\ L'_z - \omega_z\ L'_z = N_x$$

One must keep in mind that the object $[(\partial \mathbf{L}'/\partial t)_S]_x$ is "unnatural" in either notation. So G is using neither our non-swap nor our swap notation! He is in between. He uses **L** for the fixed frame **L**
and he uses just $(\partial \mathbf{L}/\partial t)$ for the "cross derivative" thing. Then he just has

$$(\partial \mathbf{L}/\partial t) + \omega_y\ L_z - \omega_z\ L_z = N_x \qquad (5\text{-}35)$$

Again, there is nothing wrong with having an object $(\partial \mathbf{L}/\partial t)_{S'}$ sitting in an equation and measuring it and working with it as he does.

Now go his next step : we had

$$\mathbf{L} = \mathbf{I}'\ \boldsymbol{\omega}$$

$$(\partial \mathbf{L}/\partial t)_{S'} = \mathbf{N} - \boldsymbol{\omega} \times \mathbf{L} \qquad \mathbf{N} = \mathbf{N}' \qquad ****$$

so insert L from the first to get

$$(\partial \mathbf{L}/\partial t)_{S'} = \mathbf{N} - \boldsymbol{\omega} \times (\mathbf{I}'\ \boldsymbol{\omega})$$

Now write *this* out in components in Frame S'

$$[(\partial \mathbf{L}/\partial t)_{S'}]'_i = N_i - \varepsilon_{ijk}\ (\omega)'_j\ (\mathbf{I}'\ \boldsymbol{\omega}\ )'_k \qquad\qquad (\mathbf{I}'\ \boldsymbol{\omega}\ )'_k\ = I'_{kn}\ (\omega)'_n$$

$$= N_i - \varepsilon_{ijk}\ (\omega)'_j\ I'_{kn}\ (\omega)'_n$$

For example

$$[(\partial \mathbf{L}/\partial t)_{S'}]'_1\ =\ N_1 - \varepsilon_{1jk}\ (\omega)'_j\ I'_{kn}\ (\omega)'_n$$

$$=\ N_1 - \varepsilon_{123}\ (\omega)'_2\ I'_{3n}\ (\omega)'_n\ - \varepsilon_{132}\ (\omega)'_3\ I'_{2n}\ (\omega)'_n$$

$$=\ N_1 - (\omega)'_2\ I'_{3n}\ (\omega)'_n\ + (\omega)'_3\ I'_{2n}\ (\omega)'_n$$

$$=\ N_1 - (\omega)'_2\ [I'_{31}\ (\omega)'_1 + I'_{32}\ (\omega)'_2 + I'_{33}\ (\omega)'_3]\ + (\omega)'_3\ [I'_{21}\ (\omega)'_1 + I'_{22}\ (\omega)'_2 + I'_{23}\ (\omega)'_3]$$

$$=\ N_1 - (\omega)'_2\ [I'_{31}\ (\omega)'_1 + I'_{32}\ (\omega)'_2 + I'_{33}\ (\omega)'_3]\ + (\omega)'_3\ [I'_{21}\ (\omega)'_1 + I'_{22}\ (\omega)'_2 + I'_{23}\ (\omega)'_3]$$

This certainly is a give mess.

Now G adds a new wrinkle. Since I'_{ij} is symmetric, it can be diagonalized, so from now on let's assume that we select the body frame to align with the principle axes which make I be diagonal. Then the above becomes

$$[(\partial \mathbf{L}/\partial t)_{S'}]'_i\ =\ N_i - \varepsilon_{ijk}\ (\omega)'_j\ I'_{kn}\ (\omega)'_n$$

$$= N_i - \varepsilon_{ijk}\ (\omega)'_j\ I'_n\delta_{kn}\ (\omega)'_n$$

$$= N_i - \varepsilon_{ijk}\ (\omega)'_j\ I'_k(\omega)'_k \qquad\qquad *****$$

Then for example

$$[(\partial \mathbf{L}/\partial t)_{S'}]'_1\ = N_1 -\ \ (\omega)'_2\ I'_3(\omega)'_3\ +\ (\omega)'_3\ I'_2(\omega)'_2$$

$$=\ N_1 - (\omega)'_2\ (\omega)'_3\ [\ I'_3 -\ I'_2]$$

Now meanwhile we have

$$\mathbf{L} = \mathbf{I}'\ \boldsymbol{\omega}$$

$$(\partial \mathbf{L}/\partial t)_{S'}\ =\ \mathbf{I}'\ \dot{\boldsymbol{\omega}} \qquad\qquad \text{// recall is the same in both frames}$$

Then

$[(\partial \mathbf{L}/\partial t)_{S'}]'_i = I'_{ii}\,(\dot{\boldsymbol{\omega}})'_i = I'_i\,(\dot{\boldsymbol{\omega}})'_i$ // no sum on i

Then we have from *****

$$I'_i\,(\dot{\boldsymbol{\omega}})'_i = N_i - \varepsilon_{ijk}\,(\omega)'_j\,I'_k(\omega)'_k$$

or

$$I'_i\,(\dot{\boldsymbol{\omega}})'_i + \varepsilon_{ijk}\,(\omega)'_j\,I'_k(\omega)'_k = N_i$$

For example

$$I'_1\,(\dot{\boldsymbol{\omega}})'_1 + (\omega)'_2\,I'_3(\omega)'_3 - (\omega)'_3\,I'_2(\omega)'_2 = N_1$$

or

$$I'_1\,(\dot{\boldsymbol{\omega}})'_1 + (\omega)'_2(\omega)'_3\,[\,I'_3 - I'_2] = N_1$$

or

$$I'_1\,(\dot{\boldsymbol{\omega}})'_x + (\omega)'_y(\omega)'_z\,[\,I'_3 - I'_2] = N_x$$

or

$$I'_1\,(\dot{\boldsymbol{\omega}})'_x - (\omega)'_y(\omega)'_z\,[\,I'_2 - I'_3] = N_x$$ // matches G page 158 A

and then G is off and running. We no longer have any $(\ )_{S'}$ mixed objects in the equation.

So I think I have figured out what G's compact notation means in that section.

Question: Have I overlooked a simple way to express fictional torques?

I have this result in frames doc

$$\mathbf{N}^{(c)} = \dot{\mathbf{L}}^{(c)}$$

$$\mathbf{N}'^{(c')}_{\mathbf{eff}} = \dot{\mathbf{L}}'^{(c')} \qquad (11.3.4)$$

$$\mathbf{N}'^{(c')}_{\mathbf{eff}} = \mathbf{N}^{(c)} + \mathbf{N}'^{(c')}_{\mathbf{fict}} \qquad \text{// defines } \mathbf{N}'^{(c')}_{\mathbf{fict}} \qquad (11.3.5)$$

$$\mathbf{N}'^{(c')}_{\mathbf{fict}} = \dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)}\ . \qquad (11.3.7)$$

It is admittedly messy when written out. Then above I have these equations,

$$\mathbf{N} = (d\mathbf{L}/dt)_S$$

$$(d\mathbf{L}/dt)_S = (d\mathbf{L}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{L} \qquad (1)$$

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}^{(0)}{}_{S'} + \boldsymbol{\omega} \times \mathbf{L}^{(0)}$$

I can write the last as

$$\dot{\mathbf{L}}^{(0)}{}_{s'} - \dot{\mathbf{L}}^{(0)} = -\boldsymbol{\omega} \times \mathbf{L}^{(0)}$$

which compare with

$$\dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)} = \mathbf{N}'^{(c')}_{fict}$$

The left hand sides are not the same !!! $\dot{\mathbf{L}}^{(0)}{}_{s'}$ is a cross -object, $\dot{\mathbf{L}}'^{(c')}$ is a natural.