Rotating Frames of Reference

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last update: February 9, 2017
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Maple code is available upon request. Comments and errata are welcome.
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The graphics look ratty in Windows Adobe PDF viewers when not scaled up, but look just fine in this excellent freeware viewer: https://www.tracker-software.com/product/pdf-xchange-editor .

The table of contents has live links, and use of a wide Bookmarks pane is recommended.

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Overview

This monograph presents an extended discussion of doing physics in non-inertial frames of reference. The first chapters present the general theory, while the latter chapters and appendices contain examples concerning ant paths on turntables, tides, pendulums, fluid flows and tethered/dumbbell satellites. The presentation is entirely self-contained with all support material provided. Where possible, connections are made to external sources on this subject. Both linear (force) and rotational (torque) viewpoints are considered. Maple is used extensively to plot particle trajectories, to obtain and plot numerical solutions of differential equations (dsolve), and to verify complicated equalities.

Our general context is an Apparatus containing a Particle observed from two frames of reference called S and S'. Frame S' is rotating and translating in some arbitrary manner with respect to Frame S as indicated in this drawing,

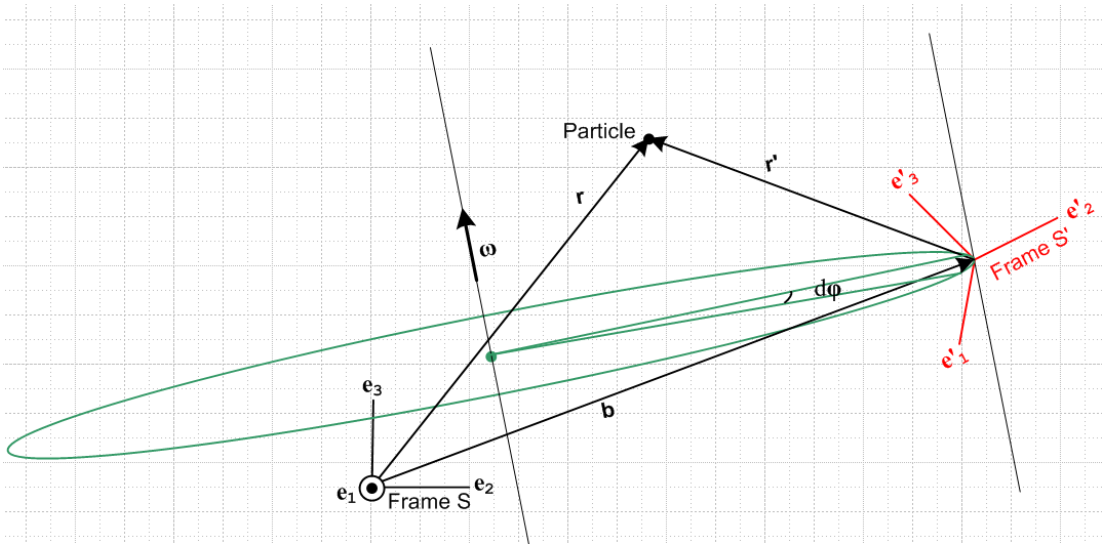


Fig 1

The Particle is located at position **r** relative to the Frame S origin, and at position **r'** relative to the Frame S' origin. These vectors are related by **r** = **r'** + **b** where **b** is a dynamic vector connecting the frame origins. Rotation is about some possibly moving axis with some angular velocity **ω** which might be changing in both direction and magnitude.

We describe the relationship between the properties of the Particle as measured in these two frames of reference. The entire discussion takes place in a non-relativistic framework where time is the same in the two frames. Even in this limited context, things are fairly complicated.

An important subtopic of the rotating frames discussion might be called "Newtonian mechanics in non-inertial frames" where one considers the fate of **F** = m**a** in a non-inertial frame. This is where the famous fictitious forces and less-famous fictitious torques appear.

Most mechanics textbooks which treat rotating frames, having a multitude of other topics to address, spend 10-20 pages on the subject with the following itinerary: state the G Rule (see below), use it to derive an inter-frame velocity and/or acceleration relation, discuss fictitious forces in a rotating frame with emphasis on the Coriolis force, do a few basic problems, and end up treating the Foucault pendulum.

A notable exception is the book of Taylor which devotes 40 pages to the subject including a nice discussion of the tides. (In his book, our frames S and S' are called S_0 and S.)

In this document, having the luxury of no space limitations, we try to probe more deeply into the technical nuts and bolts of rotating frames analysis. Almost all calculations are done in line for the reader to see.

The "papal we" mode of presentation is often used below, as if this paper had multiple authors who seem to own the equations, drawings and experiments as their personal possessions. The approved mode of course is to use passive or impersonal sentence constructions as if the author did not exist. Interestingly, Taylor uses the "I mode" which is perhaps more honest and is certainly refreshing.

Summary

Section 1 sets up the notation used throughout the document. As shown in Fig 1, a decision was made that Frame S' is the rotating frame, even though this conflicts with the choice made by many textbook authors. We refer to this notation as our **non-swap notation** and all our development work is done in this notation. One can imagine another version of Fig 1 which has S ↔ S', which is then our **swap notation**. Often it is more convenient to have Frame S be the rotating frame to avoid an avalanche of primes in the equations of interest, and in that case the "swap" notation is more useful. All key results are summarized in Sections 12 and 13 in both "non-swap" and "swap" notations.

Unit basis vectors are $\mathbf{e}_i$ and $\mathbf{e}'_i$ for Frame S and Frame S'. A generic vector **a** can be expanded on either basis. The presence of two frames of reference often associates with a vector **a** another vector **a**' which leads to the need for a compact notation which distinguishes the components $(a')_i$ and $(a)'_i$ which are often different. **The prime symbol '** plays a central role in the notation and is never used to indicate time differentiation (overdots are used for that purpose).

After considering the meaning of equality for two vectors, we develop the notion of "conical rotational motion" according to $\dot{\mathbf{a}} = \boldsymbol{\omega} \times \mathbf{a}$ and then apply that notion to the basis vectors. The need for putting a frame label on the time derivative of a vector (but not a scalar or vector component) is demonstrated. It is then shown that the Particle in Fig 1 has four distinct velocities and eight distinct accelerations, reinforcing the need for a precise notation. Finally, it is noted that the operations of computing a time derivative and taking a component generally do not commute.

Section 2 derives and discusses what we call "**the G Rule**", namely, $(d\mathbf{a}/dt)_S = (d\mathbf{a}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{a}$ where Frame S' rotates at rate $\boldsymbol{\omega}$ relative to Frame S and **a** is an arbitrary vector.

Section 3 describes an **Observer** and an **Apparatus** containing a Particle, all of which are in the rotating Frame S'.

Section 4 explains the relationship between Frames S and S' for a general placement of the instantaneous rotation axis about which Frame S' rotates at vector angular frequency $\boldsymbol{\omega}$. Two **special cases** are identified for the location of this axis, and then three applications are roughly outlined.

Section 5 (which is a very brief) states **the goal** of subsequent sections which is basically to determine how, given Particle properties in Frame S', one may determine these properties in Frame S. The results are eventually summarized in Section 12.

Section 6 derives the relationships among the four **velocities** mentioned above.

Section 7 derives the relationship among three of the eight **accelerations**. This and the preceding section are as dry as dust (mud's thirsty sister), but the results are of key importance, so everything is done step by step.

Section 8 addresses the traditional subject of **fictitious forces** and interprets them. First the centrifugal and Euler forces are interpreted with the aid of some drawings, then comes the Coriolis force. An arm-waving interpretation of this force is provided for a simple four-projectile problem (on a turntable), and an exact solution to this problem later appears in Section 15. Three **applications** involving fictitious forces are then considered, but not fully analyzed: problems with moving objects near the surface of the Earth, **tethered satellites**, and **ocean tides**.

Section 9 relates our notation to that used by **Marion** (1970) and **Thornton & Marion** [T&M] (2003). It is found that the Marion texts are very close to our "swap" notation.

Section 10 then relates our notation to that used by **Goldstein** (1950) and **Goldstein, Poole and Safko** [GPS] (2001). These texts assume that both reference frames have the same origin which simplifies their presentations.

Section 11 comments on the **angular momentum** vector **L** and its time derivative, and establishes how these vectors are related in the two frames. The notion of **fictitious torques** is introduced. It is demonstrated how both fictitious forces and torques are applied in **fluid dynamics**. Brief comments are made concerning fluid material and **control volumes** and the Reynolds Transport Theorem.

Section 12 summarizes the set of equations which fulfill the goal stated in Section 5: given properties in Frame S', what are they in Frame S? The results are given in both "non-swap" and "swap" notation.

Section 13 then considers the **Inverse Problem**: given properties in Frame S, what are they in Frame S'? The inverse equations are first obtained by laboriously inverting those presented in Section 12.1, and are then reobtained by a simple symmetry operation. The Inverse Problem equations are then summarized in both "non-swap" and "swap" notation.

Section 14 briefly adds the complication of having a different **orthogonal curvilinear coordinate** system in each of the Frames S and S'. Up to this point, only Cartesian coordinates have been used.

Section 15 treats three **"ant on turntable" problems** in some detail. In the first two problems, the ant crawls in a certain manner on the turntable as it rotates (Frame S') and the ant's position, velocity and acceleration are computed in inertial Frame S. In the third problem, the ant flies in a straight line in Frame S just over the turntable surface, and the ant's position, velocity and acceleration are computed in Frame S'. Many (hopefully entertaining) Maple trajectory plots are presented, along with the very simple code for these plots. In the final section, the projectile problem of Section 8.3 is solved using the third problem results, and it is noted that there are hard ways to solve rotation problems that can be avoided.

Appendix A computes a certain matrix R(ξ) which relates spherical unit vectors to Cartesian ones.

Appendix B derives the **G Rule** for a general tensor of rank-n.

Appendix C contains a detailed discussion of the plane (**simple**) **pendulum**, the **spherical pendulum**, and the **Foucault** mode of the spherical pendulum, including details of the **Airy precession**. A compact summary may be found at the start of Appendix C.

Appendix D defines the notion of a **center of gravity** (as distinct from a center of mass) and computes the location of the center of gravity for various orientations of a dumbbell satellite. Certain intuitive notions regarding the location of the center of gravity are seen to be not fully accurate.

Appendix E summarizes useful **facts about spherical coordinates**. An emphasis is on relations involving the spherical unit vectors including their spatial and time derivatives. The position, velocity and acceleration of a point particle are expressed in terms of these spherical unit vectors. A small section provides similar information for polar coordinates.

Appendix F undertakes a detailed study of the motion of a **dumbbell or tethered satellite** in circular orbit around the Earth. The equations of motion are obtained both from a fictitious torque analysis and a fictitious force analysis. Numerical solutions are presented for satellite librations and more general motions. A compact summary may be found at the start of Appendix F.

References are then given for all works mentioned.

1. Notation, important role of the Prime Symbol, and other Preliminaries

Note: When an equation is repeated, its equation number is put in italics.

1.1 The basis vectors $\mathbf{e}_n$ and $\mathbf{e}'_n$ and two ways in which they are related

Frame S has Cartesian basis unit vectors $\mathbf{e}_n$, while Frame S' has Cartesian basis unit vectors $\mathbf{e}'_n$. The two sets of basis vectors are related by some rotation we shall call R (repeated indices have implied sums),

$$\mathbf{e}_n = R\,\mathbf{e}'_n \qquad n = 1,2,3 \qquad \text{or} \qquad (\mathbf{e}_n)_i = R_{ij}(\mathbf{e}'_n)_j \tag{1.1.1}$$

where on the right the component subscripts refer to components in Frame S.

The above relation between $\mathbf{e}_n$ and $\mathbf{e}'_n$ can also be written in this manner

$$\mathbf{e}_n = (R^{-1})_{nm}\,\mathbf{e}'_m \qquad \text{or} \qquad \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \tag{1.1.2}$$

a fact proven below in a short series of steps. Notice that the right equation in (1.1.1) involves a sum of basis vector components, whereas the equations in (1.1.2) involve a sum of basis vectors.

Proof of (1.1.2):

Step 1: The orthonormal Cartesian basis vectors have these properties

$$\delta_{n,k} = \mathbf{e}_n \bullet \mathbf{e}_k = \mathbf{e}'_n \bullet \mathbf{e}'_k \qquad \text{and} \qquad (\mathbf{e}_n)_k = \delta_{n,k} \tag{1.1.3}$$

where $(\mathbf{e}_n)_k = \mathbf{e}_k \bullet \mathbf{e}_n$ denotes the k^{th} component of $\mathbf{e}_n$ in Frame S. The basis vectors $\mathbf{e}_n$ are axis-aligned in Frame S, while the $\mathbf{e}'_n$ are axis-aligned in Frame S'. The dot product is $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \equiv a_i b_i$.

Step 2: Note that

$$(\mathbf{e}'_n)_m = \mathbf{e}_m \bullet \mathbf{e}'_n \overset{(1.1.1)}{=} \mathbf{e}_m \bullet [R^{-1}\mathbf{e}_n] \overset{(1.1.3)}{=} (\mathbf{e}_m)_i\,(R)^{-1}_{ij}(\mathbf{e}_n)_j \overset{(1.1.3)}{=} \delta_{m,i}\,(R)^{-1}_{ij}\,\delta_{n,j}$$

$$= (R^{-1})_{mn} = (R^T)_{mn} = R_{nm} \tag{1.1.4}$$

where in the last step we use the fact that $R^{-1} = R^T$ (or $RR^T = R^TR = 1$) for any rotation. Such a matrix R is said to be "real orthogonal" (T means transpose).

Step 3: We wish to show (1.1.2) that $\mathbf{e}'_n = R_{nm}\mathbf{e}_m$. We shall verify this by showing that all three components of each side are equal in Frame S, and to do this we dot both sides into $\mathbf{e}_k$:

LHS $= \mathbf{e}_k \bullet \mathbf{e}'_n = (\mathbf{e}'_n)_k = R_{nk}$ // using (1.1.4)

RHS $= \mathbf{e}_k \bullet [R_{nm}\,\mathbf{e}_m] = R_{nm}\,\mathbf{e}_k \bullet \mathbf{e}_m = R_{nm}\,\delta_{k,m} = R_{nk}$. QED

Time dependence of the R_{ij}

If Frame S is fixed and Frame S' is rotating, then we really have $\mathbf{e}_n = R(t)\mathbf{e}'_n(t)$ where the R matrix is a function of time, so we have $R_{ij}(t)$. Similarly, if Frame S is moving and Frame S' is fixed, $\mathbf{e}_n(t) = R(t)\,\mathbf{e}'_n$ and again one has $R_{ij}(t)$. Only in the case where there is no rotation between the frames are the R_{ij} independent of time. This means that $\boldsymbol{\omega} = 0$ in Fig 1. Since our document is about "rotating frames of reference" we exclude this no-rotation case from consideration.

Comment: In the above it was convenient to represent all three Cartesian unit vectors in Frame S and Frame S' by $\mathbf{e}_i$ and $\mathbf{e}'_i$. Later when we do particular problems, we shall often revert to the following more standard physics notation:

$$x_1, x_2, x_3 = x, y, z \qquad \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \qquad \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3 = \hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}' \qquad (1.1.5)$$

Then for example $\delta_{n,k} = \mathbf{e}_n \bullet \mathbf{e}_k = \mathbf{e}'_n \bullet \mathbf{e}'_k$ for n = 1 and k = 2 says $0 = \hat{\mathbf{x}} \bullet \hat{\mathbf{y}} = \hat{\mathbf{x}}' \bullet \hat{\mathbf{y}}'$.

1.2 Expansions of a vector and use of primes and parentheses

Note: We write $(a)_i$ as a component of vector $\mathbf{a}$, but $(\mathbf{e}_n)_i$ as a component of $\mathbf{e}_n$. In the first case the a in $(a)_i$ is not bolded, but since $\mathbf{e}_n$ is decorated with a label n, it gets bolded. It is just our convention.

Any vector $\mathbf{a}$ can be expanded on either set of basis vectors so that, with implied summation on i,

$$\mathbf{a} = a_i\,\mathbf{e}_i = a'_i\,\mathbf{e}'_i \quad . \qquad a_i = \mathbf{a} \bullet \mathbf{e}_i \qquad a'_i = \mathbf{a} \bullet \mathbf{e}'_i \quad . \qquad (1.2.1)$$

If some other vector named $\mathbf{a}'$ is lurking in the wings, one might want to be more careful labeling components. A safe method would be this:

$$\begin{aligned} &\mathbf{a} = (a)_i\,\mathbf{e}_i = (a)'_i\,\mathbf{e}'_i && (a)_i = \mathbf{a} \bullet \mathbf{e}_i && (a)'_i = \mathbf{a} \bullet \mathbf{e}'_i \\ &\mathbf{a}' = (a')_i\,\mathbf{e}_i = (a')'_i\,\mathbf{e}'_i && (a')_i = \mathbf{a}' \bullet \mathbf{e}_i && (a')'_i = \mathbf{a}' \bullet \mathbf{e}'_i \quad . \end{aligned} \qquad (1.2.2)$$

Here, a prime *inside* a parentheses is part of the vector name, whereas a prime *outside* a parentheses denotes a vector component in Frame S' (whereas no prime outside means a component in Frame S). Unless the relationship between vectors $\mathbf{a}$ and $\mathbf{a}'$ has a certain simple form, it is very likely that $(a')_i \neq (a)'_i$. In this case the notation a'_i would be ambiguous since one doesn't know whether it refers to $(a')_i$ or $(a)'_i$. It is true that the notation a_i could be unambiguously identified with $(a)_i$, but we shall maintain the parentheses just to be uniform.

Example 1. As an example of the four expansions above, let us consider $\mathbf{a} = \mathbf{e}_n$:

$$\mathbf{e}_n = (\mathbf{e}_n)_i\,\mathbf{e}_i \quad \Rightarrow \quad (\mathbf{e}_n)_i = \delta_{n,i} \quad \text{// by inspection}$$

$$\mathbf{e}'_n = (\mathbf{e}'_n)'_i\,\mathbf{e}'_i \quad \Rightarrow \quad (\mathbf{e}'_n)'_i = \delta_{n,i} \quad \text{// by inspection}$$

$\mathbf{e}_n = (\mathbf{e}_n)'_i \, \mathbf{e}'_i \quad \Rightarrow \quad (\mathbf{e}_n)'_i = (R^{-1})_{ni}$ // using (1.1.2) that $\mathbf{e}_n = (R^{-1})_{ni} \, \mathbf{e}'_i$

$\mathbf{e}'_n = (\mathbf{e}'_n)_i \, \mathbf{e}_i \quad \Rightarrow \quad (\mathbf{e}'_n)_i = R_{ni}$. // using (1.1.2) that $\mathbf{e}'_n = R_{ni} \, \mathbf{e}_i$

We can now restate these results showing the dot products which represent each expansion coefficient, and in this way we obtain expressions for all the dot products,

$\mathbf{e}_n = (\mathbf{e}_n)_i \, \mathbf{e}_i \quad \Rightarrow \quad (\mathbf{e}_n)_i = \mathbf{e}_i \bullet \mathbf{e}_n = \delta_{n,i}$

$\mathbf{e}'_n = (\mathbf{e}'_n)'_i \, \mathbf{e}'_i \quad \Rightarrow \quad (\mathbf{e}'_n)'_i = \mathbf{e}'_i \bullet \mathbf{e}'_n = \delta_{n,i}$

$\mathbf{e}_n = (\mathbf{e}_n)'_i \, \mathbf{e}'_i \quad \Rightarrow \quad (\mathbf{e}_n)'_i = \mathbf{e}'_i \bullet \mathbf{e}_n = (R^{-1})_{ni} = R^{T}_{ni} = R_{in}$

$\mathbf{e}'_n = (\mathbf{e}'_n)_i \, \mathbf{e}_i \quad \Rightarrow \quad (\mathbf{e}'_n)_i = \mathbf{e}_i \bullet \mathbf{e}'_n = R_{ni}$. (1.2.3)

Notice that all these results are consistent with claims made earlier.

In passing, we note that if R is any rotation (so therefore $R^{T}R = 1$), then

$\mathbf{a} \bullet \mathbf{b} = [R\mathbf{a}] \bullet [R\mathbf{b}]$. (1.2.4)

<u>One line proof</u>:

$[R\mathbf{a}] \bullet [R\mathbf{b}] = [R\mathbf{a}]_k [R\mathbf{b}]_k = R_{ki} a_i R_{kj} b_j = (R^{T})_{jk} R_{ki} a_i b_j = (R^{T}R)_{ji} a_i b_j = \delta_{j,i} a_i b_j = a_i b_i = \mathbf{a} \bullet \mathbf{b}$

<u>Matrix Notation to show how components are related.</u>

Consider the following expansion of vector **a** on the basis vectors $\mathbf{e}'_j$,

$\mathbf{a} = (a)'_j \, \mathbf{e}'_j$ // (1.2.2)

$= (a)'_j \{ R_{ji} \, \mathbf{e}_i \}$ // (1.1.2)

$= (a)'_j \{ (R^{-1})_{ij} \, \mathbf{e}_i \}$ // $R = (R^{-1})^{T}$ real orthogonal rotation

$= \{ (R^{-1})_{ij} (a)'_j \) \mathbf{e}_i$. // reorder (1.2.5)

Comparing this to $\mathbf{a} = (a)_i \, \mathbf{e}_i$ of (1.2.2) we conclude that, since $\mathbf{e}_i$ is a complete basis,

$$(a)_i = (R^{-1})_{ij} (a)'_j \quad \Leftrightarrow \quad \begin{pmatrix} (a)_1 \\ (a)_2 \\ (a)_3 \end{pmatrix} = R^{-1} \begin{pmatrix} (a)'_1 \\ (a)'_2 \\ (a)'_3 \end{pmatrix} \qquad (1.2.6)$$

where R^{-1} is a 3x3 rotation matrix.

We can repeat the above discussion replacing **a** with **a'** with this result,

$$(a')_i = (R^{-1})_{ij}(a')'_j \qquad \Leftrightarrow \qquad \begin{pmatrix} (a')_1 \\ (a')_2 \\ (a')_3 \end{pmatrix} = R^{-1} \begin{pmatrix} (a')'_1 \\ (a')'_2 \\ (a')'_3 \end{pmatrix} . \tag{1.2.7}$$

These matrix equations are convenient for computing the components of a vector on the $\mathbf{e}_i$ basis if they are known in the $\mathbf{e}'_i$ basis (and vice versa) .

1.3 Special case where a'_i is unambiguous

We shall now examine the type of relationship between **a'** and **a** in which $(a')_i = (a)'_i$ and therefore we can use the notation a'_i without ambiguity. First, the components $(a)'_i$ and $(a)_i$ are related in the following simple manner, using (1.2.3),

$$(a)'_n = \mathbf{e}'_n \bullet \mathbf{a} = (\mathbf{e}'_n)_m (a)_m = R_{nm}(a)_m . \tag{1.3.1}$$

Now suppose we *define* a new vector **a'** in this way,

$$\mathbf{a}' \equiv R\mathbf{a} . \tag{1.3.2}$$

If a vector **a** transforms into **a'** according to (1.3.2), we say it is a "vector under rotations" which means it "transforms as a vector under rotations".

When written in Frame S components this says

$$(a')_n = R_{nm}(a)_m . \tag{1.3.3}$$

Comparison of (1.3.1) and (1.3.3) shows that

$$(a)'_n = (a')_n \tag{1.3.4}$$

and therefore in this case we can use

$$a'_n \equiv (a)'_n = (a')_n . \tag{1.3.5}$$

Thus, if the vectors **a** and **a'** are related by **a'** = R**a** where R is the rotation appearing in $\mathbf{e}_n = R\,\mathbf{e}'_n$, then we can dispense with the parentheses as shown in (1.3.4). We still have $(a')'_i$ which requires parentheses.

Example 2: Consider equation (1.1.1) ,

$$\mathbf{e}_n = R\,\mathbf{e}'_n .$$

Since this is *not* of the form $\mathbf{a}' \equiv R\mathbf{a}$, we may not dispense with the parentheses. In fact from (1.2.3) we have

$$(\mathbf{e}_n)'_i = R_{in}$$
$$(\mathbf{e}'_n)_i = R_{ni} \qquad (1.3.6)$$

and these are not the same because rotation matrices are not symmetric. Unlike normal vectors for which one writes the transform $\mathbf{a}' = R\mathbf{a}$, the basis vectors are "back-rotated" so $\mathbf{e}'_n = (R^{-1})\mathbf{e}_n$.

Active and Passive

One can think of $\mathbf{a}' = R\mathbf{a}$ as an **active** rotation of vector **a** into another vector **a'** within Frame S. In this case, the components of **a'** are $(a')_i$. The alternative is to think of vector **a** as not moving at all in Frame S, but the basis vectors are back-rotated from $\mathbf{e}_n$ to $\mathbf{e}'_n$ taking us to Frame S'. In this back-rotated basis the components of **a** are $(a)'_i$. This is the **passive** view of a rotation and is in fact the view we take throughout this document because we want to *observe* activities in Frame S from Frame S' and vice versa. Each view has its usefulness and we have just shown that if $\mathbf{a}' = R\mathbf{a}$, then $(a)'_n = (a')_n$.

In the active view, the "apparatus" is rotated and the axes stay put, while in the passive view the apparatus stays put but the axes are back-rotated. There is a third view in which the apparatus and the axes are both rotated in the same direction, and this view is useful in the discussion of covariance of equations in Lucht *Tensor*. One can regard the three views as three "experiments" one might perform. As noted, we shall work with the passive view exclusively in this document.

Example 3: Soon we shall be dealing with the Fig 1 equation $\mathbf{r}' = \mathbf{r} - \mathbf{b}$. Since this is *not* of the form $\mathbf{r}' \equiv R\mathbf{r}$, we may *not* dispense with the parentheses, and we expect that $(r')_i$ and $(r)'_i$ will be different.

Footnote: More generally, if R is the linearized version of some general transformation $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ at a point **x**, so that $d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x}$, then (1.3.2) that $\mathbf{a}' = R\mathbf{a}$ says that **a** "transforms as a contravariant vector with respect to the underlying transformation **F** ". In general $R(\mathbf{x})$ is a combination of rotation and stretch and is a function of location. In our current document we deal only with R(x) = R = a rotation that is the same at all points in space. It turns out however that the notation a'_i is unambiguous in the general case as well as we shall now show. Lucht *Tensor* uses a different notation for basis vectors, and to make the connection between our current document and *Tensor* one must take

$$\{\mathbf{e}_n, \mathbf{e}'_n\} \rightarrow \{\mathbf{u}_n, \mathbf{e}_n\} \qquad \text{Frame S} = \{\mathbf{e}_n\} \rightarrow \text{Frame S} = \{\mathbf{u}_n\}$$
$$\mathbf{e}_n = R\,\mathbf{e}'_n \rightarrow \mathbf{u}_n = R\mathbf{e}_i \qquad \text{Frame S'} = \{\mathbf{e}'_n\} \rightarrow \text{Frame S'} = \{\mathbf{e}_n\} \ .$$

Then in the language of *Tensor* where • is the "covariant dot product", one has (implied summations),

$$(a')_n = \mathbf{a}' \bullet \mathbf{u}_n = (R\mathbf{a}) \bullet (R\mathbf{e}_n) = (R\mathbf{a})_i (R\mathbf{e}_n)^i = R_i{}^j (a)_j\, R^i{}_k (\mathbf{e}_n)^k = (R_i{}^j R^i{}_k)\, (a)_j (\mathbf{e}_n)^k$$

$$= \delta^j{}_k (a)_j (\mathbf{e}_n)^k = (a)_j (\mathbf{e}_n)^j = \mathbf{a} \bullet \mathbf{e}_n = (a)'_n \ .$$

In this general case, $\mathbf{u_n}$ are still axis-aligned basis vectors, but $\mathbf{e_n}$ are generally not axis-aligned and are generally not unit vectors. For example, in spherical coordinates $\mathbf{e_1} = \hat{\mathbf{r}}$, $\mathbf{e_2} = r\,\hat{\boldsymbol{\theta}}$ and $\mathbf{e_3} = r\sin\theta\,\hat{\boldsymbol{\varphi}}$ as shown in (E.6.8).

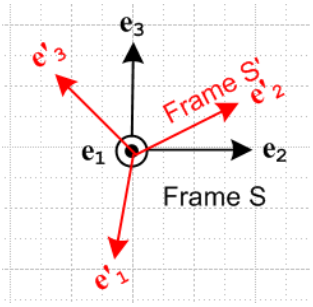
1.4 When are two vectors equal?

This topic will probably seem strange and unnecessary, but it has been a constant annoyance to the author so here are some words on the subject.

When we say two vectors **A** and **B** are the same or are equal, we mean that the two vectors have the same components in the same coordinate system and we write **A** = **B**. This does not require that vectors **A** and **B** coincide. It might be that **B** is a translated copy of **A**. To be really fussy, we could define a stronger equality $\mathbf{A} \triangleq \mathbf{B}$ to mean that not only do the vectors have the same components in the sense of **A** = **B,** but the vectors actually coincide with each other. We shall have no use for $\mathbf{A} \triangleq \mathbf{B}$ in this document. For us, two vectors are "the same" even if translated from one another.

In light of this interpretation of vectors being equal, we can examine the meaning of certain statements. For example, we normally say "a particle is located at **r** in Frame S ". This really means the particle is at *point* **r** in Frame S which has coordinates (x,y,z). What this means in terms of the graphic *vector* **r** is that *if* the vector **r** is translated so that its tail is at the origin of Frame S, *then* its tip will be at the particle location. The vector **r** can be drawn anywhere in a picture. It describes the *displacement* of a particle in Frame S from the origin in Frame S.

Example 4: When we say $\mathbf{e_n} = R\,\mathbf{e'_n}$ as in (1.1.1) above, it is understood that the tails of all vectors involved (the $\mathbf{e_n}$ and the $\mathbf{e'_n}$) are at a common location, as in this picture



(1.4.1)

even though, in our application of Fig 1, the $\mathbf{e_n}$ are drawn with their tails at the origin of Frame S while the $\mathbf{e'_n}$ are drawn with their tails at the origin of Frame S'.

Example 5: In the expansion $\mathbf{r} = (r)_i\mathbf{e_i}$ we normally think of vector **r** having its tail at the origin of Frame S, while in the expansion $\mathbf{r} = (r')'_i\mathbf{e'_i}$ one would be inclined to think of vector **r** as having its tail at the origin of Frame S'. In our stricter sense of coincidence noted above, we might say $(r)_i\,\mathbf{e_i} \triangleq (r')'_i\,\mathbf{e'_i}$ but this is not of interest. What we care about is that $(r)_i\,\mathbf{e_i} = (r')'_i\,\mathbf{e'_i}$ in the sense **A** = **B** above and we don't care if the vectors **A** and **B** are translated relative to one another. What we care about is that the vectors have the same components in any given Frame.

1.5 The small rotation of a vector about an axis

We do this first algebraically and second geometrically. In the algebraic (linear algebra) approach, the reader must accept a few facts about rotation matrices. The 3x3 matrix which rotates a vector by angle φ about rotation axis $\hat{\mathbf{n}}$ according to the right hand rule is obtained by exponentiating another 3x3 matrix,

$$R_{\hat{\mathbf{n}}}(\varphi) = \exp(-i\,\varphi\,\hat{\mathbf{n}} \bullet \mathbf{J}) \,, \tag{1.5.1}$$

where the $(J)_k$ are 3x3 matrices known as the rotation generator matrices:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \,.$$

The numbers in these three matrices can be summarized in this single statement,

$$(J_k)_{ij} = -\,i\,\varepsilon_{kij} \tag{1.5.2}$$

where ε is the totally antisymmetric permutation tensor defined by

$\varepsilon_{abc} = +1$ if abc is an obtained from 123 by an *even* number of pairwise swaps (such as 312)
$\varepsilon_{abc} = -1$ if abc is an obtained from 123 by an *odd* number of pairwise swaps (such as 213)
$\varepsilon_{abc} = 0$ otherwise (ie, when two or more indices have the same value such as 122 or 333) (1.5.3)

Note that $\varepsilon_{abc} = -\,\varepsilon_{bac}$ regardless of index value. A cross product component can be expressed in terms of the permutation tensor as $[\mathbf{a} \times \mathbf{b}]_i = \varepsilon_{ijk} a_j b_k$ with implied sums on j and k.

It is convenient to define a *vector* rotation angle in this manner

$$\boldsymbol{\varphi} \equiv \varphi\,\hat{\mathbf{n}} \tag{1.5.4}$$

and then the rotation (1.5.1) may be written in these new ways,

$$R_{\hat{\mathbf{n}}}(\varphi) = R(\boldsymbol{\varphi}) = \exp(-i\,\boldsymbol{\varphi} \bullet \mathbf{J}) \quad . \tag{1.5.5}$$

For a small rotation $d\boldsymbol{\varphi} = d\varphi\,\hat{\mathbf{n}}$, this may be approximated as (using $e^x = 1 + x + \ldots$ but applied to x = matrix)

$$R(d\boldsymbol{\varphi}) = \exp(-i\,d\boldsymbol{\varphi} \bullet \mathbf{J}) \approx 1 - i\,d\boldsymbol{\varphi} \bullet \mathbf{J} \quad . \tag{1.5.6}$$

With these preliminary remarks out of the way, we can consider the rotation of a vector **a** by a small amount as we move from time t to time t + dt ,

$$\mathbf{a}(t+dt) = R(d\boldsymbol{\varphi})\,\mathbf{a}(t) \tag{1.5.7}$$

where $d\boldsymbol{\varphi} \equiv d\varphi\ \hat{\mathbf{n}}$ is in some arbitrary direction $\hat{\mathbf{n}}$ which is unrelated to the direction of the vector $\mathbf{a}(t)$. The change in vector $\mathbf{a}$ is given by

$$d\mathbf{a} = \mathbf{a}(t+dt) - \mathbf{a}(t) = R(d\boldsymbol{\varphi})\,\mathbf{a}(t) - \mathbf{a}(t) \approx [1 - i\, d\boldsymbol{\varphi} \bullet \mathbf{J}]\,\mathbf{a}(t) - \mathbf{a}(t) = - i\, d\boldsymbol{\varphi} \bullet \mathbf{J}\,\mathbf{a}(t) = - i\, d\varphi_k\, J_k\ \mathbf{a}(t) \ .$$

Taking the i^{th} component of the above equation we get

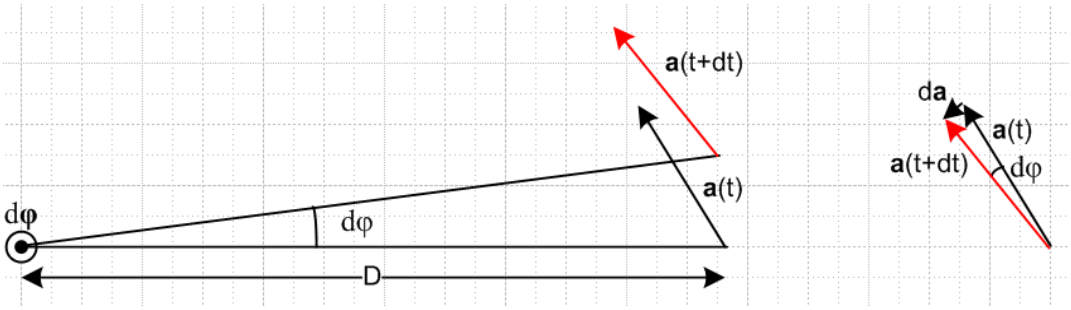
$$da_i = - i\, d\varphi_k [J_k \mathbf{a}]_i = - i\, d\varphi_k (J_k)_{ij} a_j = - i\, d\varphi_k (-i\, \varepsilon_{kij})\, a_j = - \varepsilon_{kij}\, d\varphi_k\, a_j = + \varepsilon_{ikj}\, d\varphi_k\, a_j$$

and going back to vector notation we find that

$$d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a} \qquad (1.5.8)$$

which is our main result. It tells us the change in a vector $\mathbf{a}$ under a small rotation $d\boldsymbol{\varphi}$.

Here then is a graphical derivation of this same fact for a limited geometry. Consider this picture



(1.5.9)

where the small rotation vector $d\boldsymbol{\varphi}$ points out of the plane of paper, and where $\mathbf{a}(t)$ happens to lie in the plane of paper (hence this picture does not cover the most general case). Using the right hand rule for cross products, we can see that $d\mathbf{a}$ (shown on the right) lies in the direction of $d\boldsymbol{\varphi} \times \mathbf{a}$, so we can write that $d\mathbf{a} = C\, d\boldsymbol{\varphi} \times \mathbf{a}$ where C is some constant. We can determine C from the equation $|d\mathbf{a}| = C\ |d\boldsymbol{\varphi} \times \mathbf{a}|$. Since $d\boldsymbol{\varphi}$ and $\mathbf{a}$ are at right angles, we know that $|d\boldsymbol{\varphi} \times \mathbf{a}| = |d\boldsymbol{\varphi}|\ |\mathbf{a}| = d\varphi\, a$. But from the picture on the right it seems quite clear that $|d\mathbf{a}| \approx a\, d\varphi$. Therefore

$$|d\mathbf{a}| = C\ |d\boldsymbol{\varphi} \times \mathbf{a}| \quad \Rightarrow \quad d\varphi\, a = C\, d\varphi\, a \quad \Rightarrow C = 1 \qquad (1.5.10)$$

so we end up with $d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a}$ which agrees with (1.5.8). In the case that $\mathbf{a}$ does not lie in the plane of paper, the geometric derivation takes more work, and in this case we just rely on the algebraic result. We have made free use of the notion that translated vectors are "equal".

Notice the fact that $d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a}$ does not depend on the *distance* D between the rotation axis and the tail of vector $\mathbf{a}$! The result is true even if this distance is 0 so that the tail of vector $\mathbf{a}$ lies right on the rotation axis. In this case the pair of arrows in the left picture coincides with the pair of arrows in the right picture.

Footnote: The equation $R(\boldsymbol{\varphi}) = \exp(-i\,\boldsymbol{\varphi} \bullet \mathbf{J})$ can be interpreted as a rotation in N dimensions with a set of three appropriate NxN generator matrices $(J_k)_{ij}$. Only when N = 3 is (1.5.2) valid or even meaningful. In group theoretic language, these NxN matrices J_k form an N-dimensional irreducible representation of the Lie Algebra so(N) which is $[J_a, J_b] = i\varepsilon_{abc}J_c$. When **J** is exponentiated as shown in (1.5.5), the rotations $R(\boldsymbol{\varphi})$ are then elements of the Lie Group known as the Rotation Group SO(N). The meaning is Special (det = +1), Orthogonal (as in $R^TR = 1$), and N dimensions. The vector $\boldsymbol{\varphi}$ then has N components. For N = 2, the generators are $J_k = \sigma_k/2$ where σ_k are the so-called 2x2 Pauli matrices associated with "spin 1/2".

1.6 The time rate of change of a rotating vector

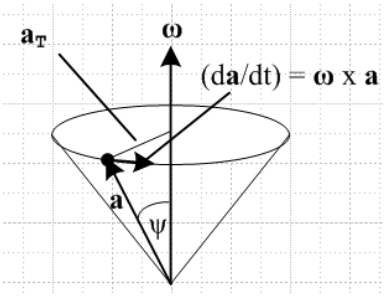
In the previous section we found that,

$$d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a} \quad . \qquad (1.5.8)$$

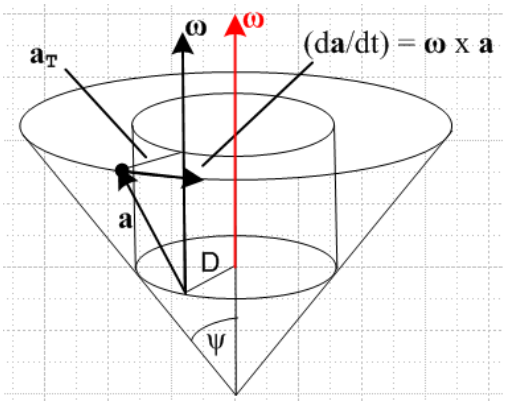
Dividing by dt gives

$$(d\mathbf{a}/dt) = \boldsymbol{\omega} \times \mathbf{a} \qquad \text{where } \boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt \quad . \qquad (1.6.1)$$

This equation is of fundamental importance in this document. It describes the rotation of vector **a** at rate ω about an axis parallel to **ω**. The equation can be represented by either of these "cone pictures" :



(a)



(b)

(1.6.2)

Comments:

1. In (b), the tip of vector **a** traverses a circular path and so does the tail. The tail is always distance D from the rotation axis. One usually sees the simpler picture (a) where D = 0. Recall from above that distance D of the tail from the rotation axis is irrelevant. The motion of the vector **a** is exactly the same in both these cone pictures. (Recall the comments in Section 1.4 about "when are two vectors equal".)

2. It is probably best to describe the rotating motion of vector **a** as a **conical motion** rather than a "circular motion", even though the *tip* of vector **a** travels in a circular motion. In the special case that $\psi = \pi/2$, meaning the tip of the cone has moved to the center of its circular end face (and $\boldsymbol{\omega} \bullet \mathbf{a} = 0$), *then* vector **a** would swing around in a true "circular motion".

3. Note that one might have $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ so the vector $\boldsymbol{\omega}$ could be changing in both direction and magnitude as time progresses. But at time t we have a definite $\hat{\boldsymbol{\omega}}(t)$, and this is sometimes referred to as the **instantaneous** direction of rotation at time t, and $\boldsymbol{\omega}(t)$ the instantaneous angular velocity. In everything below, we always think of $\boldsymbol{\omega}$ in this instantaneous sense, even though we might draw guide circles and cones to show the instantaneous motion at some instant of time t.

4. Knowledge of vector $\boldsymbol{\omega}$ does not in fact say *where* the rotation axis is located. Again we invoke the comments of Section 1.4 above. There are two translational degrees of freedom in that the rotation axis can be translated parallel to itself in two dimensions. Comparing (a) and (b) above, one sees an example of the same $\boldsymbol{\omega}$ but two different rotation axes, one translated relative to the other. In (b) the red and black $\boldsymbol{\omega}$ vectors are the same vector. One normally draws the vector $\boldsymbol{\omega}$ on the rotation axis as done in red.

5. As noted, in (a) and (b) the conical motion of vector **a** is the same and is independent of the placement of the rotation axis as long as it is parallel to $\boldsymbol{\omega}$. *However*, in Fig 1 there is a large change in the physical relationship between Frame S and Frame S' if the rotation axis is translated and $\boldsymbol{\omega}$ stays the same.

6. In the case that $\boldsymbol{\omega}$ = constant, a simple solution to (1.6.1) $\dot{\mathbf{a}} = \boldsymbol{\omega} \times \mathbf{a}$ can be obtained. This equation is a system of three coupled first-order linear ordinary differential equations. Writing $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, the three equations become $\dot{a}_x = -\omega a_y$, $\dot{a}_y = \omega a_x$, and $\dot{a}_z = 0$. Thus $\ddot{a}_x = -\omega^2 a_x$ and we end up with this general solution,

$$\begin{aligned} a_x(t) &= A\cos(\omega t) - B\sin(\omega t) & a_x(0) &= A & // \; a_x^2 + a_y^2 &= A^2 + B^2 = \text{circle} \\ a_y(t) &= A\sin(\omega t) + B\cos(\omega t) & a_y(0) &= B \\ a_z(t) &= a_z(0) = C = \text{constant} & a_z(0) &= C \quad . \end{aligned} \tag{1.6.3}$$

The vector **a** starts out at some $\mathbf{a}(0) = (A,B,C)$ and does the conical motion drawn above, not just in the instantaneous sense, but in the full sense so that **a** really goes around the entire cone.

1.7 Rate of change of the basis vectors

We can apply our rate of change rule (1.6.1) to the basis vectors $\mathbf{e}'_n$ to obtain

$$(d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n \quad .$$

We are implicitly computing this derivative while "standing" in Frame S. That is to say, looking at Fig 1 in the Introduction above, we can regard Frame S as being fixed to the paper and Frame S' is rotating and its basis vectors $\mathbf{e}'_n$ are changing as stated above. It is extremely important to make this fact explicit, so we now add a label S showing this fact,

$$(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n \quad . \tag{1.7.1}$$

Were we to compute this same derivative standing in Frame S', we would get

$$(d\mathbf{e}'_n/dt)_{S'} = 0 \tag{1.7.2}$$

because in Frame S' the basis vectors $\mathbf{e'_n}$ are not rotating at $\boldsymbol{\omega}$, but are just sitting there frozen. By the same argument, we know that

$$(d\mathbf{e_n}/dt)_{\mathbf{S}} = 0 \ . \tag{1.7.3}$$

What about the fourth possible derivative $(d\mathbf{e_n}/dt)_{\mathbf{S'}}$? We will show in (2.5) below that in fact

$$(d\mathbf{e_n}/dt)_{\mathbf{S'}} = -\boldsymbol{\omega} \times \mathbf{e_n} \ . \tag{1.7.4}$$

It seems at least reasonable that if the $\mathbf{e'_n}$ are rotating relative to the $\mathbf{e_n}$ by $\boldsymbol{\omega}$, then the $\mathbf{e_n}$ are rotating relative to the $\mathbf{e'_n}$ by $-\boldsymbol{\omega}$.

Main Conclusion: We thus arrive at the notion that, when dealing with multiple frames of reference and vectors represented in terms of their respective basis vectors, we absolutely must indicate with a label the frame in which a time derivative is being calculated.

1.8 Notations for the many time derivatives of vectors r, r', b and L

In the Overview we described **r** and **r'** as the position vectors of a Particle with respect to Frames S and S'. We can associate with these two vectors *four* different time derivatives.

$$(d\mathbf{r}/dt)_{\mathbf{S}} \qquad (d\mathbf{r}/dt)_{\mathbf{S'}} \qquad (d\mathbf{r'}/dt)_{\mathbf{S}} \qquad (d\mathbf{r'}/dt)_{\mathbf{S'}} \tag{1.8.1}$$

In the usual manner, we represent a time derivative of a vector by an over-dot, and a second time derivative by an over-double-dot. The first time derivative of position **r** is velocity **v**, and the second is acceleration **a**. The same for **r'**, **v'** and **a'** , so

$$\begin{aligned} \mathbf{v} &= \dot{\mathbf{r}} & \mathbf{v'} &= \dot{\mathbf{r}}' \\ \mathbf{a} &= \dot{\mathbf{v}} = \ddot{\mathbf{r}} & \mathbf{a'} &= \dot{\mathbf{v}}' = \ddot{\mathbf{r}}' \end{aligned} \ . \tag{1.8.2}$$

In order to save space, we can define these two *operators*

$$\partial_{\mathbf{S}} \equiv (d/dt)_{\mathbf{S}} \qquad \partial_{\mathbf{S'}} \equiv (d/dt)_{\mathbf{S'}} \tag{1.8.3}$$

Although the ∂ symbol is used in partial differentiation, our use here is exactly as defined above and so our ∂ is not a partial derivative (except in a very abstract sense).

The list of derivatives above can be written now in several ways. Within each column below, all objects are exactly the same thing, just written in different notations: (**r** in Frame S appears as **r'** in Frame S')

Table of first derivatives of **r**

$(d\mathbf{r}/dt)_S$	$(d\mathbf{r}/dt)_{S'}$
$\dot{\mathbf{r}}_S \equiv \dot{\mathbf{r}}$	$\dot{\mathbf{r}}_{S'}$
$\mathbf{v}_S \equiv \mathbf{v}$	$\mathbf{v}_{S'}$
$\partial_S \mathbf{r}$	$\partial_{S'} \mathbf{r}$

Table of first derivatives of **r'**

$(d\mathbf{r}'/dt)_S$	$(d\mathbf{r}'/dt)_{S'}$
$\dot{\mathbf{r}}'_S$	$\dot{\mathbf{r}}'_{S'} \equiv \dot{\mathbf{r}}'$
$\mathbf{v}'_S$	$\mathbf{v}'_{S'} \equiv \mathbf{v}'$
$\partial_S \mathbf{r}'$	$\partial_{S'} \mathbf{r}'$. (1.8.4)

For further clutter reduction, we have added the four new notations shown in red according to the following rule: When a vector and all its derivatives (here only one) are all in (or associated with) the same frame, we suppress the frame subscript and just let the prime or lack of it "do the talking", and refer to such as object as being "natural". There is no notational ambiguity in so doing.

The velocities shown in the center two columns above, $\mathbf{v}_{S'} = (d\mathbf{r}/dt)_{S'}$ and $\mathbf{v}'_S = (d\mathbf{r}'/dt)_S$, shall be referred to as "cross velocities", as distinct from the two "natural velocities".

What about second derivatives? Things are more complicated now because vector **r** can have *four* distinct second derivatives, and so can vector **r'**. Just as done above, we construct a table for each, and within each column of each table, all objects are the same object : (**r** in Frame S appears as **r'** in Frame S')

Table of second derivatives of **r**

$\partial_S \partial_S \mathbf{r}$	$\partial_S \partial_{S'} \mathbf{r}$	$\partial_{S'} \partial_S \mathbf{r}$	$\partial_{S'} \partial_{S'} \mathbf{r}$
$\ddot{\mathbf{r}}_{SS} \equiv \ddot{\mathbf{r}}_S \equiv \ddot{\mathbf{r}}$	$\ddot{\mathbf{r}}_{SS'}$	$\ddot{\mathbf{r}}_{S'S}$	$\ddot{\mathbf{r}}_{S'S'} \equiv \ddot{\mathbf{r}}_{S'}$
$\dot{\mathbf{v}}_{SS} \equiv \dot{\mathbf{v}}_S \equiv \dot{\mathbf{v}}$	$\dot{\mathbf{v}}_{SS'}$	$\dot{\mathbf{v}}_{S'S}$	$\dot{\mathbf{v}}_{S'S'} \equiv \dot{\mathbf{v}}_{S'}$
$\mathbf{a}_{SS} \equiv \mathbf{a}_S \equiv \mathbf{a}$	$\mathbf{a}_{SS'}$	$\mathbf{a}_{S'S}$	$\mathbf{a}_{S'S'} \equiv \mathbf{a}_{S'}$ (1.8.5)

Table of second derivatives of **r'**

$\partial_S \partial_S \mathbf{r}'$	$\partial_S \partial_{S'} \mathbf{r}'$	$\partial_{S'} \partial_S \mathbf{r}'$	$\partial_{S'} \partial_{S'} \mathbf{r}'$
$\ddot{\mathbf{r}}'_{SS} \equiv \ddot{\mathbf{r}}'_S$	$\ddot{\mathbf{r}}'_{SS'}$	$\ddot{\mathbf{r}}'_{S'S}$	$\ddot{\mathbf{r}}'_{S'S'} \equiv \ddot{\mathbf{r}}'_{S'} \equiv \ddot{\mathbf{r}}'$
$\dot{\mathbf{v}}'_{SS} \equiv \dot{\mathbf{v}}'_S$	$\dot{\mathbf{v}}'_{SS'}$	$\dot{\mathbf{v}}'_{S'S}$	$\dot{\mathbf{v}}'_{S'S'} \equiv \dot{\mathbf{v}}'_{S'} \equiv \dot{\mathbf{v}}'$
$\mathbf{a}'_{SS} \equiv \mathbf{a}'_S$	$\mathbf{a}'_{SS'}$	$\mathbf{a}'_{S'S}$	$\mathbf{a}'_{S'S'} \equiv \mathbf{a}'_{S'} \equiv \mathbf{a}'$ (1.8.6)

Here we again use the rule mentioned above that when a vector and all its derivatives are in the same frame, we let the overall prime or lack of it do the talking. We have introduced a second rule as well, which says that whenever both frame subscripts are the same, we suppress one of them to save space. In the tables above we then have two "natural" accelerations **a** and **a'**, and six "cross accelerations".

So now we have the following "natural" vectors having minimal (that is, no) frame subscript clutter:

$\dot{\mathbf{r}}$	$\mathbf{v}$	$\ddot{\mathbf{r}}$	$\dot{\mathbf{v}}$	$\mathbf{a}$	natural in Frame S
$\dot{\mathbf{r}}'$	$\mathbf{v}'$	$\ddot{\mathbf{r}}'$	$\dot{\mathbf{v}}'$	$\mathbf{a}'$	natural in Frame S' (1.8.7)

Recall from the Introduction that the vector **b** connects the origins of two frames. We can make a table of first and second derivatives for this vector as well. Although $\dot{\mathbf{b}}$ is a velocity and $\ddot{\mathbf{b}}$ is an acceleration, we shall not make up separate symbol names for these objects (though some authors do). Also, note that there is no vector in Fig 1 called **b'**, we just have **r** = **r'** + **b**. Here are the corresponding tables for first and second derivatives of **b**, where we use only the second rule above that when two frame subscripts are the same we suppress one of them.

Table of first derivatives of **b** . (Items are the same within each column.)

$(d\mathbf{b}/dt)_S$	$(d\mathbf{b}/dt)_{S'}$
$\dot{\mathbf{b}}_S$	$\dot{\mathbf{b}}_{S'}$
$\partial_S \mathbf{b}$	$\partial_{S'} \mathbf{b}$

(1.8.8)

Table of second derivatives of **b**

$\partial_S \partial_S \mathbf{b}$	$\partial_S \partial_{S'} \mathbf{b}$	$\partial_{S'} \partial_S \mathbf{b}$	$\partial_{S'} \partial_{S'} \mathbf{b}$
$\ddot{\mathbf{b}}_{SS} \equiv \ddot{\mathbf{b}}_S$	$\ddot{\mathbf{b}}_{SS'}$	$\ddot{\mathbf{b}}_{S'S}$	$\ddot{\mathbf{b}}_{S'S'} \equiv \ddot{\mathbf{b}}_{S'}$

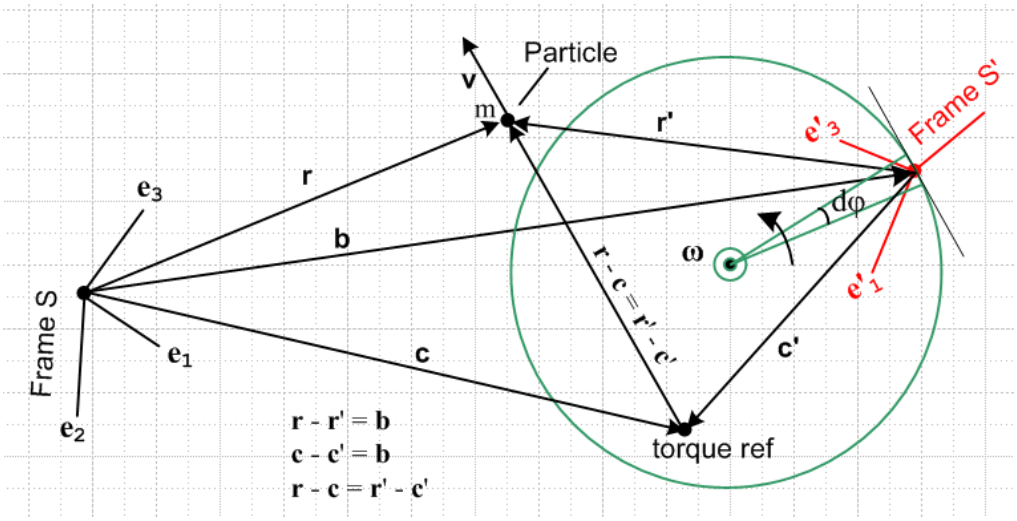
(1.8.9)

Comment: Just as a reminder, *any* derivative in the above section can be expanded onto either Frame S or Frame S' basis vectors, so *any* derivative has Frame S and Frame S' components. This statement would be true for any vector and we just remind the reader that the notation like $(d\mathbf{r}/dt)_S$ does not mean the components are evaluated only in Frame S. Just as an example, this first derivative has these two expansions where the expansion coefficients (components) are shown on the right,

$$(d\mathbf{r}/dt)_S = [(d\mathbf{r}/dt)_S]_i \, \mathbf{e}_i \qquad [(d\mathbf{r}/dt)_S]_i = (d\mathbf{r}/dt)_S \bullet \mathbf{e}_i$$
$$(d\mathbf{r}/dt)_S = [(d\mathbf{r}/dt)_S]'_i \, \mathbf{e}'_i \qquad [(d\mathbf{r}/dt)_S]'_i = (d\mathbf{r}/dt)_S \bullet \mathbf{e}'_i \quad . \tag{1.8.10}$$

1.9 Angular momentum

The drawing of interest is a reoriented Fig 1, to which we have added arbitrary points **c** and **c'**,



(1.9.1)

Whereas the linear momentum **p** = m**v** of a particle does not require a reference point, angular momentum **L** does require such a reference point. For example, the Particle in the above figure has many different values of **L** in Frame S, some of which we might denote as follows,

$$\begin{aligned}
\mathbf{L}^{(0)} &= \mathbf{r} \times m\mathbf{v} && \text{// } \mathbf{L} \text{ in Frame S with respect to Frame S origin} \\
\mathbf{L}^{(c)} &= (\mathbf{r}-\mathbf{c}) \times m\mathbf{v} && \text{// } \mathbf{L} \text{ in Frame S with respect to Frame S point } \mathbf{c} \\
\mathbf{L}^{(b)} &= (\mathbf{r}-\mathbf{b}) \times m\mathbf{v} && \text{// } \mathbf{L} \text{ in Frame S with respect to Frame S point } \mathbf{b} \\
\mathbf{L}^{(b)} &= \mathbf{r}' \times m\mathbf{v} \ . && \text{// } \mathbf{L} \text{ in Frame S with respect to Frame S' origin}
\end{aligned} \tag{1.9.2}$$

The last two lines are exactly the same since **b** is a vector between the two origins so **r'**= **r-b**.

Using the general form $\mathbf{L}^{(c)}$, we may identify the following two "natural" angular momenta in Frames S and S',

$$\mathbf{L}^{(c)}{}_S \equiv (\mathbf{r}-\mathbf{c}) \times m\mathbf{v}_S = (\mathbf{r}-\mathbf{c}) \times m\mathbf{v} = (\mathbf{r}-\mathbf{c}) \times \mathbf{p} \equiv \mathbf{L}^{(c)}$$

$$\mathbf{L}'^{(c')}{}_{S'} \equiv (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}'_{S'} = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}' = (\mathbf{r}'-\mathbf{c}') \times \mathbf{p}' \equiv \mathbf{L}'^{(c')} \ . \tag{1.9.3}$$

These definitions are analogous to **p** ≡ m**v** and **p'** ≡ m**v'** in the linear momentum world.

The time derivative of the first of these objects is given by

$$\begin{aligned}
\dot{\mathbf{L}}^{(c)} &= \dot{\mathbf{L}}^{(c)}{}_S = \partial_S \mathbf{L}^{(c)}{}_S = \partial_S [\, (\mathbf{r}-\mathbf{c}) \times m\mathbf{v}_S] = (\mathbf{v}_S - \dot{\mathbf{c}}_S) \times m\mathbf{v}_S + (\mathbf{r}-\mathbf{c}) \times \partial_S(m\mathbf{v}_S) \\
&= -\dot{\mathbf{c}}_S \times m\mathbf{v}_S + (\mathbf{r}-\mathbf{c}) \times m\mathbf{a}_S \\
&= -\dot{\mathbf{c}} \times m\mathbf{v} + (\mathbf{r}-\mathbf{c}) \times m\mathbf{a} \ .
\end{aligned}$$

A similar result is obtained by priming everything on the above line, so we end up with these four equations:

$$\mathbf{L}^{(c)} = \mathbf{(r-c)} \times m\mathbf{v} \tag{1.9.4}$$

$$\dot{\mathbf{L}}^{(c)} = \mathbf{(r-c)} \times m\mathbf{a} - \dot{\mathbf{c}} \times m\mathbf{v} \tag{1.9.5}$$

$$\mathbf{L}'^{(c')} = \mathbf{(r'-c')} \times m\mathbf{v}' \tag{1.9.6}$$

$$\dot{\mathbf{L}}'^{(c')} = \mathbf{(r'-c')} \times m\mathbf{a}' - \dot{\mathbf{c}}' \times m\mathbf{v}' \quad . \tag{1.9.7}$$

If one thinks of $\mathbf{L}^{(c)} = \mathbf{(r-c)} \times m\mathbf{v}$ just as a cross product of two vectors to get a third vector, one can apply our Theorem (A.20) to write $(\mathbf{L}^{(c)})' = \mathbf{(r'-c')} \times m\mathbf{v}'$ and therefore $(\mathbf{L}^{(c)})' = \mathbf{L}'^{(c')}$. It is just another notation.

The discussion of angular momentum continues in Section 11.

1.10 No frame label is needed for d/dt of a scalar function

If one is differentiating a specific *component* of a vector, such as $a_i(t) = 3t^2$, there is no need to add the frame label since the derivative of this function is 6t no matter what frame it is computed in. This is true when differentiating any component of any tensor. That is to say, if $T_{ij..}(t)$ is a component of a tensor, then

$$(dT_{ij..}(t)/dt)_S = (dT_{ij..}(t)/dt)_{S'} = (dT_{ij..}(t)/dt) \quad . \tag{1.10.1}$$

We would like simply to say that the d/dt derivative of any scalar function does not need a label S or S', but the word "scalar" has multiple meanings. In one meaning, any single function f(t) is a scalar function since it is a 1-tuple of functions, but in another meaning (tensorial scalar), only a function which is a rotational scalar is a scalar function, and this would rule out the component of a vector as being a scalar function. We refer to the first meaning in the title of this subsection.

1.11 When do operations d/dt and taking a component "commute" ?

We claim the following theorem (to be proved below) :

Commutation Theorem: (1.11.1)

If the time derivative ($\partial_{\mathbf{x}}$) is computed in the same frame (Frame X) in which components are taken, then the operations of taking the time derivative and taking the component can be done in either order with the same result, so these operations are said to commute. Otherwise the operations do not commute. Thus

$$(\partial_{\mathbf{s}}\mathbf{a})_{\mathbf{j}} = \partial_{\mathbf{s}}(a)_{\mathbf{j}} \qquad \text{but} \qquad (\partial_{\mathbf{s}}\mathbf{a})'_{\mathbf{j}} \neq \partial_{\mathbf{s}}(a)'_{\mathbf{j}}$$
$$(\partial_{\mathbf{s}'}\mathbf{a})'_{\mathbf{j}} = \partial_{\mathbf{s}'}(a)')_{\mathbf{j}} \qquad (\partial_{\mathbf{s}'}\mathbf{a})_{\mathbf{j}} \neq \partial_{\mathbf{s}'}(a)_{\mathbf{j}} \ .$$

In each of these four equations, on the left we compute $\partial_{\mathbf{x}}\mathbf{a}$ and then we take a component, while on the right side we take a component and then compute $\partial_{\mathbf{x}}$ on that component.

Goldstein makes a point about the non-commuting cases as a "word of caution" on the bottom of page 133 with an example on the top of page 134. In Goldstein, Poole and Safko the caution is stated on page 173 below (4.86), but the example has been removed.

As a preliminary to proving the theorem, we note that for a scalar u and vector **v** one can write,

$$(d[u\mathbf{v}]/dt)_{\mathbf{s}} = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_{\mathbf{s}} \qquad \text{S}$$
$$(d[u\mathbf{v}]/dt)_{\mathbf{s}'} = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_{\mathbf{s}'} \qquad \text{S'}$$

or

$$\partial_{\mathbf{s}}(u\mathbf{v}) = (\partial_{\mathbf{s}}u)\,\mathbf{v} + u\,(\partial_{\mathbf{s}}\mathbf{v}) \qquad \text{S}$$
$$\partial_{\mathbf{s}'}(u\mathbf{v}) = (\partial_{\mathbf{s}'}u)\,\mathbf{v} + u\,(\partial_{\mathbf{s}'}\mathbf{v}) \qquad \text{S'} \qquad (1.11.2)$$

where recall that $\partial_{\mathbf{s}} = \partial_{\mathbf{s}'} = \partial_{\mathbf{t}}$ when applied to a scalar like u. Within either frame this is just the calculus Leibniz product rule applied to as function u(t) times a vector function **v**(t),

$$\partial_{\mathbf{t}}(u\mathbf{v}) = (\partial_{\mathbf{t}}u)\,\mathbf{v} + u\,(\partial_{\mathbf{t}}\mathbf{v}) \ ,$$

so (1.11.2) is just a statement of this known rule in the two frames.

One can trivially generalize (1.11.2) to a sum of the form $u_{\mathbf{i}}\mathbf{v}_{\mathbf{i}}$ (implied sum on i) to get

$$\partial_{\mathbf{s}}(u_{\mathbf{i}}\mathbf{v}_{\mathbf{i}}) = (\partial_{\mathbf{s}}u_{\mathbf{i}})\,\mathbf{v}_{\mathbf{i}} + u_{\mathbf{i}}\,(\partial_{\mathbf{s}}\mathbf{v}_{\mathbf{i}}) \qquad \text{S}$$
$$\partial_{\mathbf{s}'}(u_{\mathbf{i}}\mathbf{v}_{\mathbf{i}}) = (\partial_{\mathbf{s}'}u_{\mathbf{i}})\,\mathbf{v}_{\mathbf{i}} + u_{\mathbf{i}}\,(\partial_{\mathbf{s}'}\mathbf{v}_{\mathbf{i}}) \qquad \text{S'} \qquad (1.11.3)$$

Reader Exercise: Show that $\partial_{\mathbf{x}}(\mathbf{a} \times \mathbf{b}) = (\partial_{\mathbf{x}}\mathbf{a}) \times \mathbf{b} + \mathbf{a} \times (\partial_{\mathbf{x}}\mathbf{b})$ for X = S or S' (more Leibniz). (1.11.4)

Proof of the Commutation Theorem: In the proof we use all the following facts developed above and gathered below for convenience. Notice that it is *not* assumed that **a'** = R**a** and in fact the vector **a'** does not appear anywhere.

$(R^{-1})_{ij} = (R^{T})_{ij} = R_{ji}$ rotation is real orthogonal *(1.1.4)* (1.11.5)

$R_{ij}(t)$ = function of time so $(\partial_t R_{ij}(t)) \neq 0$ and $(\partial_t (R^{-1})_{ij}(t)) \neq 0$ $(\boldsymbol{\omega} \neq 0)$ "rotating frames"

$\mathbf{a} = (a)_i \, \mathbf{e}_i = (a)'_i \, \mathbf{e}'_i$ expansions in the two frames *(1.2.2)*

$(\mathbf{e}_n)_i = \delta_{n,i}$ $(\mathbf{e}'_n)'_i = \delta_{n,i}$
$(\mathbf{e}_n)'_i = (R^{-1})_{ni}$ $(\mathbf{e}'_n)_i = R_{ni}$ basis vector components *(1.2.3)*

$(a)'_i = R_{ij}(a)_j$. components of **a** in the two frames *(1.3.1)*

$\partial_S \mathbf{e}_n = 0$ and $\partial_{S'} \mathbf{e}'_n = 0$ frame definitions *(1.7.3), (1.7.2)*

$\partial_S f(t) = \partial_{S'} f(t) = \partial_t f(t)$ time derivative of a component *(1.10.1)*

$\partial_S(u_i \mathbf{v}_i) = (\partial_S u_i) \, \mathbf{v}_i + u_i \, (\partial_S \mathbf{v}_i)$ product rule in frame S
$\partial_{S'}(u_i \mathbf{v}_i) = (\partial_{S'} u_i) \, \mathbf{v}_i + u_i \, (\partial_{S'} \mathbf{v}_i)$ product rule in frame S' *(1.11.3)*

Here then is the detailed proof of the above theorem. First, consider ∂_S with $\mathbf{a} = (a)_i \, \mathbf{e}_i$:

$(\partial_S \mathbf{a})_j = (\partial_S [(a)_i \mathbf{e}_i])_j = (\partial_S (a)_i) \, (\mathbf{e}_i)_j = (\partial_t (a)_i) \, \delta_{i,j} = \partial_t (a)_j$
$\partial_S (a)_j = \partial_t (a)_j$
$\Rightarrow \partial_S (a)_j = (\partial_S \mathbf{a})_j$ // commutes (∂_S with Frame S components)

$(\partial_S \mathbf{a})'_j = (\partial_S [(a)_i \mathbf{e}_i])'_j = (\partial_S (a)_i) \, (\mathbf{e}_i)'_j = (\partial_t (a)_i) \, R_{ji} = R_{ji} (\partial_t (a)_i)$
$(\partial_S (a)'_j) = \partial_S [R_{ji} (a)_i] = R_{ji} (\partial_S (a)_i) + (\partial_S R_{ji}) \, (a)_i = R_{ji} (\partial_t (a)_i) + (\partial_t R_{ji}) \, (a)_i$
$\Rightarrow (\partial_S (a)'_j) = (\partial_S \mathbf{a})'_j + (\partial_t R_{ji}) \, (a)_i$
$\Rightarrow (\partial_S (a)'_j) \neq (\partial_S \mathbf{a})'_j$ // does not commute (∂_S with Frame S' components)

Next consider $\partial_{S'}$ with $\mathbf{a} = (a)'_i \, \mathbf{e}'_i$:

$(\partial_{S'} \mathbf{a})_j = (\partial_{S'} [(a)'_i \mathbf{e}'_i])_j = (\partial_{S'} (a)'_i) \, (\mathbf{e}'_i)_j = (\partial_{S'} (a)'_i) \, R_{ij} = (R^{-1})_{ji} \, (\partial_t (a)'_i)$
$(\partial_{S'} (a)_j) = \partial_{S'} [(R^{-1})_{ji} (a)'_i] = (\partial_t (R^{-1})_{ji}) (a)'_i + (R^{-1})_{ji} (\partial_t (a)'_i)$
$\Rightarrow (\partial_{S'} (a)_j) = (\partial_{S'} \mathbf{a})_j + (\partial_t (R^{-1})_{ji}) (a)'_i$
$\Rightarrow (\partial_{S'} (a)_j) \neq (\partial_{S'} \mathbf{a})_j$ // does not commute ($\partial_{S'}$ with Frame S components)

$(\partial_{S'} \mathbf{a})'_j = (\partial_S [(a)'_i \mathbf{e}'_i])'_j = (\partial_{S'} (a)'_i) \, (\mathbf{e}_i')'_j = (\partial_{S'} (a)'_i) \, \delta_{i,j} = (\partial_{S'} (a)'_j) = (\partial_t (a)'_j)$
$\partial_{S'} (a)'_j = \partial_t (a)'_j$
$\Rightarrow \partial_{S'} (a)'_j = (\partial_{S'} \mathbf{a})'_j$ // commutes ($\partial_{S'}$ with Frame S' components)

2. The G Rule for arbitrary vector a and its derivation

In this section, **a** is a generic vector -- it could be *any* vector.

The "G Rule for vector **a**" is this (written various equivalent ways)

$$\begin{aligned}
\partial_S \mathbf{a} &= \partial_{S'} \mathbf{a} + \boldsymbol{\omega} \times \mathbf{a} \\
(d\mathbf{a}/dt)_S &= (d\mathbf{a}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{a} \\
\dot{\mathbf{a}}_S &= \dot{\mathbf{a}}_{S'} + \boldsymbol{\omega} \times \mathbf{a} \\
(\dot{\mathbf{a}}_S)_i &= (\dot{\mathbf{a}}_{S'})_i + i\varepsilon_{ijk}\omega_j a_k \quad . \qquad \text{// components}
\end{aligned} \tag{2.1}$$

In the first line we use the abbreviations $\partial_X = (d/dt)_X$ where X is a frame of reference. G is in honor of Herbert Goldstein since the Rule appears on page 133 of his classic 1950 book *Classical Mechanics* (*and* he uses the vector **G** instead of our **a**). Perhaps another name for this rule would be "the rule which relates the time derivatives of a vector taken in two frames of reference S and S' where Frame S' is rotating relative to Frame S at instantaneous vector angular velocity **ω** about some unspecified rotation axis which is parallel to the vector **ω**".

Equation (2.1) goes by a few obscure names, some people calling it a "transport theorem" or a "basic kinematic equation", but most authors who use it give it no name.

Comment: The content of (2.1) must have been known to Coriolis (see Gaspard-Gustave) in 1835, but the cross product notation was not in use at that time. According to Crowe, the cross product idea arose gradually from the work of Hamilton (quaternions) and Grassmann (vector areas) as early as the 1840's, and later from work of Gibbs in the 1880's, with the possible involvement of a certain Reverend O'Brien in 1852. In any event, the cross product notation (and bolded vector notation in general) was first introduced to the textbook-reading public when Gibbs gave his student Wilson the task of publishing and improving his notes. Their book *Vector Analysis*, published in 1901, was reprinted 7 times and was then made a Dover book in 1960 and can be found online (see Refs). It happens, however, that this book makes no mention of the G Rule or Coriolis forces or "frames of reference". It is a math book, not a physics book.

Since the G Rule applies to any vector **a**, sometimes the vector is left out and one writes this operator equation,

$$(d/dt)_S = (d/dt)_{S'} + \boldsymbol{\omega} \times \quad . \tag{2.2}$$

Here is where and how the G rule appears in some popular mechanics texts:

$(d\mathbf{G}/dt)_{space} = (d\mathbf{G}/dt)_{body} + \boldsymbol{\omega} \times \mathbf{G}$	// Goldstein p 133 (4-100) // Goldstein Poole Safko p 172 (4.82)
$(d\mathbf{Q}/dt)_{fixed} = (d\mathbf{Q}/dt)_{rotating} + \boldsymbol{\omega} \times \mathbf{Q}$	// Marion p 342 (11.7) // Thornton Marion p 390 (10.12)
$(d\mathbf{Q}/dt)_{S_0} = (d\mathbf{Q}/dt)_S + \boldsymbol{\omega} \times \mathbf{Q}$	// Taylor p 342 (9.30)
$(d/dt)_{space} = (d/dt)_{body} + \boldsymbol{\omega} \times$	// Goldstein p 133 (4-102)
$(d/dt)_S = (d/dt)_r + \boldsymbol{\omega} \times$	// Goldstein Poole Safko p 173 (4.86)

Proof of the G Rule :

We want to show that

$$\partial_S \mathbf{a} = \partial_{S'} \mathbf{a} + \boldsymbol{\omega} \times \mathbf{a} \quad . \qquad (2.1)$$

Start with the known fact (1.7.1),

$$\partial_S \mathbf{e'}_i = \boldsymbol{\omega} \times \mathbf{e'}_i \quad . \qquad (1.7.1)$$

Then using the above equation and facts collected in (1.11.5) we find,

$$\partial_S \mathbf{a} = \partial_S[(a)'_i \mathbf{e'}_i] = (\partial_S (a)'_i)\mathbf{e'}_i + (a)'_i(\partial_S \mathbf{e'}_i) = (\partial_S (a)'_i)\mathbf{e'}_i + (a)'_i \boldsymbol{\omega} \times \mathbf{e'}_i$$

$$= (\partial_S (a)'_i)\mathbf{e'}_i + \boldsymbol{\omega} \times [(a)'_i \mathbf{e'}_i] = (\partial_S (a)'_i)\mathbf{e'}_i + \boldsymbol{\omega} \times \mathbf{a} \quad . \qquad (2.3)$$

But,

$$\partial_{S'} \mathbf{a} = \partial_{S'}[(a)'_i \mathbf{e'}_i] = (\partial_{S'} (a)'_i)\, \mathbf{e'}_i \qquad \text{since} \qquad \partial_{S'} \mathbf{e'}_i = 0 \quad . \qquad (2.4)$$

Inserting this into (2.3) then gives

$$\partial_S \mathbf{a} = \partial_{S'} \mathbf{a} + \boldsymbol{\omega} \times \mathbf{a}$$

which is the desired G rule (2.1). QED

Comment: The G Rule involves two frames of reference called Frame S and Frame S'. It is true that in Fig 1 and Fig (1.9.1) we think of Frame S as being "glued to the paper", but there is nothing that says the paper might not be rotating about some other axis $\boldsymbol{\omega}'$. In other words, even if Frame S and Frame S' are both non-inertial frames, the G Rule is still valid because nothing in its derivation requires that either frame be inertial. Inertial only matters when we later start talking about $\mathbf{F} = m\mathbf{a}$. The G Rule involves Frame S' rotating by ω *relative to* Frame S. An implication is that any equation obtained below by use of the G Rule is also valid even if both Frame S and Frame S' are non-inertial.

Example 1: (reversing the above) Suppose $\mathbf{a} = \mathbf{e'}_i$. Then our rule (2.1) says

$$(d\mathbf{e'}_i/dt)_S = (d\mathbf{e'}_i/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{e'}_i \quad .$$

But we noted in (1.7.2) the obvious fact that $(d\mathbf{e'}_i/dt)_{S'} = 0$, so the above becomes

$$(d\mathbf{e'}_i/dt)_S = \boldsymbol{\omega} \times \mathbf{e'}_i$$

which agrees with (1.7.1).

Example 2: Suppose $\mathbf{a} = \mathbf{e}_i$. Then rule (2.1) says

$$(d\mathbf{e}_i/dt)_S = (d\mathbf{e}_i/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{e}_i \quad .$$

But we noted in (1.7.3) the obvious fact that $(d\mathbf{e}_i/dt)_S = 0$, so the above becomes

$$(d\mathbf{e}_i/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_i \qquad (2.5)$$

and we have now derived the claim made in (1.7.4).

Example 3: Suppose $\mathbf{a} = \boldsymbol{\omega}$. Then rule (2.1) says

$$(d\boldsymbol{\omega}/dt)_S = (d\boldsymbol{\omega}/dt)_{S'} + \boldsymbol{\omega} \times \boldsymbol{\omega} = (d\boldsymbol{\omega}/dt)_{S'}$$

so for this one vector $\boldsymbol{\omega}$ both derivatives are the same and we write

$$(d\boldsymbol{\omega}/dt)_S = (d\boldsymbol{\omega}/dt)_{S'} \equiv d\boldsymbol{\omega}/dt = \dot{\boldsymbol{\omega}} \quad . \qquad (2.6)$$

Example 4. Using the dot notation of (1.8.2), and assuming there are two related vectors $\mathbf{a}$ and $\mathbf{a}'$, the G Rule states

$$\dot{\mathbf{a}}_S = \dot{\mathbf{a}}_{S'} + \boldsymbol{\omega} \times \mathbf{a} \qquad \text{that is} \qquad \partial_S \dot{\mathbf{a}} = \partial_{S'} \dot{\mathbf{a}} + \boldsymbol{\omega} \times \mathbf{a}$$
$$\dot{\mathbf{a}}'_S = \dot{\mathbf{a}}'_{S'} + \boldsymbol{\omega} \times \mathbf{a}' \qquad \text{that is} \qquad \partial_S \dot{\mathbf{a}}' = \partial_{S'} \dot{\mathbf{a}}' + \boldsymbol{\omega} \times \mathbf{a}' \quad . \qquad (2.7)$$

Example 5. We can apply this rule to any of the vectors listed in Section 1.8. Here are a few examples:

$$\dot{\mathbf{b}}_S = \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times \mathbf{b} \qquad (2.8)$$

$$\dot{\mathbf{r}}_S = \dot{\mathbf{r}}_{S'} + \boldsymbol{\omega} \times \mathbf{r} \qquad \text{or} \qquad \mathbf{v}_S = \mathbf{v}_{S'} + \boldsymbol{\omega} \times \mathbf{r}$$
$$\dot{\mathbf{r}}'_S = \dot{\mathbf{r}}'_{S'} + \boldsymbol{\omega} \times \mathbf{r}' \qquad \text{or} \qquad \mathbf{v}'_S = \mathbf{v}'_{S'} + \boldsymbol{\omega} \times \mathbf{r}' \qquad (2.9)$$
$$\dot{\mathbf{v}}_S = \dot{\mathbf{v}}_{S'} + \boldsymbol{\omega} \times \mathbf{v}_S \qquad \text{or} \qquad \mathbf{a}_S = \mathbf{a}_{S'} + \boldsymbol{\omega} \times \mathbf{v}_S$$
$$\dot{\mathbf{v}}'_S = \dot{\mathbf{v}}'_{S'} + \boldsymbol{\omega} \times \mathbf{v}'_S \qquad \text{or} \qquad \mathbf{a}'_S = \mathbf{a}'_{S'} + \boldsymbol{\omega} \times \mathbf{v}'_S \quad . \qquad (2.10)$$

Question: Are there any restrictions on the vector $\mathbf{a}$ for which the G Rule applies? The only fact used above about $\mathbf{a}$ is that $\mathbf{a}$ can be expanded on Cartesian basis vectors as $\mathbf{a} = a'_i \, \mathbf{e}'_i$ and that the component functions $a'_i(t)$ are differentiable. If the tail of vector $\mathbf{a}$ does not lie at the origin of Frame S', we just translate $\mathbf{a}$ such that this is the case, as per Section 1.4.

Significance of the G Rule: In all the computations below, basically the only operations done are these:

- Apply the G Rule to some vector
- Apply ∂_S to both sides of some equation
- Apply $\partial_{S'}$ to both sides of some equation

Note: In Appendix B we generalize the G Rule to tensors of arbitrary rank, so the G Rule of this Section is the general case applied to rank 1 tensors (vectors).

3. The Apparatus and its Observer at Rest in Frame S'

In our general "experiment" to be described below, Frame S' is rotating with respect to Frame S. This does not necessarily imply that Frame S is "at rest", but Frame S is at rest with respect to the paper on which we draw Fig 1. If Frame S is truly at rest with respect to the stars, then Frame S is called an inertial frame, and in such a frame Newton's 2nd Law **F** = m**a** is valid. We do not in general assume that S is such an inertial frame.

Imagine now that we have some Apparatus sitting in Frame S' which contains a Particle which undergoes some motion. The Particle might be a mass on one or more springs, or it might be a Particle in ballistic flight, or it might be a Particle of matter in a gear wheel which is turning in some complicated machine, or it might be a Particle of a fluid or of an elastic solid.

An Observer also sitting (at rest) in Frame S' has some measurement equipment, can see the axes of Frame S' of course, and does certain measurements on the Particle *while Frame S' is rotating* with respect to Frame S. The Particle is located at position **r** in Frame S and position **r'** in Frame S'. The Observer can see Frame S and is aware of both **r** and **r'** and can make measurements of them both.

For example, although **r** is the position vector of the Particle relative to the Frame S origin, our Observer measures its components in Frame S' this way

$$\mathbf{r} = (r)'_i \mathbf{e}'_i \qquad\qquad (r)'_i = \mathbf{r} \bullet \mathbf{e}'_i$$

Meanwhile, measurements of **r'** reveal these different components

$$\mathbf{r}' = (r')'_i \mathbf{e}'_i \qquad\qquad (r')'_i = \mathbf{r}' \bullet \mathbf{e}'_i \ .$$

However, the Observer can only measure frame-S' derivatives. From our list we then select some items :

$$\mathbf{v}_{S'} = \dot{\mathbf{r}}_{S'} = (d\mathbf{r}/dt)_{S'} \qquad \text{// cross velocity}$$

$$\mathbf{v}'_{S'} = \dot{\mathbf{r}}'_{S'} = (d\mathbf{r}'/dt)_{S'} \equiv \mathbf{v}' \qquad \text{// natural} \qquad (1.8.4)$$

$$\mathbf{a}_{S'} = \dot{\mathbf{v}}_{S'} = (d\mathbf{v}_{S'}/dt)_{S'} = (d^2\mathbf{r}/dt^2)_{S'} \qquad \text{// cross acceleration}$$

$$\mathbf{a}'_{S'} = \dot{\mathbf{v}}'_{S'} = (d\mathbf{v}'_{S'}/dt)_{S'} = (d^2\mathbf{r}'/dt^2)_{S'} \equiv \mathbf{a}' \qquad \text{// natural} \qquad (1.8.6)$$

$$\dot{\mathbf{b}}_{S'} = (d\mathbf{b}/dt)_{S'} \qquad \text{rate of change of } \mathbf{b} \text{ as viewed from Frame S' .} \qquad (1.8.9)$$

For example, the Observer might measure the frame-S Particle location **r**(t) at time t, wait one time tick dt, then measure it again to get **r**(t+dt). Then (the components of **r**(t) are $[\mathbf{r}(t)]'_i$),

$$\mathbf{v}_{S'}(t) = [\mathbf{r}(t+dt) - \mathbf{r}(t)]/(dt) \qquad \text{or} \qquad \mathbf{v}_{S'}(t) = (d\mathbf{r}/dt)_{S'} \qquad \text{// cross velocity}$$

Alternatively, the Observer could measure **r'**(t) and **r'**(t+dt) and get

$$\mathbf{v}'_{S'}(t) = [\mathbf{r}'(t+dt) - \mathbf{r}'(t)]/(dt) \qquad \text{or} \qquad \mathbf{v}'_{S'}(t) = (d\mathbf{r}'/dt)_{S'} = \mathbf{v}' \ . \quad \text{// natural}$$

Waiting another dt tick, one could measure $\mathbf{r}(t+2dt)$ and $\mathbf{r}'(t+2dt)$ and then deduce $\mathbf{v}_{\mathbf{S}'}(t+dt)$ and $\mathbf{v}'_{\mathbf{S}'}(t+dt)$. From these the Observer could determine

$\mathbf{a}_{\mathbf{S}'}(t) = [\mathbf{v}_{\mathbf{S}'}(t+dt) - \mathbf{v}_{\mathbf{S}'}(t)]/(dt)$ or $\mathbf{a}_{\mathbf{S}'}(t) = (d\mathbf{v}_{\mathbf{S}'}/dt)_{\mathbf{S}'}$ // cross

$\mathbf{a}'_{\mathbf{S}'}(t) = [\mathbf{v}'_{\mathbf{S}'}(t+dt) - \mathbf{v}'_{\mathbf{S}'}(t)]/(dt)$ or $\mathbf{a}'_{\mathbf{S}'}(t) = (d\mathbf{v}'_{\mathbf{S}'}/dt)_{\mathbf{S}'} = \mathbf{a}'$. // natural

As a final example, here the Observer measures a cross angular momentum and a natural one :

$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times m\mathbf{v}$ *(1.9.4)*

$\dot{\mathbf{L}}^{(c)}_{\mathbf{S}'} = \partial_{\mathbf{S}'}\mathbf{L}^{(c)} = [\mathbf{L}^{(c+dt)}(t+dt) - \mathbf{L}^{(c)}(t)]/(dt)$ // cross

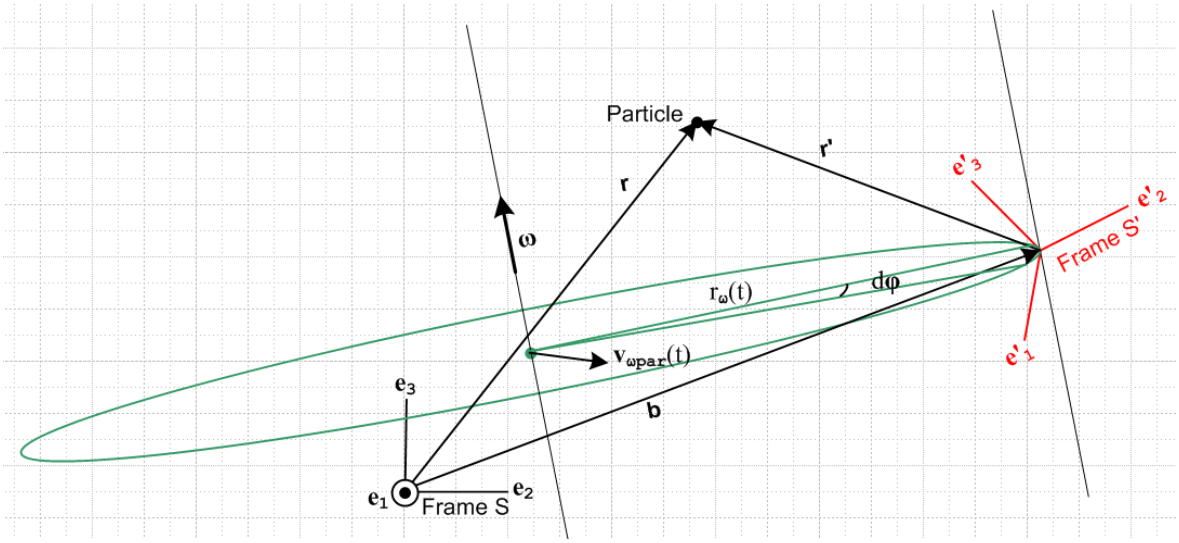
$\mathbf{L}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}'$ *(1.9.6)*

$\dot{\mathbf{L}}'^{(c')}_{\mathbf{S}'} = \partial_{\mathbf{S}'}\mathbf{L}'^{(c')} = [\mathbf{L}'^{(c'+dt)}(t+dt) - \mathbf{L}'^{(c')}(t)]/(dt) = \dot{\mathbf{L}}'^{(c')}$. // natural

4. The Relationship between the Two Frames S and S'

4.1 Explanation of Fig (4.1.1): Frame S in the plane of paper

The relation between the two frames is shown in this picture, a snapshot at some time t :



(4.1.1)

There is a lot to be said about this picture (which is the same as Fig 1 in the Overview) .

The axes $\mathbf{e}_2$ and $\mathbf{e}_3$ of Frame S lie in the plane of paper and are "aligned with paper" as shown and remain fixed relative to paper, so the Frame S origin lies in the plane of paper. The origin of Frame S' is displaced by amount **b** from the origin of Frame S, and this S' origin does not lie in the plane of paper. The location of the Particle does not lie in the plane of paper, so vectors **r**, **r'** and **b** , although coplanar with each other, are each not in the plane of paper. Similarly, the **ω** rotation axis does not lie in the plane of paper nor is it parallel to it.

Frame S' is instantaneously rotating about some axis indicated by **ω**(t). Each of the basis vectors $\mathbf{e'}_n$ is rotating according to (1.7.1), $(d\mathbf{e'}_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e'}_n$, as the Frame S' moves rigidly in rotation about the **ω** rotation axis. The origin of Frame S' is instantaneously rotating along the green circle of some instantaneous radius r_ω which has its center on the rotation axis at a green dot. This green dot, meanwhile, is moving at some velocity $\mathbf{v}_{\omega par}$ in the plane of the green circle relative to Frame S. This indicates the motion of the rotation axis parallel to itself.

Suppose Frame S' contains a rigid object fixed relative to the Frame S' axes. If we were to select some point P in that rigid object, that point P would be instantaneously rotating about the **ω**(t) axis along a circle similar to the one shown above, but which has a different radius and a different center point along the same rotation axis.

If $\hat{\boldsymbol{\omega}}$ were constant in time and if $\mathbf{v}_{\omega par}$ were 0, the origin of Frame S' really would move along the full green circle shown, but we have in mind that **ω** = **ω**(t) and this varies in time, both in magnitude and direction. Thus, the green circle is itself tilting to stay in a plane perpendicular to **ω**(t).

Viewed from Frame S, the unit vectors of Frame S' are oriented and move according to several equations we have already dealt with,

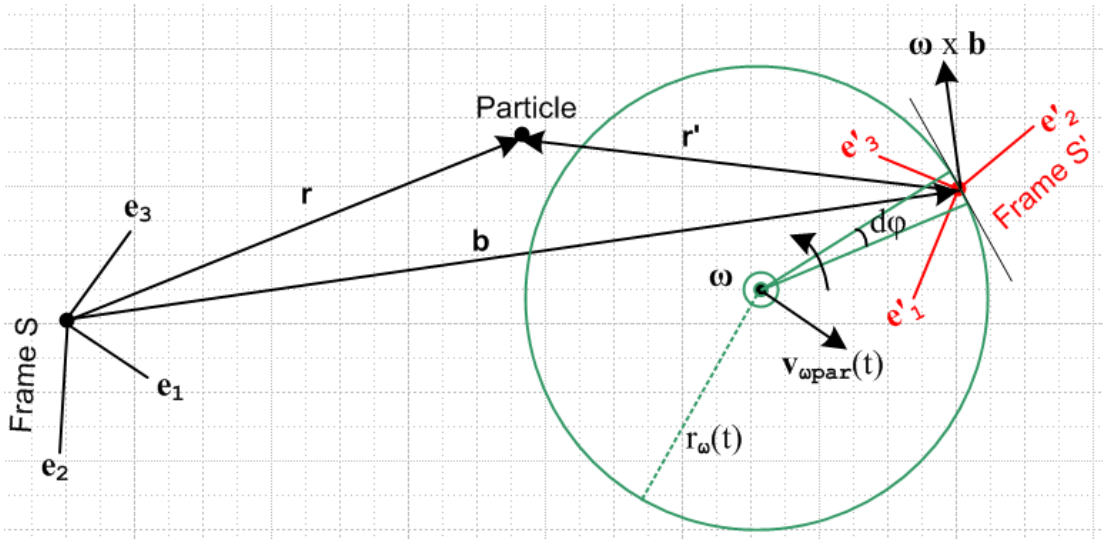
$$\mathbf{e'}_n(t) = R^{-1}(t)\,\mathbf{e}_n \qquad (1.1.1)$$

$$\mathbf{e'}_n(t) = R_{nm}(t)\,\mathbf{e}_m \qquad (1.1.2)$$

$$(d\mathbf{e'}_n/dt)_S = \boldsymbol{\omega}(t) \times \mathbf{e'}_n(t) \quad . \qquad (1.7.1)$$

4.2 Explanation of Fig (4.2.1) : Vector ω pointing directly out of paper

We now draw Fig (4.1.1) from a different perspective. The reader will hopefully forgive the "artist" for not attempting to draw Fig (4.2.1) as a precision 3D rotated version of Fig (4.1.1), but hopefully the general features of the drawing are sufficient for our purposes below.



(4.2.1)

Frame S remains fixed relative to paper as time varies, but is now rotated relative to Fig (4.1.1) and the S origin no longer lies in the plane of paper. At time t for which the picture is drawn, the origin of Frame S' and the green circle and its center *do* lie in the plane of paper. The **ω** vector points straight out at the viewer as indicated by the circled dot. Vectors **r**, **r'** and **b** do not lie in the plane of paper.

Fig (4.1.1) is meant to give the general lay of the land, but Fig (4.2.1) is the one we will work with below.

4.3 Comments on $\dot{\mathbf{b}}_S$ and $\dot{\mathbf{b}}_{S'}$

The quantity $\dot{\mathbf{b}}_S = (d\mathbf{b}/dt)_S$ describes the instantaneous velocity of the Frame S' origin relative to that of Frame S. When **ω** = 0, Frame S' merely translates relative to Frame S with position **b**, velocity $\dot{\mathbf{b}}_S$ and acceleration $\ddot{\mathbf{b}}_S$. When **ω** ≠ 0, however, for a general placement of the rotation axis one can regard **b**,

$\dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_S$ as quantities derived from the location and movement of that axis and from the value of $\boldsymbol{\omega}$, all of which one imagines are controlled by some mechanical outside agency. An exception is Special Case #2 below where the rotation axis passes through the Frame S' origin.

The quantity $\dot{\mathbf{b}}_{S'} = (d\mathbf{b}/dt)_{S'}$ is a "cross velocity" in the sense of Section 1.8 and is difficult to interpret. In Special Case #1 below, however, the vector $\mathbf{b}$ is effectively glued to the Frame S' axes and $\dot{\mathbf{b}}_{S'} = 0$, causing simplification of several equations which will be obtained below.

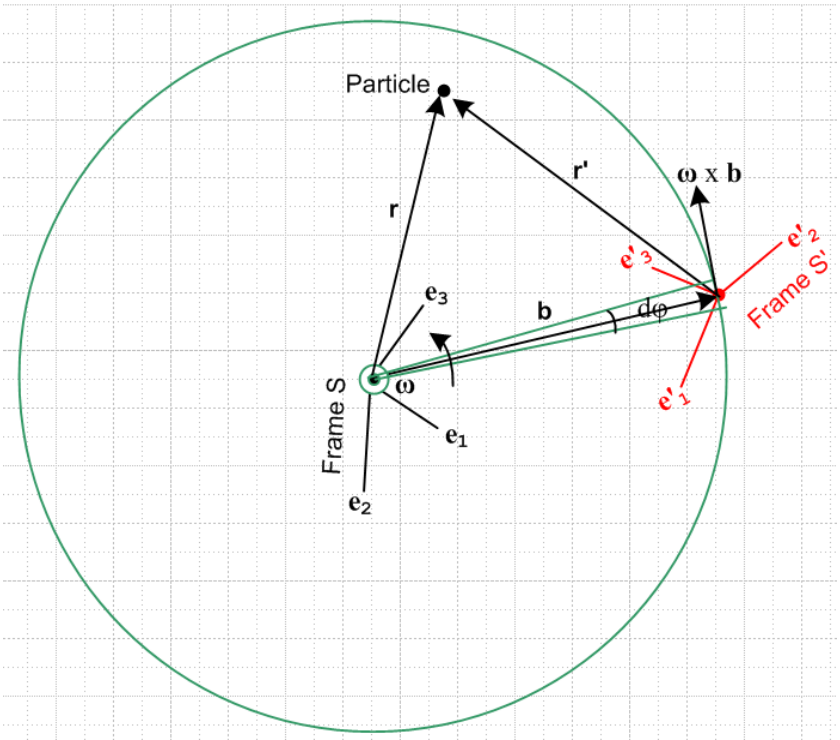
The connection between $\dot{\mathbf{b}}_S$ and $\dot{\mathbf{b}}_{S'}$ is provided by the G Rule (2.1) applied to vector $\mathbf{b}$,

$$\dot{\mathbf{b}}_S = \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times \mathbf{b} \quad . \tag{4.3.1}$$

In Section 7.5 we derive the relationship between accelerations $\ddot{\mathbf{b}}_S = \partial_S \dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_{S'} = \partial_{S'} \dot{\mathbf{b}}_{S'}$.

4.4 Special Case #1 : ω axis through Frame S origin

The rotation axis always passes through the origin of Frame S, so Fig (4.2.1) appears as follows,



Special Case #1
(4.4.1)

The Frame S' origin and the green circle are still in the plane of paper, but the Frame S origin is in general not, so vector $\mathbf{b}$ is not in the plane of paper. In this situation, the vector $\mathbf{b}$ and the vectors $\mathbf{e}'_n$ all rotate

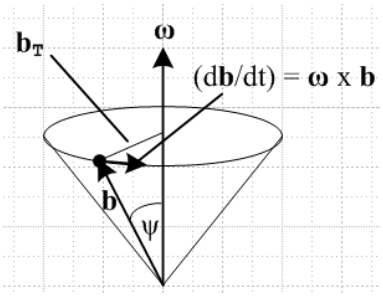
together as if they were thin metal rods soldered together. If $\mathbf{b}'' = R\mathbf{b}$ and $\mathbf{e}''_n = R\mathbf{e}'_n$ then angles are fixed, as indicated by $\mathbf{b}'' \bullet \mathbf{e}''_n = [R\mathbf{b}] \bullet [R\mathbf{e}'_n] = \mathbf{b} \bullet \mathbf{e}'_n$ using (1.2.4) and $R = R_{\omega}(d\varphi)$. See Fig (1.5.9).

This has the immediate implication that, just as $(d\mathbf{e}'_n/dt)_{S'} = 0$ in (1.7.2), here we have

$$\dot{\mathbf{b}}_{S'} \equiv (d\mathbf{b}/dt)_{S'} = 0 \ . \tag{4.4.2}$$

Therefore, from (4.3.1), we have instantaneous conical motion for **b** as seen in Frame S,

$$\dot{\mathbf{b}}_{S} \equiv (d\mathbf{b}/dt)_{S} = \boldsymbol{\omega} \times \mathbf{b} \ . \qquad \text{// Special Case \#1} \tag{4.4.3}$$

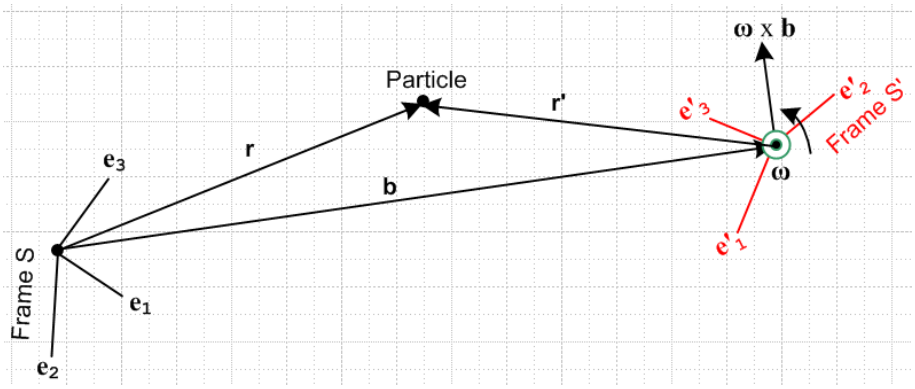


(4.4.4)

This says that, as seen in Frame S, the change in **b** is always perpendicular to **b,** so the length of **b** does not change. We still allow $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ and, as $\hat{\boldsymbol{\omega}}(t)$ changes, the tip of vector **b** (which is the origin of Frame S') describes some path on the surface of a sphere of radius b in Frame S. Here we allow $\hat{\boldsymbol{\omega}}(t)$ to change, but the rotation axis must always pass through the Frame S origin.

4.5 Special Case #2 : ω axis through Frame S' origin

Here the rotation axis always passes through the origin of Frame S' :



Special Case #2
(4.5.1)

In this case our Fig (4.2.1) green circle has shrunk around the Frame S' origin. In this situation, we can think of the "driving parameters" being the three parameters of **b** and the three parameters of **ω** giving the 6 independent (Galilean) parameters defining the instantaneous relationship between the frames. Think of

Frame S' as a camera platform which is supported on a boom $\mathbf{b}(t)$ and which independently controls its own orientation and rotation $\boldsymbol{\omega}(t)$. Then $\dot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_S$ are determined by $\mathbf{b}(t)$, and $\dot{\mathbf{b}}_{S'}$ is given by (4.3.1) as

$$\dot{\mathbf{b}}_{S'} = \dot{\mathbf{b}}_S - \boldsymbol{\omega} \times \mathbf{b} \quad . \tag{4.5.2}$$

We now look at three sample applications. The first two fall into the Special Case #1 category, while the third is Special Case #2.

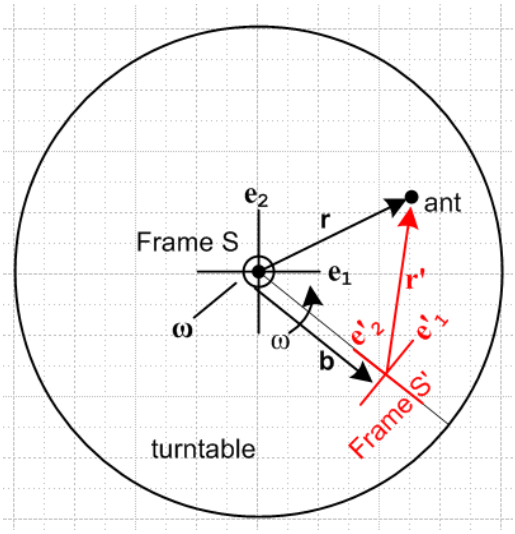
4.6 The Turntable

Although obsolete, the phonograph turntable continues to provide an excellent visualization of rotating frames. It turns slowly enough for one to actually see it turn, it is fairly large, and the surface is not shiny and completely featureless. It is believed that throughout history such turntables normally rotated clockwise in both hemispheres of the Earth, but in our drawings below we shall think of our turntable as turning counterclockwise relative to the little spindle sticking up.



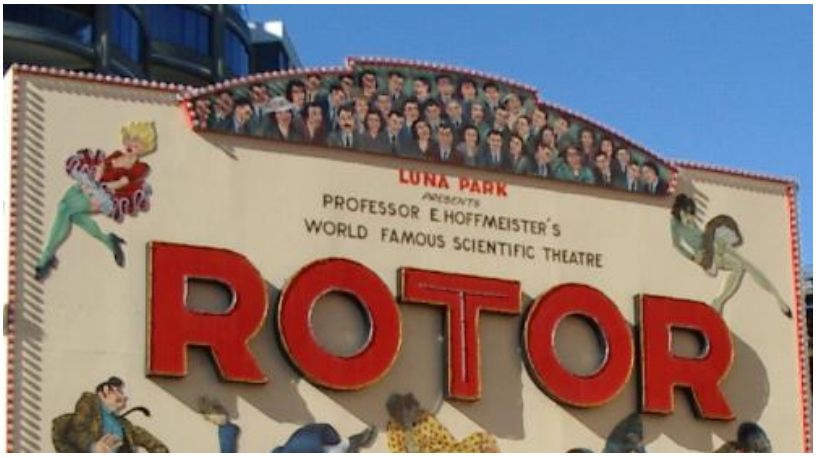
(4.6.1)

An ant (Particle) is crawling around on such a turntable which is rotating with some $\boldsymbol{\omega}(t)$. The Frame S' is set up some distance $b = |\mathbf{b}(t)|$ from the spindle, and has its $\mathbf{e}'_1$ axis pointing "to the right" and its $\mathbf{e}'_2$ axis pointing to the spindle. That is to say, $\mathbf{e}'_1 = \hat{\boldsymbol{\theta}}$ and $\mathbf{e}'_2 = -\hat{\mathbf{r}}$ if we think of r,θ as polar coordinates for fixed Frame S. In this application, $\boldsymbol{\omega}(t) = \omega(t)\mathbf{e}_3$ and the rotation axis passes though the origin of Frame S, so this is a Special Case #1 situation with $\hat{\boldsymbol{\omega}}(t) = \mathbf{e}_3$ being constant. Conversely, vector $\mathbf{b}(t)$ maintains its magnitude b, but $\hat{\mathbf{b}}(t)$ rotates about the spindle at angular rate ω(t) as seen from Frame S which is at rest.



(4.6.2)

This type of Frame S' designation would be familiar to merry-go-round riders hanging on and facing the center. While once in Sydney, the author rode "The Rotor" for which Frame S' can be regarded as being at the cylindrical wall to which the rider is glued by centrifugal force (the circular floor then drops away).





(wiki) (4.6.3)

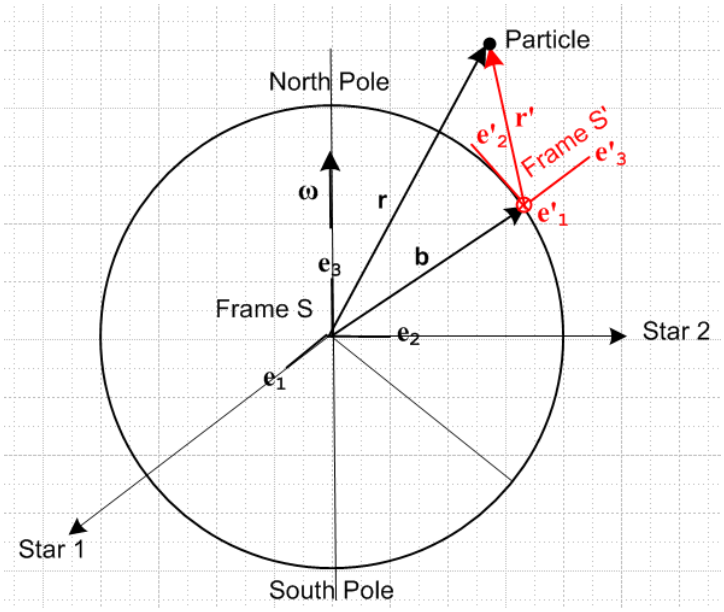
In Section 15 we shall solve three "ant on turntable" problems using the above geometry.

4.7 The Earth

The non-rotating Cartesian Frame S is located at Earth center with $\mathbf{e}_3$ pointing to the North Pole. The $\mathbf{e}_1$ axis points out through a point on the equator to some fixed distant star (Star 1), and then $\mathbf{e}_2$ is the third Cartesian axis which points out through a different point on the equator to Star 2. If we ignore the Earth's orbiting around the Sun, and the motion of the Orion spiral galactic arm containing the Sun, and the motion of the Milky Way galactic center away from the "Dipole repeller" and so on, then Frame S is an inertial frame.

Rotating Frame S' has its origin at some arbitrary fixed point on the surface of the Earth. For this system, the $\mathbf{e'_3}$ axis points "up", meaning in the $\hat{\mathbf{r}}$ direction for Frame S spherical coordinates. The $\mathbf{e'_1}$ axis points to the east, and the $\mathbf{e'_2}$ axis points north.

The Earth rotates at some $\boldsymbol{\omega} = \omega\mathbf{e_3}$ with $\omega > 0$. Since the $\boldsymbol{\omega}$ axis passes through the Frame S origin, this is another Special Case #1 application.



(4.7.1)

Figure (4.7.1) and the discussion of this section are in the "non-swap" notation mentioned in the Section 1 summary at the start of this document. In practice, one usually uses the "swap" notation S↔S' for Earth problems in order to avoid the appearance of primes in equations. This will be done in Appendix C concerning the Foucault pendulum.

4.8 The Flying Camera Platform

Some Apparatus is located in Frame S instead of S', and Frame S' is a "camera platform" which flies around in some complicated way and observes the activity in Frame S. In this case, both $\boldsymbol{\omega}(t)$ and $\mathbf{b}(t)$ would be under the command of the pilot of the camera platform. One would set this up as a Special Case #2 situation, so $\boldsymbol{\omega}$ passes through the Frame S' origin and $\dot{\mathbf{b}}_\mathbf{S}$ is the velocity of the platform origin and $\mathbf{b}(t)$ is its location relative to Frame S. One would use the "inverse problem" equations of Section 13 to get the primed quantities in terms of the unprimed ones.

5. The Goal of the next two sections

The symbols appearing here are defined in Section 1.8.

An Observer in Frame S' measures various properties of a Particle in motion,

$$\mathbf{r}' \qquad \mathbf{v}' \qquad \mathbf{a}' \qquad \mathbf{L}'^{(c')} \qquad \dot{\mathbf{L}}'^{(c')}$$

We want to know how these properties of the Particle appear in Frame S,

$$\mathbf{r} \qquad \mathbf{v} \qquad \mathbf{a} \qquad \mathbf{L}^{(c)} \qquad \dot{\mathbf{L}}^{(c)}$$

and we want to know the various other **v** and **a** forms of Section 1.8 in terms of the basic Frame S' objects listed above.

Later in Section 13 we will want to know how to solve "the inverse problem" of finding the primed quantities if the unprimed ones are known.

6. Determination of velocities

The notations used here are described in Section 1.8. There are four distinct velocities, and we want to express three of them in terms of the fourth which is the natural velocity in Frame S',

$$(d\mathbf{r}'/dt)_{S'} = \mathbf{v}'_{S'} \equiv \mathbf{v}' \quad .$$

We shall make frequent use of the Fig 1 relationship between **r** and **r'**,

$$\mathbf{r} = \mathbf{r}' + \mathbf{b} \tag{6.1}$$

as well as the G Rule for vector **b**, as in (4.3.1)

$$\dot{\mathbf{b}}_S = \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times \mathbf{b} \quad . \tag{6.2a}$$

Inserting (6.1) as **b** = **r** - **r'** into (6.2a) gives identity

$$\dot{\mathbf{b}}_S + \boldsymbol{\omega} \times \mathbf{r}' = \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times \mathbf{r} \quad . \tag{6.2b}$$

6.1 Velocity $\mathbf{v}_{S'}$

Apply $(d/dt)_{S'}$ to (6.1) to get (6.3a), then use (6.2a) to get (6.3b) :

$$\mathbf{v}_{S'} = \mathbf{v}' + \dot{\mathbf{b}}_{S'} \tag{6.3a}$$

$$\mathbf{v}_{S'} = \mathbf{v}' + \dot{\mathbf{b}}_S - \boldsymbol{\omega} \times \mathbf{b} \quad . \tag{6.3b}$$

6.2 Velocity $\mathbf{v} \equiv \mathbf{v}_S$

Apply $(d/dt)_S$ to (6.1) to get $\partial_S\mathbf{r} = \partial_S\mathbf{r}' + \partial_S\mathbf{b}$ or

$$\mathbf{v} = \mathbf{v}'_S + \dot{\mathbf{b}}_S \quad . \tag{6.4}$$

Now use the G Rule (2.1) for vector **r'**, $\partial_S\mathbf{r}' = \partial_{S'}\mathbf{r}' + \boldsymbol{\omega} \times \mathbf{r}'$, to get

$$\mathbf{v}'_S = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \quad . \tag{6.5}$$

Insert (6.5) into (6.4) to get the first line below. The second and third lines make use of (6.2a) and (6.1) .

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \tag{6.6a}$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times \mathbf{b} \tag{6.6b}$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \quad . \tag{6.6c}$$

6.3 Velocity $\mathbf{v'_S}$

Solve (6.4) for $\mathbf{v'_S}$,

$$\mathbf{v'_S} = \mathbf{v} - \dot{\mathbf{b}}_\mathbf{S} \quad . \tag{6.7}$$

Then solve (6.6a) for $\mathbf{v} - \dot{\mathbf{b}}_\mathbf{S}$ and install into (6.7) to get the first line below,

$$\mathbf{v'_S} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} \tag{6.8a}$$

$$\mathbf{v'_S} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r} - \boldsymbol{\omega} \times \mathbf{b} \tag{6.8b}$$

$$\mathbf{v'_S} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{\mathbf{S}'} - \dot{\mathbf{b}}_\mathbf{S} \quad . \tag{6.8c}$$

The remaining two lines come from using (6.1) and (6.2a).

6.4 Velocity Summary

$$\mathbf{v} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_\mathbf{S} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{\mathbf{S}'} \qquad \textit{(6.6a,c)}$$

$$\mathbf{v}_{\mathbf{S}'} = \mathbf{v'} + \dot{\mathbf{b}}_\mathbf{S} - \boldsymbol{\omega} \times \mathbf{b} = \mathbf{v'} + \dot{\mathbf{b}}_{\mathbf{S}'} \qquad \textit{(6.3b,a)}$$

$$\mathbf{v'_S} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} \qquad \textit{(6.8a)} \tag{6.9}$$

Thus we have expressed the three velocities $\mathbf{v}_\mathbf{S} = \mathbf{v}$, $\mathbf{v}_{\mathbf{S}'}$ and $\mathbf{v'_S}$ in terms of $\mathbf{v'}_{\mathbf{S}'} = \mathbf{v'}$.

6.5 Velocities for Special Cases

These cases were discussed above in Section 4.4 and 4.5.

For Special Case #1, where the $\boldsymbol{\omega}$ axis passes through the Frame S origin, in any equations above set

$$\dot{\mathbf{b}}_\mathbf{S'} = 0 \qquad \text{// (4.4.2)}$$

$$\dot{\mathbf{b}}_\mathbf{S} = \boldsymbol{\omega} \times \mathbf{b} = \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{r'}) = \boldsymbol{\omega} \times \mathbf{r} - \boldsymbol{\omega} \times \mathbf{r'} \quad . \qquad \text{// (4.4.3)} \tag{6.10}$$

Then the Section 6.4 summary becomes

$\mathbf{v} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r}$	Special Case #1 only	*(6.6a)*	
$\mathbf{v}_{\mathbf{S}'} = \mathbf{v'}$	Special Case #1 only	*(6.3b)*	
$\mathbf{v'_S} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'}$.	general	*(6.8a)*	(6.11)

For Special Case #2, where the $\boldsymbol{\omega}$ axis passes through the Frame S' origin, $\dot{\mathbf{b}}_\mathbf{S}$ is a driving parameter, so we select just the $\dot{\mathbf{b}}_\mathbf{S}$ forms from the summary

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad \text{general} \qquad \textit{(6.6a)}$$

$$\mathbf{v}_{S'} = \mathbf{v}' + \dot{\mathbf{b}}_S - \boldsymbol{\omega} \times \mathbf{b} \qquad \text{general} \qquad \textit{(6.3b)}$$

$$\mathbf{v}'_S = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \quad . \qquad \text{general} \qquad \textit{(6.8a)} \qquad (6.12)$$

6.6 Comments

1. Consider these two results from above (picked more or less at random)

$$\mathbf{r} = \mathbf{r}' + \mathbf{b} \qquad \textit{(6.1)}$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \quad . \qquad \textit{(6.6c)}$$

Either equation can be "evaluated" in either Frame S or Frame S'. Evaluation in Frame S gives

$$(r)_i = (r')_i + (b)_i$$

$$(v)_i = (v')_i + \varepsilon_{ijk}(\omega)_j(r)_k + (\dot{b}_{S'})_i$$

while evaluation in Frame S' gives

$$(r)'_i = (r')'_i + (b)'_i$$

$$(v)'_i = (v')'_i + \varepsilon_{ijk}(\omega)'_j(r)'_i + (\dot{b}_{S'})'_i \quad . \qquad // \; (\omega)'_j = (\omega)_j$$

This is a situation where we fully expect to have $(r')_i \neq (r)'_i$ and $(v')_i \neq (v)'_i$ as mentioned in Section 1.2, so the careful placement of primes is important. This is just a reminder.

2. Based on the equations above, it is clear that we have

$$\mathbf{r}' \neq R\mathbf{r} \qquad \mathbf{v}' \neq R\mathbf{v}$$

where R is the rotation appearing in (1.1.1) which relates our two frames, $\mathbf{e}_n = R\ \mathbf{e}'_n$. Therefore, in general the pairs of vectors (**r**,**r**') and (**v**,**v**') are *not* "vectors under rotations" in the sense of (1.3.2).

3. On the other hand, the vectors **ω** and **b** appearing in these formulas *are* normal "vectors under rotations" in the sense of (1.3.2), since we just define **b**' and **ω**' by these equations,

$$\mathbf{b}' = R\mathbf{b} \qquad \boldsymbol{\omega}' = R\boldsymbol{\omega}$$

so

$$(b')_i = R_{ij}(b)_j = (b)'_i = b'_i \qquad (\omega')_i = R_{ij}(\omega)_j = (\omega)'_i = \omega'_i \qquad (6.13)$$

since, according to (1.3.4), we can dispense with parentheses for such vectors. These equations then give us the components of **b** and **ω** in Frame S'. For example, $\mathbf{e}'_i \bullet \mathbf{b} = (b)'_i = b'_i$.

4. In deriving the various velocity relations above, we have basically only used $\mathbf{r} = \mathbf{r}' + \mathbf{b}$ and the G Rule. Both the fact that $\mathbf{r} = \mathbf{r}' + \mathbf{b}$ and the G Rule are valid even if both Frame S and Frame S' are non-inertial frames, so one could think of "the paper" to which Frame S is glued in Fig (4.2.1) as possibly rotating. The implication is that the velocity relations above do not require that either frame be inertial. This same comment applies to the acceleration relations derived in Section 7 below. It is only when we later add the equation $\mathbf{F} = m\mathbf{a}$ to the story in Section 8 that we have to regard Frame S as being an inertial frame.

7. Determination of accelerations

The notations used here are described in Section 1.8. There are eight distinct accelerations of interest, and we could express seven of them in terms of the eighth which is the natural acceleration in Frame S',

$$(d\mathbf{a}'/dt)_{S'} = \mathbf{a}'_{S'} \equiv \mathbf{a}' \ .$$

We shall only express the three accelerations $\mathbf{a}_S = \mathbf{a}$, $\mathbf{a}_{S'}$ and $\mathbf{a}'_S$ in terms of $\mathbf{a}'_{S'} = \mathbf{a}'$. Below (6.9) one sees that this is exactly what we did with the velocities. Here, however, we examine $\mathbf{a}'_S$ first.

As noted in the Section 7 Summary, "dry as dust", and our apologies.

7.1 Acceleration $\mathbf{a}'_S$

The G Rule (2.1) for $\mathbf{v}'_S$ says

$$\partial_S \mathbf{v}'_S = \partial_{S'} \mathbf{v}'_S + \boldsymbol{\omega} \times \mathbf{v}'_S$$

or

$$\mathbf{a}'_S = \mathbf{a}'_{S',S} + \boldsymbol{\omega} \times \mathbf{v}'_S \ . \tag{7.1}$$

Notice the unusual cross derivative $\mathbf{a}'_{S',S}$ which involves both S and S'. This is one those cross accelerations appearing in (1.8.6). To compute this, we must go back to the G Rule (2.1) for $\mathbf{r}'$,

$$\mathbf{v}'_S = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \ . \qquad // \ \mathbf{v}'_S = \partial_S \mathbf{r}', \quad \mathbf{v}' = \partial_{S'} \mathbf{r}' \tag{6.8a}$$

Apply $\partial_{S'} \equiv (d/dt)_{S'}$ to both sides to get

$$\partial_{S'} \mathbf{v}'_S = \partial_{S'} \mathbf{v}' + \partial_{S'} (\boldsymbol{\omega} \times \mathbf{r}') = \partial_{S'} \mathbf{v}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\partial_{S'} \mathbf{r}') \qquad // \text{ see (1.11.4) and (2.6)}$$

or

$$\mathbf{a}'_{S',S} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times \mathbf{v}' \ . \tag{7.2}$$

Here we used (2.6) that $\partial_S \boldsymbol{\omega} = \partial_{S'} \boldsymbol{\omega} = \dot{\boldsymbol{\omega}}$. Now install (7.2) for $\mathbf{a}'_{S',S}$ into (7.1) to get

$$\mathbf{a}'_S = [\mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times \mathbf{v}'] + \boldsymbol{\omega} \times \mathbf{v}'_S \ . \tag{7.3}$$

Now replace $\mathbf{v}'_S$ in the last term using the G Rule for $\mathbf{r}'$ [(6.8a) a few lines above]

$$\mathbf{a}'_S = [\mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times \mathbf{v}'] + \boldsymbol{\omega} \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \ ]$$

or

$$\mathbf{a}'_S = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\, \boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \ . \tag{7.4}$$

The famous **"Coriolis factor of 2"** has now appeared and will be trivially transferred into $\mathbf{a}_S$ in the next section. It is useful to review the steps above to see where this factor of 2 comes from :

1. Write the G Rule for $\mathbf{v'_S}$	$\mathbf{a'_S} = \mathbf{a'_{S'S}} + \boldsymbol{\omega} \times \mathbf{v'_S}$	
2. Write the G Rule for $\mathbf{r'}$	$\mathbf{v'_S} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'}$	
3. Insert 2 into 1	$\mathbf{a'_S} = \mathbf{a'_{S'S}} + \boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'})$	// 1st $\boldsymbol{\omega} \times \mathbf{v'}$ term
4. Apply $\partial_{S'}$ to 2 to get	$\mathbf{a'_{S'S}} = \mathbf{a'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'} + \boldsymbol{\omega} \times \mathbf{v'}$	// 2nd $\boldsymbol{\omega} \times \mathbf{v'}$ term,
5. Install 4 into 3:	$\mathbf{a'_S} = [\mathbf{a'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'} + \boldsymbol{\omega} \times \mathbf{v'}] + \boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'})$	

and now we have two $\boldsymbol{\omega} \times \mathbf{v'}$ terms and only natural Frame S' objects $\mathbf{r'}$, $\mathbf{v'}$ and $\mathbf{a'}$.

7.2 Acceleration $\mathbf{a} \equiv \mathbf{a_S}$

Start with (6.4) which is ∂_S applied to $\mathbf{r} = \mathbf{r'} + \mathbf{b}$,

$$\mathbf{v} = \mathbf{v'_S} + \dot{\mathbf{b}}_S \quad . \tag{6.4}$$

Apply ∂_S again to get

$$\mathbf{a} = \mathbf{a'_S} + \ddot{\mathbf{b}}_S \quad . \tag{7.5}$$

Then insert (7.4) for $\mathbf{a'_S}$ into (7.5) to get the same result as (7.4) but with $\ddot{\mathbf{b}}_S$ tacked on,

$$\mathbf{a} = \mathbf{a'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + \ddot{\mathbf{b}}_S \quad . \tag{7.6a}$$

S S' Euler Coriolis centripetal frame

We have attached a name to each contribution to $\mathbf{a}$ and will discuss these terms below. Since we really want all primed objects on the right side, we can anticipate the result (7.12) derived in Section 7.5 below,

$$\ddot{\mathbf{b}}_S = \ddot{\mathbf{b}}_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{b} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \tag{7.12}$$

to get this alternate form for $\mathbf{a}$,

$$\mathbf{a} = \mathbf{a'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + [\ddot{\mathbf{b}}_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{b} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \;]$$

$$= \mathbf{a'} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \ddot{\mathbf{b}}_{S'} \tag{7.6b}$$

where we think here of $\mathbf{r}$ as just a shorthand for $\mathbf{b} + \mathbf{r'}$ to reduce the number of terms.

7.3 Acceleration $\mathbf{a}_{S'}$

Start with (6.3a) which is $\partial_{S'}$ applied to $\mathbf{r} = \mathbf{r}' + \mathbf{b}$,

$$\mathbf{v}_{S'} = \mathbf{v}' + \dot{\mathbf{b}}_{S'} \quad . \qquad \textit{(6.3a)}$$

Apply $\partial_{S'}$ again to get

$$\partial_{S'}\mathbf{v}_{S'} = \partial_{S'}\mathbf{v}' + \ddot{\mathbf{b}}_{S'}$$

or

$$\mathbf{a}_{S'} = \mathbf{a}' + \ddot{\mathbf{b}}_{S'} \quad . \qquad (7.7)$$

7.4 Acceleration Summary

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \qquad \textit{(7.6a)}$$

$$\mathbf{a}'_S = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad \textit{(7.4)}$$

$$\mathbf{a}_{S'} = \mathbf{a}' + \ddot{\mathbf{b}}_{S'} \qquad \textit{(7.7)} \qquad (7.8)$$

7.5 Relation between $\ddot{\mathbf{b}}_S$ and $\ddot{\mathbf{b}}_{S'}$

Write the G Rule (2.1) for $\dot{\mathbf{b}}_{S'}$ then solve it for $\ddot{\mathbf{b}}_{S'}$

$$\partial_S \dot{\mathbf{b}}_{S'} = \ddot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'}$$

$$\ddot{\mathbf{b}}_{S'} = \partial_S \dot{\mathbf{b}}_{S'} - \boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} \quad . \qquad (7.9)$$

Now apply ∂_S to (6.2a) then solve for $\partial_S \dot{\mathbf{b}}_{S'}$,

$$\ddot{\mathbf{b}}_S = \partial_S \dot{\mathbf{b}}_{S'} + \partial_S (\boldsymbol{\omega} \times \mathbf{b})$$

$$\partial_S \dot{\mathbf{b}}_{S'} = \ddot{\mathbf{b}}_S - \partial_S (\boldsymbol{\omega} \times \mathbf{b}) \quad . \qquad (7.10)$$

Insert (7.10) into (7.9) to get the first line below, then use (6.2a) to get the second line,

$$\ddot{\mathbf{b}}_{S'} = [\ddot{\mathbf{b}}_S - \partial_S (\boldsymbol{\omega} \times \mathbf{b})] - \boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'}$$

$$= \ddot{\mathbf{b}}_S - \partial_S (\boldsymbol{\omega} \times \mathbf{b}) - \boldsymbol{\omega} \times [\dot{\mathbf{b}}_S - \boldsymbol{\omega} \times \mathbf{b}]$$

$$= \ddot{\mathbf{b}}_S - \dot{\boldsymbol{\omega}} \times \mathbf{b} - 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \quad . \qquad (7.11)$$

The inversion of this equation may be found by using (6.2a), $\dot{\mathbf{b}}_S = \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times \mathbf{b}$,

$$\ddot{\mathbf{b}}_{S'} = \ddot{\mathbf{b}}_S - \dot{\boldsymbol{\omega}} \times \mathbf{b} - 2\boldsymbol{\omega} \times [\dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times \mathbf{b}] + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$$

$$= \ddot{\mathbf{b}}_S - \dot{\boldsymbol{\omega}} \times \mathbf{b} - 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$$

so then

$$\ddot{\mathbf{b}}_S = \ddot{\mathbf{b}}_{S'} + \dot{\boldsymbol{\omega}} \times \mathbf{b} + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \quad . \tag{7.12}$$

In a Special Case #1 problem we have $\dot{\mathbf{b}}_{S'} = 0$ from (6.10), hence $\ddot{\mathbf{b}}_{S'} = 0$, so that

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \quad . \qquad \text{// Special Case \#1} \tag{7.13}$$

8. The Fictitious Forces

8.1 Development of the Fictitious Forces

We start with (7.6a) which says

$$\underset{S}{\mathbf{a}} = \underset{S'}{\mathbf{a}'} + \underset{\text{Euler}}{\dot{\boldsymbol{\omega}} \times \mathbf{r}'} + \underset{\text{Coriolis}}{2\,\boldsymbol{\omega} \times \mathbf{v}'} + \underset{\text{centripetal}}{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')} + \underset{\text{frame}}{\ddot{\mathbf{b}}_S} \quad . \qquad \textit{(7.6a)} \qquad (8.1.1)$$

Now for the first time we assume that Frame S is an inertial reference frame. Newton's law in inertial Frame S says **(** m = mass of the Particle)

$$\mathbf{F} = m\mathbf{a} \qquad (8.1.2)$$

with **a** given as above in (8.1.1), so that, reordering the 5 terms,

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{b}}_S + m\mathbf{a}' + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \quad . \qquad (8.1.3)$$

Now suppose we imagine an "effective" version of Newton's Law that works in rotating Frame S',

$$\mathbf{F}'_{\mathtt{eff}} = m\mathbf{a}' \quad . \qquad (8.1.4)$$

Solving (8.1.3) for m**a**' tells us that (second line uses (8.1.2))

$$m\mathbf{a}' = \mathbf{F}'_{\mathtt{eff}} = \mathbf{F} - m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad (8.1.5)$$

$$m\mathbf{a}' = \mathbf{F}'_{\mathtt{eff}} = m\mathbf{a} - m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \quad . \qquad (8.1.6)$$

We can write the second equality in (8.1.5) as

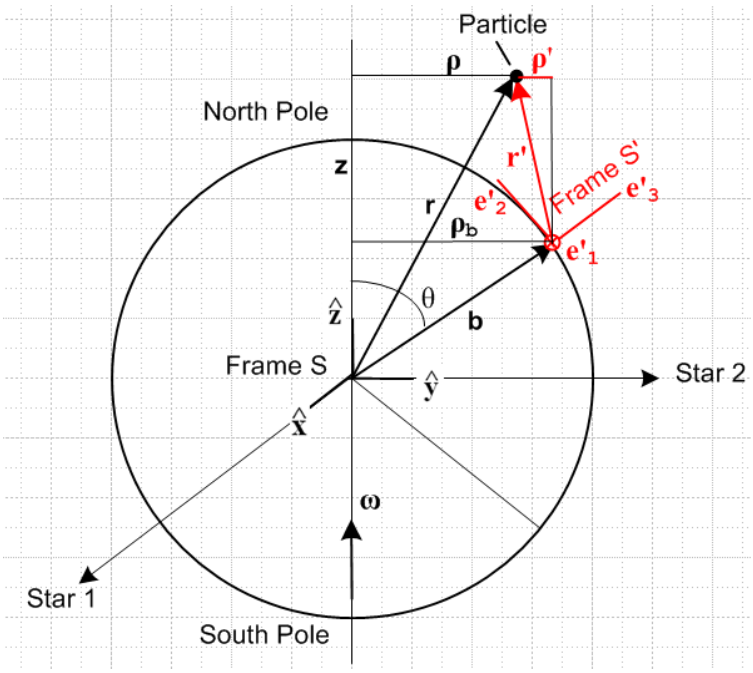
$$\mathbf{F}'_{\mathtt{eff}} = \mathbf{F} + \mathbf{F}'_{\mathtt{fict}} \qquad \text{where} \qquad (8.1.7)$$

$$\mathbf{F}'_{\mathtt{fict}} = \underset{\text{frame}}{-m\ddot{\mathbf{b}}_S} \underset{\text{centrifugal}}{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')} \underset{\text{Coriolis}}{- 2m\,\boldsymbol{\omega} \times \mathbf{v}'} \underset{\text{Euler}}{- m\dot{\boldsymbol{\omega}} \times \mathbf{r}'} \quad . \qquad (8.1.8)$$

Here $\mathbf{F}'_{\mathtt{fict}}$ represents "fictitious" forces ("pseudo" forces) that mysteriously have to be added to "real forces" **F** to make our bogus (8.1.4) "Newton's Law" $\mathbf{F}'_{\mathtt{eff}} = m\mathbf{a}'$ be valid in Frame S'. We put a prime on $\mathbf{F}'_{\mathtt{fict}}$ as a reminder that it is a force which appears in non-inertial Frame S'

8.2 Interpretation of the Centrifugal and Euler Fictitious Forces

To emphasize the long and short vector idea in our interpretation of the centrifugal and Euler fictitious forces, we take the Earth scenario of Fig (4.7.1) rather than a more general situation like Fig (4.2.1) :



(8.2.1)

Suppose in the above picture $\mathbf{v}' = 0$ so $\mathbf{r}'$ is static in Frame S'. Then (8.1.8) becomes

$$\mathbf{F'}_{\mathtt{fict}} = \underbrace{-m\ddot{\mathbf{b}}_S}_{\text{frame}} \underbrace{-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')}_{\text{centrifugal}} \underbrace{-m\dot{\boldsymbol{\omega}} \times \mathbf{r}'}_{\text{Euler}} \quad . \qquad \textit{(8.1.8)} \text{ with } \mathbf{v}' = 0 \tag{8.2.2}$$

In this expression it is the *short* vector $\mathbf{r}'$ which appears in both the centrifugal and Euler terms. Since Fig (8.2.1) is a Special Case #1 situation, we know from (7.13) that,

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \quad . \qquad \text{// Special Case \#1} \qquad \textit{(7.13)} \tag{8.2.3}$$

Inserting this into (8.2.2) gives,

$$\begin{aligned} \mathbf{F'}_{\mathtt{fict}} &= -m\,[\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) + \dot{\boldsymbol{\omega}} \times \mathbf{b}\,] - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \\ &= -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{b} + \mathbf{r}']) - m\dot{\boldsymbol{\omega}} \times [\mathbf{b} + \mathbf{r}'] \\ &= \underbrace{-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{centrifugal}} \underbrace{-m\dot{\boldsymbol{\omega}} \times \mathbf{r}}_{\text{Euler}} \quad . \end{aligned} \tag{8.2.4}$$

Now the frame term $-m\ddot{\mathbf{b}}_S$ is gone and it is the *long* vector $\mathbf{r}$ which appears in the two fictitious force terms. Equation (8.2.4) is the conventional form for $\mathbf{F'}_{\mathtt{fict}}$. The centrifugal term may be written

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = (\mathbf{A} \bullet \mathbf{C})\mathbf{A} - A^2\mathbf{C}$$

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -m\omega^2[\,\hat{\mathbf{z}} \times (\hat{\mathbf{z}} \times \mathbf{r})] = -m\omega^2\,[\,(\hat{\mathbf{z}} \bullet \mathbf{r})\hat{\mathbf{z}} - \mathbf{r}\,] = -m\omega^2\,[\,\mathbf{z} - \mathbf{r}\,]$$

$$= +m\omega^2\boldsymbol{\rho} \tag{8.2.5}$$

which is the usual way one states this centrifugal force. The vector **r** in Fig (8.2.1) is doing conical motion about the **ω** vector, and the tip of **r** executes circular motion with radius vector **ρ**.

If the Earth rotation were varying only in magnitude, one would have for the Euler force in (8.2.4),

$$-m\dot{\boldsymbol{\omega}} \times \mathbf{r} = -m\dot{\omega} r\, \hat{\mathbf{z}} \times \hat{\mathbf{r}} = -m\dot{\omega} r \sin\theta\, \hat{\boldsymbol{\varphi}} \qquad \text{// (E.2.15)}$$

$$= -m\dot{\omega}\rho\, \hat{\boldsymbol{\varphi}} = -ma_{\varphi}\, \hat{\boldsymbol{\varphi}} \qquad \text{// (E.3.6)} \qquad (8.2.6)$$

and this is the expected fictitious linear force directed in the - $\hat{\boldsymbol{\varphi}}$ direction. Accelerate your Persian Rug to the right, you feel a fictitious force to the left. We end up then with

$$\mathbf{F'}_{\mathtt{fict}} = \underbrace{+m\omega^2\boldsymbol{\rho}}_{\text{centrifugal}} \underbrace{-ma_{\varphi}\, \hat{\boldsymbol{\varphi}}}_{\text{Euler}} . \qquad (8.2.7)$$

Looking at the expression (8.2.3) for the frame acceleration component $\ddot{\mathbf{b}}_{\mathtt{S}}$, we can break it down in similar fashion to get (see Fig (8.2.1),

$$-m\ddot{\mathbf{b}}_{\mathtt{S}} = -m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) - m\dot{\boldsymbol{\omega}} \times \mathbf{b}$$

$$= \underbrace{m\omega^2\boldsymbol{\rho}_{\mathtt{b}}}_{\text{centrifugal}} \underbrace{- m\dot{\omega}\rho_{\mathtt{b}}\, \hat{\boldsymbol{\varphi}}}_{\text{Euler}} . \qquad (8.2.8)$$

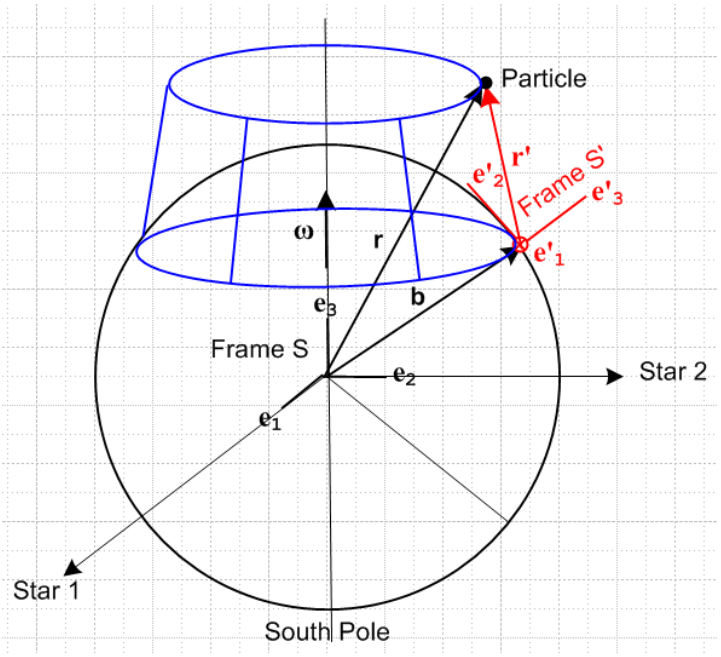
where these terms are for the origin of Frame S' located at **b** in Frame S. The vector **b** in Fig (8.2.1) is doing conical motion about the **ω** vector, and the tip of **b** executes circular motion with radius vector $\boldsymbol{\rho}_{\mathtt{b}}$.

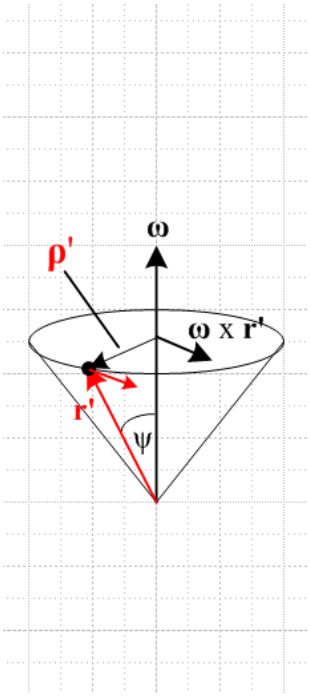
Finally, we return to the two short-vector terms in (8.2.2),

$$\Delta\mathbf{F'}_{\mathtt{fict}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

$$= \underbrace{m\omega^2\boldsymbol{\rho}'}_{\text{centrifugal}} \underbrace{- m\dot{\omega}\rho'\, \hat{\boldsymbol{\varphi}}}_{\text{Euler}} . \qquad (8.2.9)$$

These terms describe the two fictitious forces *relative to* the Frame S' origin. The vector **r'** in Fig (8.2.1) is doing conical motion "about the **ω** vector", and the tip of **r'** executes "circular motion" with radius vector **ρ'**. To see this conical motion more clearly, we show on the left below how the static (in Frame S') vector $\mathbf{r}'$ moves on the blue lampshade surface as the Earth rotates. When the tails of the **r'** vectors on this lampshade are translated to a common point, one obtains the conical motion shown on the right. The angle ψ is determined by $\mathbf{r}' \bullet \hat{\mathbf{z}} = r'\cos\psi$.





(8.2.10)

The point of this interpretative section is that, since **b** + **r'** = **r**, the centrifugal and Euler forces are additive: those *of* the Frame S' origin (**b**) plus those *relative to* the Frame S' origin (short **r'**) add up those *relative to* the center of the Earth (long **r**).

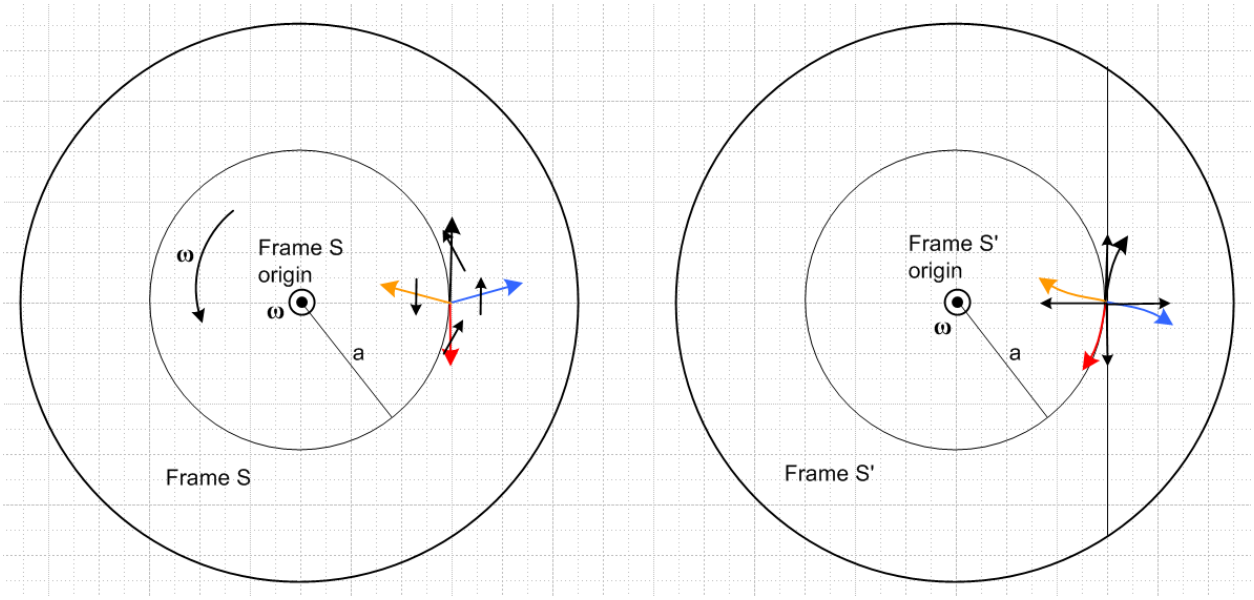
8.3 Interpretations of the Coriolis Fictitious Force

Qualitative Arm-Waving Interpretation of the Coriolis Force

The Coriolis fictitious term is always the main topic of any textbook or web page which deals with motion in rotating frames, so we won't have much to say about it other than to give a popular *qualitative* explanation of the direction of the effect. We did show very carefully how the factor of 2 arises in the derivation of the term which is $\mathbf{F'}_{cor} = -2m\,\boldsymbol{\omega} \times \mathbf{v'}$, and indeed, the expression itself was derived in full. The careful reader will notice a certain amount of "free play" in the workings of what follows.

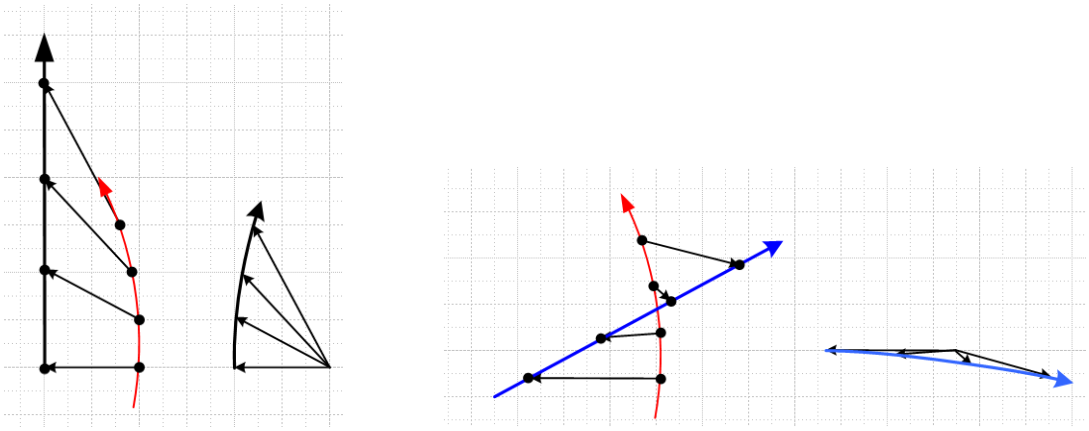
Consider the pictures below where we launch four colored projectiles horizontally from a launch platform that is screwed to a frictionless turntable surface (perhaps the projectiles are hockey pucks). Fixed Frame S and rotating Frame S' have a common origin at the spindle. The launching is done in rotating Frame S' and the projectiles are sent off in the four directions of the compass at equal speeds V. These velocities are represented by the four black equal-length arrows in the right side picture below. The colored arrows on the left show the initial Frame S velocities of these projectiles. In Frame S the initial **v**'s are not the same size because the turntable adds an upward tangential amount $\mathbf{v}_t$ to each ($v_t = a\omega$). Since Frame S is an inertial frame, we can regard the colored arrows on the left as also representing the straight-line trajectories of the projectiles in Frame S.

So, each projectile is launched with the same initial adder $\mathbf{v_t}$ in Frame S and, being in free flight, maintains that $\mathbf{v_t}$ during its flight. Notice that three of the projectiles move into to a region of larger radius, while the orange one heads to a smaller radius region. When a projectile moves to a larger radius, the particles of the turntable move faster CCW than the projectile's $\mathbf{v_t}$ causing a velocity differential between the turntable and the projectile. We want now to examine this differential in the four cases. The short black arrows on the left show the motion of the turntable particles relative to the projectile, as will now be reviewed.



(8.3.1)

For the **black** projectile in mid flight, the turntable particles under the projectile are moving to the northwest relative to the projectile, so the projectile is seen by the turntable particles to be drifting to the right. Hence the curved black trajectory path on the right of Fig (8.3.1). On the left below is a crude strobe picture where a turntable particle's path in Frame S is shown in red and the arrows are then transferred to the right with a common tail to show what projectile motion the turntable particle sees in its rest frame.



black blue (8.3.2)

For the **blue** projectile in mid flight, the turntable particles under the projectile are moving to the north relative to the projectile, so the projectile is seen to be drifting south. Hence the curved blue trajectory path on the right of Fig (8.3.1). The right strobe picture above shows this effect.

For the **red** projectile in mid flight, the turntable particles under the projectile are moving to the northeast relative to the projectile, so the projectile is seen to be drifting to the left. Hence the curved red trajectory path on the right.

For the **orange** projectile in mid flight, the turntable particles under the projectile are moving to the south relative to the projectile (these particles are at a smaller radius and move more slowly than v_t), so the projectile is seen to be drifting to the north. Hence the curved orange trajectory path on the right.

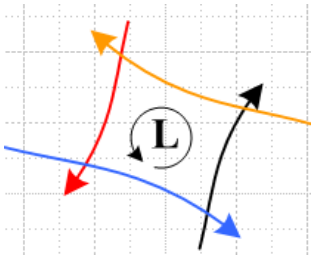
Viewed from the direction of launch in Frame S', all four trajectories drift "to the right". This is in agreement with the right hand rule applied to our expression $\mathbf{F'_{cor}} = -2m\,\boldsymbol{\omega} \times \mathbf{v'} = +2m\,\mathbf{v'} \times \boldsymbol{\omega}$. What is not particularly obvious from the above qualitative discussion is that the $|\mathbf{F'_{cor}}|$ is exactly the same for all four projectiles, and indeed for a projectile launched in *any* direction.

Comments:

1. If the turntable were going CW instead of CCW, the drift directions would all be reversed, both by the qualitative argument, and by the $\mathbf{F'_{cor}}$ expression. Projectiles would drift to the left instead of to the right.

2. One can think of Fig (8.3.1) as a view of the Earth from the North Pole. In this case, the vectors do not lie in the plane of paper, but qualitatively the conclusion is the same: projectiles drift to the right in the Northern Hemisphere. A view from the South Pole would then show drift to the left in the Southern Hemisphere since $\boldsymbol{\omega}$ is reversed.

3. The colored trajectories on the right in Fig (8.3.1), when applied to air masses moving into a region of Low pressure, look like this





(8.3.3)

and explain why lower pressure regions are CCW cyclonic in the Northern Hemisphere. Since lows often drift to the east in the western US, warm Mexican air is felt prior to the low's arrival, and cool Canadian air is felt afterwards.

Superposition Interpretation of the Coriolis Force

Recall now the fictitious forces seen by the projectiles in Fig 8.5,

$$\mathbf{F'}_{fict} = \underbrace{-m\ddot{\mathbf{b}}_S}_{\text{frame}} \underbrace{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'})}_{\text{centrifugal}} \underbrace{- 2m\,\boldsymbol{\omega} \times \mathbf{v'}}_{\text{Coriolis}} \underbrace{- m\dot{\boldsymbol{\omega}} \times \mathbf{r'}}_{\text{Euler}} \quad . \qquad (8.1.8)$$

Since the Frame S and Frame S' origins align, $\mathbf{b} = 0$ and $\ddot{\mathbf{b}}_S = 0$. If we assume $\boldsymbol{\omega}$ = constant, and express the centrifugal term in the simpler form of (8.2.9), then the projectiles are controlled by

$$\mathbf{F'}_{fict} = \underbrace{m\omega^2\mathbf{r'}}_{\text{centrifugal}} \underbrace{- 2m\,\boldsymbol{\omega} \times \mathbf{v'}}_{\text{Coriolis}} \quad . \qquad (8.3.4)$$

[To obtain the first term, set $\psi = 90°$ in the cone picture of Fig (8.2.10) so then $\boldsymbol{\rho}' = \mathbf{r}'$.]

One can think of the above as the superposition of two problems. The centrifugal term alone accelerates the projectiles radially outward in proportion to their distances from the turntable center, so the effect of this term is different for the four projectiles as they progress in flight. The Coriolis term alone causes each projectile to deflect to its right ($\omega > 0$) along a path that is part of a *circle* as we now show. Recall the bogus Newton's Law $\mathbf{F'}_{fict} = m\mathbf{a'}$ from (8.1.4) where $\mathbf{a'} = (d\mathbf{v'}/dt)_{S'}$. If $\mathbf{F'}_{fict} = -2m\,\boldsymbol{\omega} \times \mathbf{v'}$ alone, then we have $\mathbf{a'} = -2\boldsymbol{\omega} \times \mathbf{v'}$ or

$$(d\mathbf{v'}/dt)_{S'} = \boldsymbol{\Omega} \times \mathbf{v'} \qquad \text{where} \quad \boldsymbol{\Omega} = (-2\boldsymbol{\omega}) \quad . \qquad (8.3.5)$$

According to (1.6.1) and Fig (1.6.2) vector $\mathbf{v'}$ must rotate on a cone whose axis is $\boldsymbol{\Omega}$. For the turntable situation, however, $\mathbf{v'}$ is always in the plane of the turntable, so that cone must be flat, so vector $\mathbf{v'}$ goes in a circle at rate Ω. The trajectory $\mathbf{r'}(t)$ is then also circular so the Coriolis deflection is part of a circle. Here is some Maple code illustrating this fact for the blue projectile. We enter the circular $\mathbf{v'}$ (called $\mathbf{v}$) and integrate to get the trajectory.

```
Omega := -2*omega;
```

$$\Omega := -2\,\omega$$

```
vx := (t) -> V*cos(Omega*t);      # velocity goes in a circle
```

$$vx := t \to V\cos(\Omega\, t)$$

```
vy := (t) -> V*sin(Omega*t);
```

$$vy := t \to V\sin(\Omega\, t)$$

```
x := x0 + int(vx(T),T=0..t);      # integrate to get position
```

$$x := x0 + \frac{1}{2}\frac{V\sin(2\,\omega\, t)}{\omega}$$

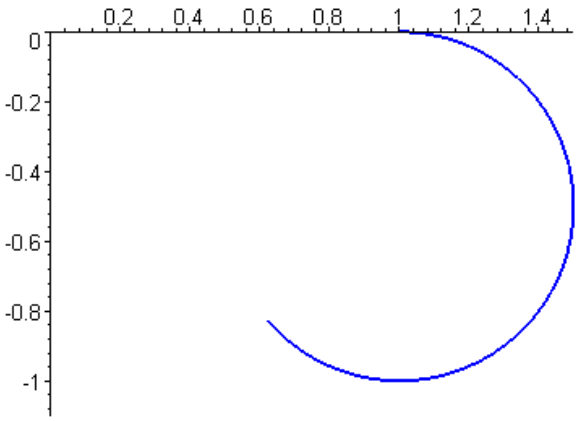
```
y := y0 +int(vy(T),T=0..t);
```

$$y := y0 + \frac{1}{2}\frac{V(\cos(2\,\omega\, t) - 1)}{\omega}$$

```
x0 := 1: y0 := 0: omega := 1: V := 1:
```

We set parameters $\omega = +1$, $a = 1$, $V = 1$ with initial condition $\mathbf{v'}(0) = V\hat{\mathbf{x}}'$ and then create a plot,

```
x0 := 1: y0 := 0: omega := 1: V := 1:
plot([x,y,t=0..2],scaling=CONSTRAINED,color=blue,thickness=2,
 view=[0..1.5,-1.1..0]);
```



(8.3.6)

The projectile is deflected "to the right" in this case since $\omega > 0$.

The actual deflection of the four projectiles is then a superposition of the centrifugal and Coriolis motions and is therefore not perfectly circular. We shall solve this problem exactly in Section 15.5 and plot the all four projectile trajectories.

8.4 Special Case #1 Problems

Recall from (8.1.5) that

$$\mathbf{F'_{eff}} = \mathbf{F} - m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \quad . \qquad (8.1.5)$$

frame centrifugal Coriolis Euler

If the rotation axis passes through the center of Frame S, we know from (7.13) that

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad \text{Special Case \#1} \qquad (7.12)$$

so that, using $\mathbf{r} = \mathbf{r}' + \mathbf{b}$ (6.1) in the third line below,

$$\mathbf{F'_{eff}} = \mathbf{F} - m[\dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})] - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$$

$$= \mathbf{F} - m\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{b} + \mathbf{r}']) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times [\mathbf{b} + \mathbf{r}'] \qquad // = \mathbf{F} + \mathbf{F'_{fict}}$$

$$= \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad . \qquad \text{Special Case \#1} \qquad (8.4.1)$$

so

$$\mathbf{F'_{fict}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{Special case \#1} \qquad (8.4.2)$$

This is similar to (8.2.4) but now we have included the Coriolis term. Here $\mathbf{r}$ is the long vector, but $\mathbf{v}'$ is the rate of change of the short vector $\mathbf{r}'$.

Problems involving the motion of objects on or near the Earth's surface, or of objects in orbit around the Earth, fall into Special Case #1. In the first problem class, Frame S' can be defined as shown in Fig (4.7.1).

8.5 Problems on the surface of the Earth

Let $\mathbf{F}_0 = m\mathbf{g}_0$ where $\mathbf{g}_o = -g_0\hat{\mathbf{r}}$ is a vector pointing to the center of the Earth, and $g_0 = GM_E/R_E^2$. Then for problems involving motions of objects near the surface of the Earth, one has these real forces,

$$\mathbf{F} = m\mathbf{g}_0 + \text{possible other real forces} \quad . \tag{8.5.1}$$

Possible other real forces might include air friction, wind, the action of magnetic fields on charged particles, etc.

Let $T = 24*60*60 = 86400 \sim 10^5$ sec be the nominal period of a day. The day has a seasonal variation in its duration of roughly (see e.g. https://www.iers.org/IERS/EN/Science/EarthRotation/LODplot.html)

$$dT/dt \sim 0.5 \text{ msec/day} \sim 10^{-3} * 10^{-5} \sim 10^{-8} \qquad \text{// rms} \tag{8.5.2}$$

with smaller short-term variations. From this we compute a rough value for $\dot{\omega}$,

$$|\dot{\omega}| = |d(2\pi/T)/dt| = 2\pi T^{-2}\,(dT/dt) \sim 6 * 10^{-10} * 10^{-8} \sim 10^{-17}\ \text{sec}^{-2}\ . \tag{8.5.3}$$

For activities on the surface of the Earth, $r \approx R_E \sim 10^7$ m so,

$$\dot{\omega} \times r \sim 10^{-17} * 10^7 \sim 10^{-10} \sim 10^{-9}\ g \quad . \qquad \text{// max value at the equator} \tag{8.5.4}$$

Thus in (8.4.1) we neglect the Euler term $-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ to get

$$\mathbf{F'}_{\mathbf{eff}} = (m\mathbf{g}_0 + \text{possible other real forces}) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' \quad . \tag{8.5.5}$$

Carrying out a static experiment ($\mathbf{v}' = 0$) to measure the $\mathbf{g}$ vector at some location on the Earth (no other forces in this experiment), one finds from (8.5.5),

$$\mathbf{F'}_{\mathbf{eff}} = m\mathbf{g}_0 - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad \mathbf{r} = R_E\hat{\mathbf{r}}$$

$$\equiv m\mathbf{g} \quad , \tag{8.5.6}$$

where $\mathbf{g}$ is then the *local* gravity vector (which does not quite point to Earth center). In terms of this $\mathbf{g}$, one then has

$$\mathbf{F'}_{\mathbf{eff}} = (m\mathbf{g} + \text{possible other real forces}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' \tag{8.5.7}$$

so only the Coriolis fictitious force is left -- the centrifugal term has been absorbed into m**g**.

By how much do $\mathbf{g}_0$ and **g** differ? We can write

$$\mathbf{g} - \mathbf{g}_0 = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \omega^2 r \cos\theta\, \hat{\boldsymbol{\rho}} \qquad r = R_E \tag{8.5.8}$$

where θ is latitude measured from the equator and $\hat{\boldsymbol{\rho}}$ is the usual cylindrical unit vector pointing away from the rotation axis of the Earth at our point on the surface. So we see the direction of the difference to be $\hat{\boldsymbol{\rho}}$ for any θ. The magnitude is roughly

$$\omega = 7.27 \times 10^{-5}\ \text{sec}^{-1} \qquad // \ \omega_E = 2\pi \text{ radians} / [\ 24*60*60 \text{ sec}\ ]$$

$$r \approx 6.37 \times 10^{6} \qquad // = R_E \tag{8.5.9}$$

$$\omega^2 r \cos\theta = (7.27)^2(6.37)\ 10^{-4}\cos\theta = 337 \times 10^{-4}\cos\theta \sim 3 \times 10^{-2}\ \text{m/sec}^2 \cos\theta \sim (3/1000) g_0 \cos\theta$$

so the difference is small but not zero. Presumably the local surface of the Earth is perpendicular to **g** and not $\mathbf{g}_0$, and one would certainly expect this to be true for a quiet ocean surface.

We now repeat the above discussion in the "swap" notation mentioned at the start of the Summary.

Swap Notation. The meaning of "swap notation" is that, in Fig 1, the vectors **ω** and **b** stay put, but all other vectors undergo **V**↔**V'**. This latter group includes basis vectors $\mathbf{e}_i \leftrightarrow \mathbf{e'}_i$, **r** ↔ **r'** , **v** ↔ **v'**, **a** ↔ **a'** and of course Frame S ↔ Frame S'. This is nothing more than a change of the way things are labeled. If a non-swap equation has the number (x.x.x), then the corresponding equation in swap notation will be given the number $(x.x.x)_s$ (8.5.10)

Here we rewrite equations (8.5.5), (8.5.6), (8.5.7) and (8.5.8) in swap notation.

$$\mathbf{F}_{eff} = (m\mathbf{g}_0 + \text{possible other real forces}) - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) - 2m\, \boldsymbol{\omega} \times \mathbf{v} \quad . \tag{8.5.5}_s$$

Carrying out a static experiment (**v** = 0) to measure the **g** vector at some location on the Earth (no other forces in this experiment), one finds from (8.5.10),

$$\mathbf{F}_{eff} = m\mathbf{g}_0 - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) \qquad \mathbf{r'} = R_E \hat{\mathbf{r}}'$$

$$\equiv m\mathbf{g} \ , \tag{8.5.6}_s$$

where **g** is the *local* gravity vector (which does not quite point to Earth center). In terms of this **g**, one then has

$$\mathbf{F}_{eff} = (m\mathbf{g} + \text{possible other real forces}) - 2m\, \boldsymbol{\omega} \times \mathbf{v} \tag{8.5.7}_s$$

so only the Coriolis fictitious force is left -- the centrifugal term has been absorbed into m**g**. Finally,

$$\mathbf{g} - \mathbf{g}_0 = -\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) = \omega^2 r' \cos\theta'\, \hat{\boldsymbol{\rho}}' \quad . \qquad r' = R_E \tag{8.5.8}_s$$

8.6 Tethered satellites and Tidal Forces

Consider a satellite, consisting of a pair of radially-aligned tethered masses m_1 and m_2, which orbits the Earth at angular frequency ω. Inertial Frame S is fixed with its origin at the center of the Earth, while Frame S' is fixed to the tethered satellite and so Frame S' rotates at ω as the satellite moves around the Earth.

The tethered masses are assumed to be at rest in Frame S' and are radially aligned with $r_1 > r_2$. One can show (see Appendix F) that such a system is stable due to a restoring torque and, in the stable position, the line between the masses points to the center of the Earth as the satellite orbits. It might take some damping effort to achieve this stable configuration. This restoring torque plus damping over time is what caused the slightly distorted Moon to present its same face to the Earth as it orbits (apart from a small residual libration).

Since we have a Special Case #1 situation (rotation axis through Frame S origin) we use (8.4.1) for the effective force acting on a mass m in Frame S' (**F** is the real force),

$$\mathbf{F'_{eff}} = \underset{\text{real}}{\mathbf{F}} \; \underset{\text{centripetal}}{-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})} \; \underset{\text{Coriolis}}{-2m\,\boldsymbol{\omega} \times \mathbf{v'}} \; \underset{\text{Euler}}{-m\dot{\boldsymbol{\omega}} \times \mathbf{r}} \quad . \qquad \text{Special Case \#1} \qquad (8.4.1)$$

where **r** is a long vector from the center of the Earth to mass m.

In their stable positions, the two masses have $\mathbf{v_i'} = 0$ so there is no Coriolis term. The orbit has a constant ω, so there is no Euler term. Thus the effective force on a static mass m in Frame S' is given by,

$$\begin{aligned} \mathbf{F'_{eff}} &= \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = \mathbf{F} - m[(\boldsymbol{\omega}\bullet\mathbf{r})\boldsymbol{\omega} - \omega^2\mathbf{r}] \\ &= \mathbf{F} + m\omega^2 r\,\hat{\mathbf{r}} \quad . \end{aligned} \qquad (8.6.1)$$

The real force **F** is the force of gravity plus the force of the tether T > 0. We then write for our two tethered masses, recalling that $r_1 > r_2$,

$$\begin{aligned} \mathbf{F'_{eff,1}} &= -(Gm_E m_1/r_1^2)\,\hat{\mathbf{r}} - T\hat{\mathbf{r}} + m_1\omega^2 r_1\,\hat{\mathbf{r}} = [\,-(Gm_E m_1/r_1^2) + m_1\omega^2 r_1 - T\,]\,\hat{\mathbf{r}} \\ \mathbf{F'_{eff,2}} &= -(Gm_E m_2/r_2^2)\,\hat{\mathbf{r}} + T\hat{\mathbf{r}} + m_2\omega^2 r_2\,\hat{\mathbf{r}} = [\,-(Gm_E m_2/r_2^2) + m_2\omega^2 r_2 + T\,]\,\hat{\mathbf{r}} \end{aligned} \qquad (8.6.2)$$

where M_E = mass of Earth, G = gravitational constant. It is convenient to define,

$$f(r,m) \equiv -mM_E G/r^2 + m\omega^2 r \qquad // \;\; f'(r,m) = 2mMG/r^3 + m\omega^2 \qquad (8.6.3)$$

so

$$\begin{aligned} \mathbf{F'_{eff,1}} &= [\,f(r_1, m_1) - T\,]\,\hat{\mathbf{r}} \\ \mathbf{F'_{eff,2}} &= [\,f(r_2, m_2) + T\,]\,\hat{\mathbf{r}} \quad . \end{aligned} \qquad (8.6.4)$$

Since both masses are at rest in Frame S', their accelerations are $\mathbf{a'_i} = 0$ so $\mathbf{F'_{eff,i}} = m_i\mathbf{a'_i} = 0$, and we conclude that the tether tension is given by

$$T = f(r_1, m_1) = - f(r_2.m_2) \quad . \tag{8.6.5}$$

We now borrow a picture from page 120 of the *Tethers in Space Handbook* (Cosmo and Lorenzini) which shows the tethered satellite situation. The upper mass is at r_1, the lower at r_2, and the "center of gravity" is at r_0, all measured from the center of the Earth. We refer to the two masses as m_1 and m_2 and to ω_0 as ω.

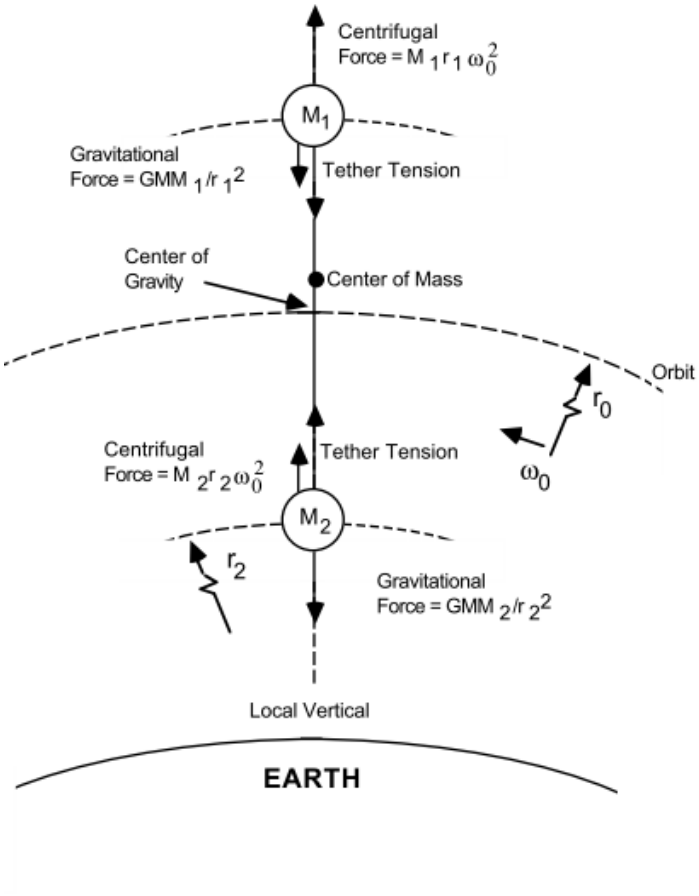


Figure 4.1 Forces on Tethered Satellites

(8.6.6)

If the entire satellite were suddenly compressed to a point mass $M = m_1 + m_2$ and if this point mass were placed at the center-of-gravity location, that point mass M would continue to orbit the Earth at radius r_0 and frequency ω and and nothing exciting happens. For either of these "systems" we have the following balance between gravitational force and centrifugal force,

$$(GM_E M/r_0^2) = M\omega^2 r_0 \qquad \text{or} \qquad GM_E/r_0^2 = \omega^2 r_0 \quad . \tag{8.6.7}$$

Comment: There is a very small distance between the satellite's center of mass and its center of gravity. See Appendix D on this subject and in particular the numerical examples of (D.3.16).

At this point to make things simple we shall assume $m_1 = m_2 = m$. Then from (8.6.5),

$T = f(r_1) = - f(r_2)$

$f(r) \equiv -mM_E G/r^2 + m\omega^2 r$ // gravitational force + centrifugal force

$f'(r) = 2mMG/r^3 + m\omega^2$. (8.6.8)

Note that $f(r_0) = 0$ according to (8.6.7). Setting $r_1 = r_0 + \Delta r$ in (8.6.8),

$T = f(r_0 + \Delta r) = -mMG/ (r_0+\Delta r)^2 + m\omega^2(r_0+\Delta r)$

$= -(mMG/r_0^2) (1 + \Delta r/ r_0)^{-2} + m\omega^2(r_0+\Delta r)$ // assume $\Delta r << r_0$

$\approx -(mMG/r_0^2) (1 - 2\ \Delta r/ r_0) + m\omega^2(r_0+\Delta r)$ // $(1+x)^n \sim 1 + nx$ for small x

$= - m\omega^2 r_0 (1 - 2\ \Delta r/ r_0) + m\omega^2(r_0+\Delta r)$ // using (8.6.7)

$= 3m\omega^2(\Delta r)$

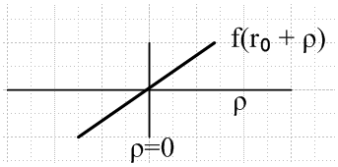
$= 3\ (mMG/r_0^3)(\Delta r)$. (8.6.9)

The real gravitational force gives 2/3 of this result, while the fictitious centrifugal term gives 1/3 of the result, a fact we will note again several times below.

If ρ represents an arbitrary vertical displacement away from r_0, so $r = r_0 + \rho$, then the above says,

$f(r_0 + \rho) = 3m\omega^2\rho$ (8.6.10)

which looks like this near $\rho=0$



(8.6.11)

This $f(r_0+\rho)$ represents the radial force on any untethered Particle that might be present in Frame S' located a radial distance ρ from r_0. A particle at $r > r_0$ is pushed up in Frame S', while a particle at $r < r_0$ is pushed down. Again, this force in Frame S' is due to the combination of gravitational (2/3) and centrifugal (1/3) forces.

Another way to view the above calculation is this,

$T = [f(r_0 + \Delta r) - f(r_0)] + f(r_0)$

$\approx f'(r_0)\ \Delta r + f(r_0)$ // $f'(r_0)$ is the gradient of the force f(r) at $r=r_0$

$= f'(r_0)\ \Delta r$ // $f(r_0) = 0$ from (8.6.7)

$= (2mMG/r_0^3 + m\omega^2)\ \Delta r$ // $f'(r_0)$ from (8.6.8)

$= (2\ m\omega^2 + m\omega^2)\ \Delta r$ // (8.6.7) again

$= 3m\omega^2\ \Delta r$. (8.6.12)

So the tension in the tether due to tidal force is equal to Δr times the derivative of f(r) evaluated at $r = r_0$,

$$T \approx \Delta r\, f'(r_0) = \Delta r\, 3m\omega^2 = 3\, \Delta r\, (mGM/r_0^3) \quad . \tag{8.6.13}$$

Again, the tidal force acting up the upper mass is T pushing up, and the tidal force acting on the lower mass is T pulling down, all in Frame S'. One might then write

$$\text{tidal force per unit mass} = \pm\, 3\, \Delta r\, (GM/r_0^3) \quad . \qquad \text{// tidal acceleration} \tag{8.6.14}$$

Once again, 2/3rds of the tidal force arises from the gravitational gradient at the satellite while 1/3 arises from the centrifugal gradient. The result (8.6.14) appears in Cosmo and Lorenzini p 123 with Δr=L and $\omega=\omega_0$.

Comments:

1. One may regard tension T as an example of a "tidal force" which tends to "rip apart" objects in orbit around a central force. In our example, the tidal force is $T = f'(r_0)\, \Delta r = 3\, (mMG/r_0^3)\, \Delta r$. For objects in orbit around the Earth, this is a small or moderate force, but for objects orbiting massive black holes, the force is strong enough to rip apart all known materials, a process called "spaghettification", a sort of cosmic disposal.

2. According to (8.6.14), an untethered water droplet on the surface of the upper mass will migrate to the upper extremity of that mass, while a water droplet on the surface of the lower mass will migrate to the lower extremity of that mass. A crude intuition would seem to say that the Earth ought to pull both water drops to the lower extremity of each mass, but that is not what happens. One must get into the rotating Frame S' to see what happens. This is essentially why the usual tides on the Earth "bulge" on the side facing the Moon *and* on the side facing away from the Moon. The Sun also has its smaller effect. This subject is considered in the next section.

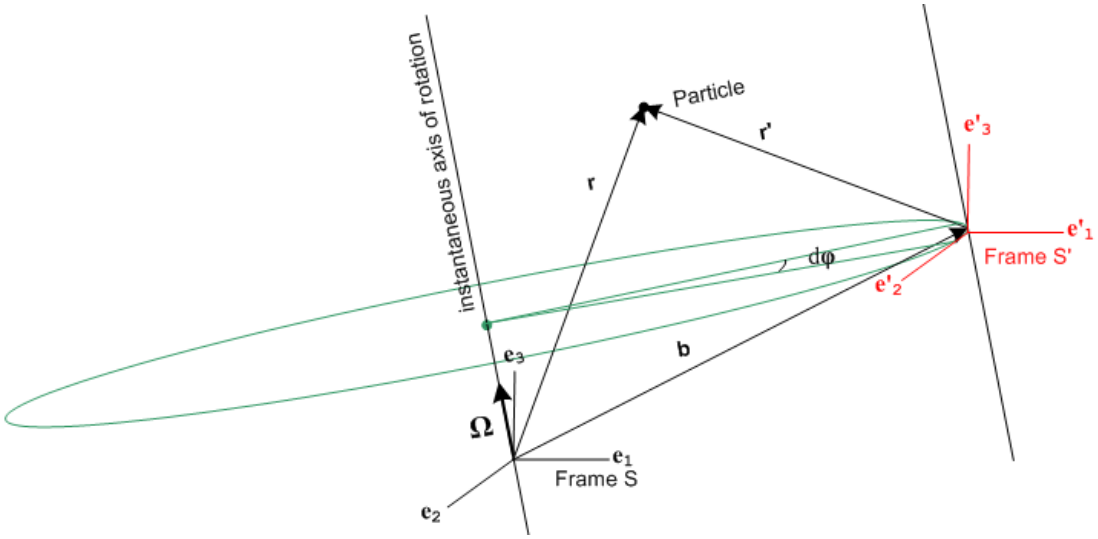
3. See comments below (8.8.26) for a discussion of why the tether tidal force (8.6.14) has a leading factor of 3, whereas the lunar Earth tide (8.8.26) has a leading factor of 2.

4. Appendix F presents a more complete analysis of the tether or "dumbbell" satellite, allowing for an arbitrary 3D orientation of the masses and analyzing their motion when not in the stable vertical orientation. For example, a general dynamic expression for the tether tension T appears in (F.6.26). For the tether vertically aligned and at rest the limit of this expression is (F.6.30) which is the same as (8.6.12) above. The reader is warned that Appendix D and F are both written in "swap" notation where prime and noprime labels are swapped relative to the no-swap notation used above. This is done to reduce the annoying number of primes that would otherwise be present in many expressions.

5. Real tethered satellites have seen action since the 1960's (including snapped tethers), see the brief history of Chen et al. [2013]. Tether applications include: (1) Power generation and thrust using a conducting tether in solar and planetary magnetic fields and plasmas; (2) lifting, stabilizing and moving objects around in space (et, reboosting tired satellites); (3) research in controlled-gravity environments (distance down the tether see (8.6.12)); (4) gravity-wave and conventional antenna experiments. Tether lengths have varied from tens of meters to tens of kilometers.

8.7 Special Case #3

Consider the following situation which we shall call Special Case #3 where we have replaced the rotation vector **ω** by **Ω** where now Ω = dφ/dt :



(8.7.1)

This resembles Special Case #1 because the rotation axis passes through the origin of Frame S. However, in this figure we intend that the axes of Frame S' always line up with those of Frame S, so the only thing that varies is the vector **b**(t). The axes $\mathbf{e'}_i$ are no longer "soldered" to the **b** vector, and $\mathbf{e'}_i = \mathbf{e}_i$ at all times. Although the axes of Frame S' do not rotate relative to those of Frame S, we still put this case into our "rotating frames" basket because the origin of Frame S' is instantaneously rotating about the **Ω** axis.

In Special Case #3 there is no distinction between $\partial_S \mathbf{a}$ and $\partial_{S'} \mathbf{a}$ for any vector **a** :

$$\partial_S \mathbf{a} = \partial_S [a_i \mathbf{e}_i] = (\partial_S a_i)\, \mathbf{e}_i = (\partial_t a_i)\, \mathbf{e}_i$$

$$\partial_{S'} \mathbf{a} = \partial_{S'} [a_i \mathbf{e}_i] = (\partial_{S'} a_i)\, \mathbf{e}_i = (\partial_t a_i)\, \mathbf{e}_i \,. \qquad (8.7.2)$$

One can *interpret* $\partial_S \mathbf{a} = \partial_{S'} \mathbf{a}$ as being the G Rule $\partial_S \mathbf{a} = \partial_{S'} \mathbf{a} + \boldsymbol{\omega} \times \mathbf{a}$ with **ω**= 0.

Thus we can just write $\dot{\mathbf{a}} = \partial_S \mathbf{a} = \partial_{S'} \mathbf{a}$. The complicated analysis of Sections 6,7 and 8 is now much simpler:

$$\mathbf{r} = \mathbf{r'} + \mathbf{b}$$

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}' + \dot{\mathbf{b}} \quad \Rightarrow \quad \mathbf{v} = \mathbf{v}' + \dot{\mathbf{b}}$$

$$\dot{\mathbf{v}} = \dot{\mathbf{v}}' + \ddot{\mathbf{b}} \quad \Rightarrow \quad \mathbf{a} = \mathbf{a}' + \ddot{\mathbf{b}} \,. \qquad (8.7.3)$$

Newton's Law in Frame S' now has only one fictitious force,

$$m\mathbf{a}' = m\mathbf{a} - m\ddot{\mathbf{b}}$$

$$\mathbf{F}'_{\mathtt{eff}} = \mathbf{F} - m\ddot{\mathbf{b}}$$

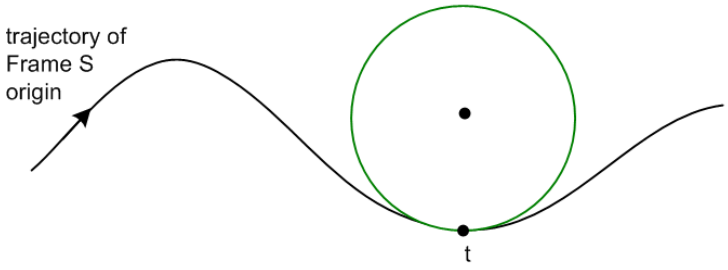
$$\mathbf{F}'_{\mathtt{fict}} = -m\ddot{\mathbf{b}} \ . \qquad \text{// Special Case \#3} \qquad (8.7.4)$$

Recall that our general case fictional force expression was

$$\mathbf{F}'_{\mathtt{fict}} = \underset{\text{frame}}{-m\ddot{\mathbf{b}}_{\mathtt{S}}} \underset{\text{centrifugal}}{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')} \underset{\text{Coriolis}}{- 2m\,\boldsymbol{\omega} \times \mathbf{v}'} \underset{\text{Euler}}{- m\dot{\boldsymbol{\omega}} \times \mathbf{r}'} \ . \qquad \textit{(8.1.8)}$$

In Special Case #3 only the "frame" portion of the fictitious force exists and we can interpret this as being the full fictional force in which we set $\boldsymbol{\omega} = 0$ and $\dot{\boldsymbol{\omega}} = 0$.

Regardless of how the Frame S' origin moves through space, we can always interpret its motion as an instantaneous rotation, as suggested by this drawing,



(8.7.5)

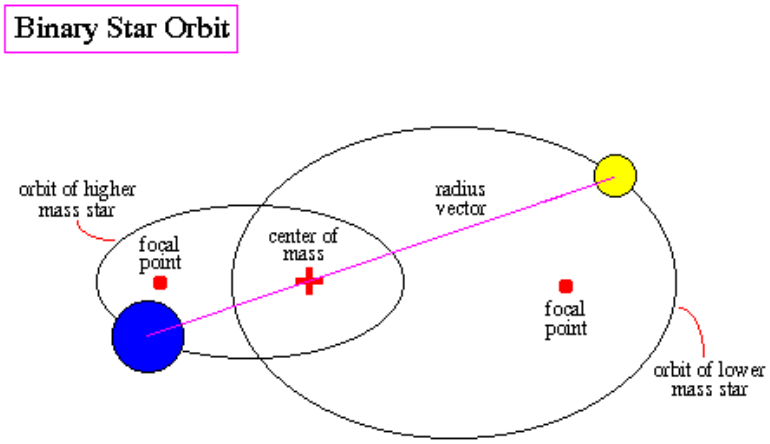
At time t the Frame S' origin is rotating along the green circle shown.

Comment: Historical authors have compared the behavior of Frame S' to a frying pan in the fast-moving hands of a busy cook who always maintains the pan's orientation. We are instead reminded of the behavior of a gimbaled alcohol stove often used in sailboats, or better yet, a gimbaled sailboat compass.

8.8 Tides on the Earth

The basic picture

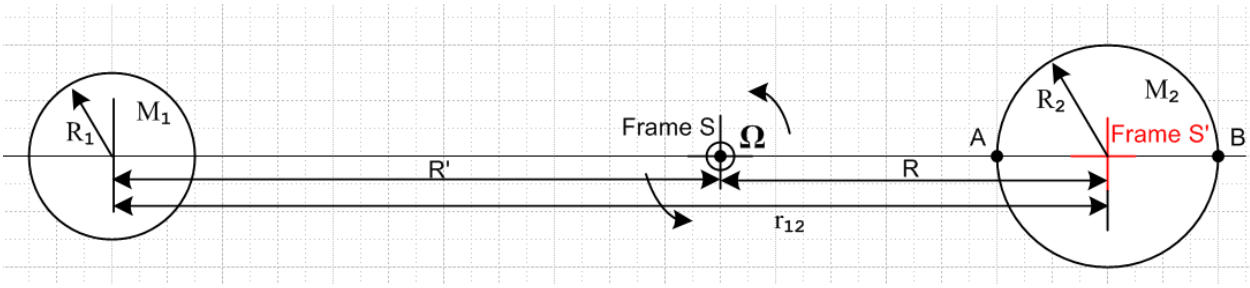
In general, the orbital pattern of a binary gravitational system has this planar appearance, where each object traverses its own ellipse



(8.8.1)

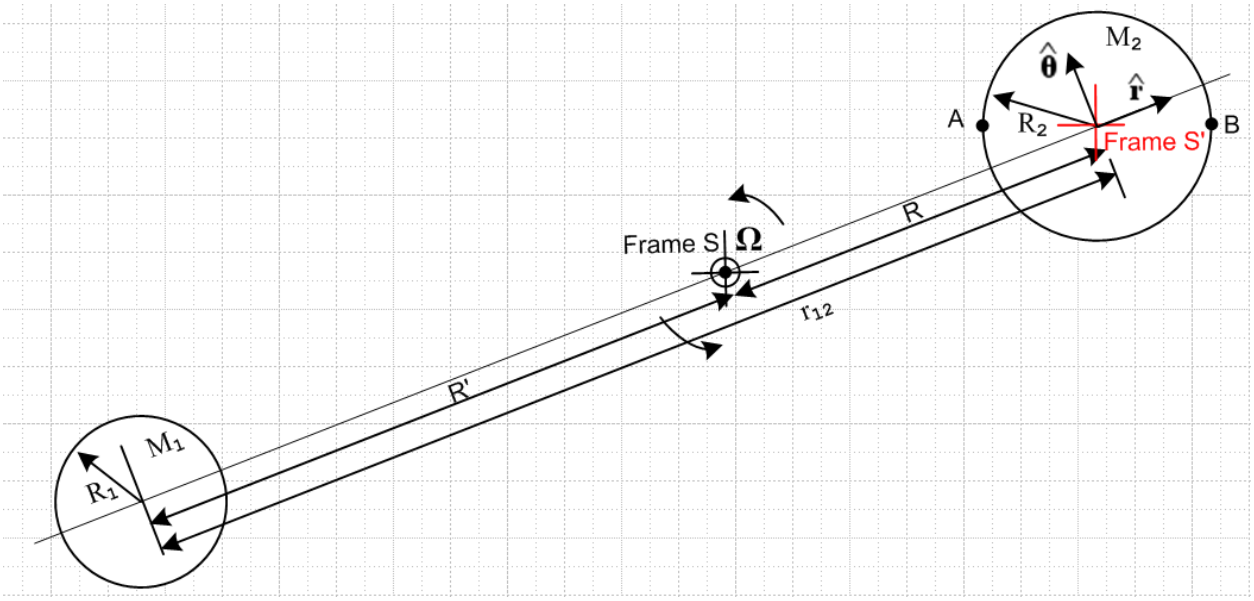
http://abyss.uoregon.edu/~js/ast122/lectures/lec10.html

In this section, however, we restrict our interest to a special case where each mass traces out a circular path, not an elliptical one. The picture is this, where we define R_1,R_2, M_1,M_2, r_{12},R,R' as shown:



(8.8.2)

Each object is assumed to be a spherically symmetric mass distribution and can thus be treated as a point mass at its center for gravitational purposes (see Section D.5). An inertial Frame S has its origin at the center-of-mass point, and the binary system rotates in the plane of paper at angular frequency Ω about this Frame S origin. Frame S' is attached to sphere 2 and we assume that sphere 2 maintains its orientation "relative to the stars" so that at some later time we have this picture



(8.8.3)

The situation with Frame S and Frame S' is then exactly that of Special Case #3 described in the previous section, where b = R and the only fictitious force in Frame S' is $\mathbf{F'_{fict}} = -m\ddot{\mathbf{b}}$. Points A and B are two fixed points on the surface of sphere 2. To apply Fig (8.8.3) to the Moon-Earth system, we temporarily turn off the rotation of the Earth (M_2) about its axis (not shown), so the this new Earth has a fixed orientation relative to the stars.

Distance R can be found from the usual center-of-mass equation (viewed from Frame S),

$$\mathbf{0} = [M_2 R\hat{\mathbf{r}} - M_1 R'\hat{\mathbf{r}}]/(M_1+M_2) \qquad R+R' = r_{12} \tag{8.8.4}$$

which reports out the obvious fact that

$$R = r_{12}\,[M_1/(M_1+M_2)]$$
$$R' = r_{12}\,[M_2/(M_1+M_2)]\ . \tag{8.8.5}$$

We collect here some data on the Sun, Earth and Moon :

$M_S = 1.989 \times 10^{30}$ kg $R_S = 695{,}700$ km $G = 6.674 \times 10^{-11}$ m^3/(kg-sec^2)
$M_E = 5.972 \times 10^{24}$ kg $R_E = 6371$ km (8.8.6)
$M_M = 7.342 \times 10^{22}$kg $R_M = 1737$ km $T_M = 27.32$ days (sidereal)

$r_{ES} = 1.496 \times 10^8$ km // average e = .0167
$r_{ME} = 384{,}400$ km // average e = .0549

We then compute,

1 ------- 2

Earth-Sun $\quad R/R_2 = r_{12}\,[M_1/(M_1+M_2)]\,/R_2 = r_{ES}\,[M_E/(M_E+M_S)]\,/R_S = .000646$

Moon-Earth $\quad R/R_2 = r_{12}\,[M_1/(M_1+M_2)]\,/R_2 = r_{ME}\,[M_M/(M_M+M_E)]\,/R_E = 0.737$

```
MS  := 1.989e30*kg:      RS := 695.7e3*km:
ME  := 5.972e24*kg:      RE := 6371*km:
MM  := 7.342e22*kg:      RM := 1737*km:
rES := 1.496e8*km:
rME := 384.4e3*km:       G :=  6.674e-11*m^3/(kg*sec^2):

rES * (ME/(ME+MS))/RS;
                                    .0006456442715
rME * (MM/(MM+ME))/RE;
                                    .7327632424
```

(8.8.7)

So for the Earth-Sun system, the center of mass is basically at the center of the Sun, while for the Moon-Earth system, the center of mass lies at a point 3/4 the radius of the Earth from the center. We could redraw our figure for these two cases, but the kinematics does not change, so we won't bother.

Force Equations

Newton's Law for a mass m in non-inertial Frame S' is, according to (8.7.4),

$$\mathbf{F'}_{\mathtt{eff}} = m\mathbf{a'} \qquad \text{where} \quad \mathbf{F'}_{\mathtt{eff}} = \mathbf{F} - m\ddot{\mathbf{b}} \tag{8.8.8}$$

and **F** is the sum of all Frame S forces acting on mass m.

It is clear from Fig (8.8.3) that (since R is a constant)

$$\mathbf{b} = R\hat{\mathbf{r}} \qquad d\hat{\mathbf{r}}/dt = \Omega\hat{\boldsymbol{\theta}} \qquad d\hat{\boldsymbol{\theta}}/dt = -\Omega\hat{\mathbf{r}} \qquad \text{// see (E.5.6)}$$

$$\dot{\mathbf{b}} = R\, d\hat{\mathbf{r}}/dt = R\Omega\hat{\boldsymbol{\theta}}$$

$$\ddot{\mathbf{b}} = R\Omega\, d\hat{\boldsymbol{\theta}}/dt = -R\Omega^2\hat{\mathbf{r}} \quad . \tag{8.8.9}$$

Therefore (8.8.8) becomes

$$m\mathbf{a'} = \mathbf{F'}_{\mathtt{eff}} = \mathbf{F} + m R\Omega^2\hat{\mathbf{r}} \quad . \tag{8.8.10}$$

For a Particle in or on the Earth, the real forces are

$$\mathbf{F} = \mathbf{F}_{g1} + \mathbf{F}_{g2} + \mathbf{F}_{ng} \quad \text{where}$$

$\mathbf{F}_{g1}$ = the gravitational force due to sphere 1 (the Moon)
$\mathbf{F}_{g2}$ = the gravitational force due to sphere 2 (the Earth)
$\mathbf{F}_{ng}$ = any non-gravitational forces (8.8.11)

and then (8.8.10) may be written

$$m\mathbf{a}' = \mathbf{F}_{g1} + \mathbf{F}_{g2} + \mathbf{F}_{ng} + m R\Omega^2 \hat{\mathbf{r}} \quad . \tag{8.8.12}$$

This is the effective Newton's Law for a mass m in Frame S'. If a mass m is at rest on the surface of the Earth (in Frame S') , then $\mathbf{a}' = 0$ and we find

$$\mathbf{F}_{ng} = - \mathbf{F}_{g1} - \mathbf{F}_{g2} - m R\Omega^2 \hat{\mathbf{r}} \, . \tag{8.8.13}$$

If there are no other non-gravitational forces affecting mass m, then $\mathbf{F}_{ng}$ is just the force of the Earth's surface pushing up on mass m to hold it in place so it has $\mathbf{a}' = 0$.

The relation between r_{12} and Ω

If we were to replace sphere 2 (the Earth) with a point mass M_2 at its center, nothing would change in our orbiting picture. This point mass does a circular orbit around the binary center of mass with radius R and angular frequency Ω. The usual rule for circular motion of a point particle says that the gravitational force balances the centrifugal force, so

$$M_1 M_2 G / r_{12}^2 = M_2 \Omega^2 R$$

or

$$M_1 G / r_{12}^2 = \Omega^2 R \quad . \tag{8.8.14}$$

Now we do a thought experiment. We imagine the Earth's core to be solid and we grind out a small spherical cavity around the Earth's center point. We take the ground-out material and compress it into a point mass m and we place that mass m at the center of the cavity. The orbit of the Earth-Moon system is unaffected by this alteration. In effect we now have two point masses in identical orbits with the moon: the Earth of mass M_2-m and the central particle of mass m. For each we have $M_1 G / r_{12}^2 = \Omega^2 R$. The claim then is that the point mass m simply *floats* in the center of the cavity. In Frame S' there is no total force acting on this mass m to cause it to move from its position. This total force of 0 is the sum of the gravitational force pulling it to the Moon, and the centrifugal force pushing it away from the Moon,

$$\mathbf{0} = - (\mathbf{F}_{g1} + m R\Omega^2 \hat{\mathbf{r}})$$

or

$$m M_1 G / r_{12}^2 = m R\Omega^2$$

or

$$M_1 G / r_{12}^2 = \Omega^2 R$$

which is the same as (8.8.14) above. We can replace the R in (8.8.14) with the R of (8.8.5) to get

$$M_1 G / r_{12}^2 = \Omega^2 \, r_{12} \, [M_1/(M_1 + M_2)]$$

or

$$(M_1 + M_2) G / r_{12}^3 = \Omega^2 \tag{8.8.15}$$

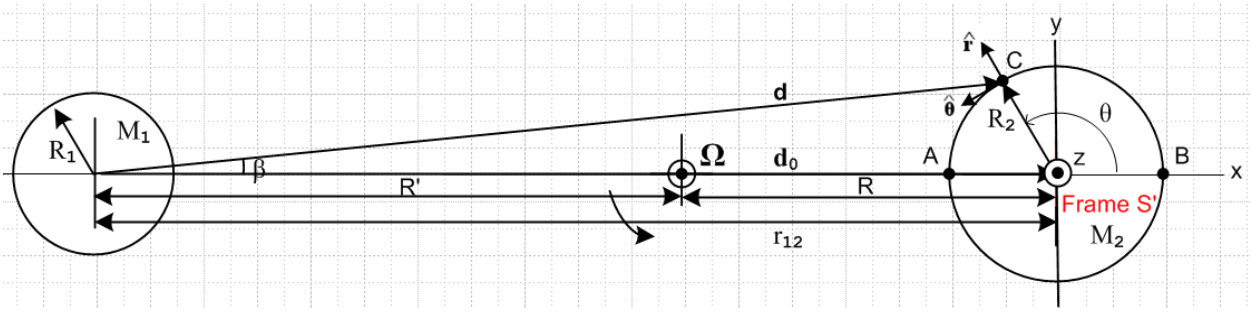
and this is the relationship between r_{12} and Ω for given masses M_1 and M_2.

Using (8.8.14) in (8.8.12) one finds,

$$m\mathbf{a}' = \mathbf{F}_{g1} + \mathbf{F}_{g2} + \mathbf{F}_{ng} + (mM_1G/r_{12}^2)\ \hat{\mathbf{r}} \quad . \tag{8.8.16}$$

Tidal Force at an arbitrary point on the Earth

Now consider a particle of mass m at some arbitrary location C on the surface of the Earth. We define angles θ and β as shown, where β is very small,



(8.8.17)

Here we show a brand new $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ which have nothing to do with those used in Fig (8.8.3). The old $\hat{\mathbf{r}}$ in (8.8.16) is now $\hat{\mathbf{d}}_0$, a unit vector to the right in (8.8.17). Vector $\mathbf{d}$ points to point C from the center of sphere 1 while vector $\mathbf{d}_0$ links the two object centers, so $d_0 = r_{12}$. We can then write

$$\mathbf{F}_{g1} = -(M_1mG/d^2)\,\hat{\mathbf{d}} \tag{8.8.18}$$

and so (8.8.16) is recast once again as

$$\begin{aligned} m\mathbf{a}' &= \mathbf{F}_{g1} + \mathbf{F}_{g2} + \mathbf{F}_{ng} + (mM_1G/r_{12}^2)\ \hat{\mathbf{d}}_0 \\ &= -(M_1mG/d^2)\,\hat{\mathbf{d}} + \mathbf{F}_{g2} + \mathbf{F}_{ng} + (mM_1G/d_0^2)\ \hat{\mathbf{d}}_0 \\ &= \mathbf{F}_{g2} + \mathbf{F}_{ng} + mM_1G\,(\hat{\mathbf{d}}_0/d_0^2 - \hat{\mathbf{d}}/d^2) \\ &= \mathbf{F}_{g2} + \mathbf{F}_{ng} + \mathbf{F}_{tid}, \end{aligned} \tag{8.8.19}$$

where

$$\mathbf{F}_{tid} \equiv mM_1G\,(\hat{\mathbf{d}}_0/d_0^2 - \hat{\mathbf{d}}/d^2) \quad . \qquad \text{// tidal force} \tag{8.8.20}$$

For a mass m at the center of the Earth, $\mathbf{d} = \mathbf{d}_0$ so $\mathbf{F}_{tid} = 0$ in agreement with our thought experiment above.

It is the fact that sphere 1's gravitational field varies slightly (in direction and magnitude) at different points on sphere 2 (as shown in (8.8.18)) which results in the tidal force. Equation (8.8.20) appears in Taylor p 332 as equation (9.12) and in Butikov as equation (2). We have tried to match Taylor's notation.

Comment: If the rotating Moon-Earth system in Fig (8.8.17) were replaced by a static non-rotating system in which the Earth and Moon were held apart by a very long, stiff (10^{20} N) rod, would the tidal force be the same as shown in (8.8.20)? Or would the water bulge only on the side of the Earth facing the Moon? In this case we have in inertial Frame S' the following force on a mass m on the surface of the Earth

$$\mathbf{F}' = m\mathbf{a}' = \mathbf{F}_{g1} + \mathbf{F}_{g2} + \mathbf{F}_{ng} \qquad \text{// Earth, Moon and Stick}$$

Since the gravitational force $\mathbf{F}_{g1}$ of M_1 on a mass m at point A is larger than at point B in Fig (8.8.17), it seems likely that water would only bulge on the A side of the Earth. So it is not *just* the non-uniformity of the gravitational field that causes the double-bulge tide on the real Earth, it is this non-uniformity in combination with the balance provided by the rotation which causes there to be zero force on a particle at the center of the Earth.

Tidal force in Cartesian coordinates

It is useful to express $\mathbf{F}_{tid}$ in both Cartesian and polar coordinates with the approximation that $R_2/d_0 << 1$. In that case, from the Law of Cosines and Fig (8.8.17),

$$d^2 = R_2^2 + d_0^2 - 2d_0R_0\cos(\pi-\theta) = R_2^2 + d_0^2 + 2d_0R_0\cos\theta = d_0^2[\,1 + (R_2/d_0)^2 + 2\,(R_2/d_0)\cos\theta\,]$$

$$\approx d_0^2[\,1 + 2\,(R_2/d_0)\cos\theta\,] \qquad (8.8.21)$$

so that

$$d^{-3} \approx d_0^{-3}\,[\,1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3}\,[\,1 - 3(R_2/d_0)\cos\theta\,]\,. \qquad (8.8.22)$$

Armed with this fact, we next write, again looking at Fig (8.8.17),

$$\mathbf{d}_0 = d_0\hat{\mathbf{x}}$$
$$\mathbf{d} = (d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}\,. \qquad (8.8.23)$$

We assume for now the usual x axis to the right and y axis up, though this will be changed below. Then,

$$\mathbf{F}_{tid} = mM_1G\,\{\,\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\,\}$$

$$= mM_1G\,\{\,d_0\hat{\mathbf{x}}/d_0^3 - [(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}]\,d_0^{-3}\,[\,1 - 3(R_2/d_0)\cos\theta\,]\}$$

$$= (mM_1G/d_0^2)\,\{\,\hat{\mathbf{x}} - [(1 + (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} + (R_2/d_0)\sin\theta\,\hat{\mathbf{y}}]\,[\,1 - 3(R_2/d_0)\cos\theta\,]\,\}$$

$$= (mM_1G/d_0^2)\,\{\,\hat{\mathbf{x}} - \hat{\mathbf{x}} + 3(R_2/d_0)\cos\theta\,\hat{\mathbf{x}} - (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta)\,\hat{\mathbf{y}}\,\} + O((R_2/d_0))^2$$

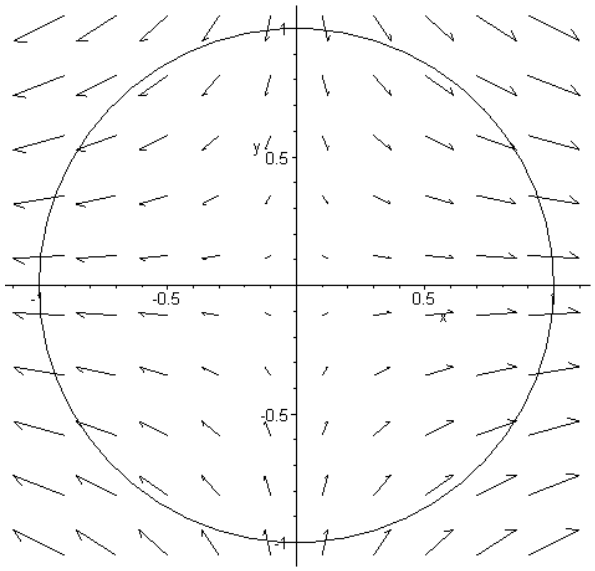
$$\approx (mM_1G/d_0^2)\,\{2(R_2/d_0)\cos\theta\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta)\,\hat{\mathbf{y}}\,\}$$

$$= (mM_1G/d_0^3)\ \{2R_2\cos\theta\ \hat{x} - R_2\sin\theta)\ \hat{y}\ \}$$

$$= (mM_1G/d_0^3)\ (\ 2x\ \hat{x} - y\ \hat{y}\ )\quad . \tag{8.8.24}$$

Using this simple form, it is easy to plot the tidal force field $\mathbf{F}_{tid}(x,y)$ in the region of the Earth,

```
with(plots):with(plottools):
s1 := fieldplot([2*x,-y],x=-1..1,y=-1..1,grid=[10,10]):
s2 := circle([0,0],1):
display(s1,s2,scaling=constrained);
```



(8.8.25)

We can evaluate $\mathbf{F}_{tid}$ in (8.8.24) at the left, right, top and bottom of the Earth:

left: A	$(x,y) = (-R_2,0)$	$\mathbf{F}_{tid} = -2R_2(mM_1G/d_0^3)\hat{x}$	points left
right: B	$(x,y) = (R_2,0)$	$\mathbf{F}_{tid} = 2R_2(mM_1G/d_0^3)\hat{x}$	points right
top:	$(x,y) = (0,R_2)$	$\mathbf{F}_{tid} = -R_2(mM_1G/d_0^3)\hat{y}$	points down
bottom:	$(x,y) = (0,-R_2)$	$\mathbf{F}_{tid} = R_2(mM_1G/d_0^3)\hat{y}$	points up

(8.8.26)

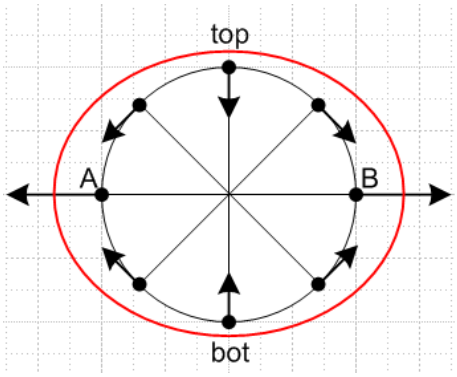
and these results seem in agreement with the above field plot (zero force at Earth center).

2 versus 3

Question: For the Earth tide situation we have just shown that at point B farthest from the Moon there is a total force $|\mathbf{F}_{tid}| = (2mM_1G/d_0^3)R_2$. For the tether analysis of (8.6.9) setting $\Delta r = R_2$ produces a tidal force of $|\mathbf{F}_{tid}| = T = (3mMG/r_0^3)R_2$. Why are the two red integers different?

Answer: For the tether satellite, the axes of the satellite Frame S' are rotating so the fictitious force set includes both the frame and "local" centrifugal contributions $-m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ of (8.1.5). It was shown that the centrifugal term accounts for 1/3 of the tidal force. But for the Earth tide analysis, the axes of Frame S' do not rotate, so only $-m\ddot{\mathbf{b}}_S$ exists so that extra 1/3 is missing. We might expect that extra 1/3 to reappear if we did an analysis of tides on an Earth which is phase locked to always have the same face pointing toward the Moon (Reader Exercise). On the other hand, for a tethered satellite that is dropping straight down toward the Earth, we would expect the 1/3 to be missing (Reader Exercise).

One gets the general impression that the water surface might have the following shape, where the black arrows show the magnitude of the tidal force at various locations, in agreement with (8.8.25) and (8.8.26),



(8.8.27)

The tidal acceleration is very weak compared to the local gravitational force on the Earth. For the lunar tidal case, continuing the Maple code above we find for the tidal acceleration at point B,

$$a_B = 2R_2(M_1G/d_0^3) = 2(M_1G/d_0^2)(R_2/d_0) = 2(M_MG/r_{M\text{-}E}^2)(R_E/r_{M\text{-}E})$$

```
rME_m := rME*(1000*m/km);   # convert to meters
```

$$rME_m := .3844000\ 10^{9}\ m$$

```
aTidB := 2*(MM*G/rME_m^2)*(RE/rME);
```

$$aTidB := .1098569855\ 10^{-5}\ \frac{m}{sec^2}$$

(8.7.28)

so basically the tidal acceleration is 10^{-7} the size of g = 9.8. It is rather amazing what such a small force can do when it is differentially applied to a lot of water.

Tidal force in polar coordinates

To express the tidal force in polar coordinates, we use (E.5.4) with $\rho \rightarrow r = R_2$ and $\varphi \rightarrow \theta$ to get

$$\hat{\mathbf{x}} = \cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}} \qquad r = R_2$$
$$\hat{\mathbf{y}} = \sin\theta\, \hat{\mathbf{r}} + \cos\theta\, \hat{\boldsymbol{\theta}} \ . \tag{8.8.29}$$

Then,

$$\mathbf{F}_{tid} = (mM_1G/d_0^3)\ \{\ 2x\ \hat{\mathbf{x}} - y\ \hat{\mathbf{y}}\ \}$$

$$= (mM_1G/d_0^3)\ \{\ 2r\cos\theta\ [\cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}] - r\sin\theta\ [\sin\theta\ \hat{\mathbf{r}} + \cos\theta\ \hat{\boldsymbol{\theta}}]$$

$$= (mM_1G/d_0^3)\ \{\ (2\cos^2\theta - \sin^2\theta)\hat{\mathbf{r}} + (\ -3\cos\theta\sin\theta)\hat{\boldsymbol{\theta}}\ \}\ . \tag{8.8.30}$$

The coefficients of the unit vectors can be simplified,

$$2\cos^2\theta - \sin^2\theta = 2(1/2)*(1+\cos2\theta) - (1/2)(1-\cos2\theta) = (1/2)\ [\ 2 + 2\cos2\theta - 1 + \cos2\theta\ ]$$

$$= (1/2)\ [\ 1 + 3\cos2\theta] = (3/2)\ [\ \cos2\theta + 1/3\ ]$$

$$-3\cos\theta\sin\theta = (-3/2)\ 2\sin\theta\cos\theta = -(3/2)\sin2\theta\ . \tag{8.8.31}$$

Therefore the tidal force in the polar coordinates appearing in Fig (8.8.17) is,

$$\mathbf{F}_{tid} = (3/2)\ (mM_1G\ R_2/d_0^3)\ [\ (\cos2\theta + 1/3)\ \hat{\mathbf{r}} - \sin2\theta\ \hat{\boldsymbol{\theta}}\ ]\ . \tag{8.8.32}$$

Since the term with the 1/3 is radially symmetric around the Earth, it really has no effect on tides and is usually just dropped. This last result appears in Butikov as (5), (6) and (7).

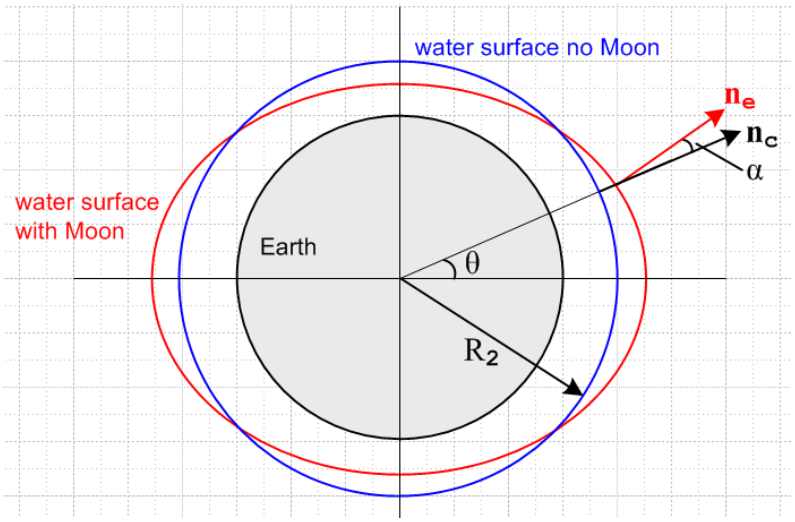
Equation of the water surface

Here we follow the development of Butikov. We first make an ansatz that the water surface on an idealized non-rotating water-covered Earth is described by the following simple equation,

$$r(\theta) = R_2 + a\cos2\theta\ . \qquad a > 0 \tag{8.8.33}$$

The constant term is R_2 so that $\langle r(\theta)\rangle = R_2$. One chooses the angle 2θ to get the shape suggested in Fig (8.8.27). The problem then is to determine the constant a which will be $<< R_2$. This is not the true equation of an ellipse, but we shall call it an ellipse anyway.

Consider the following picture of the water on the Earth,



(8.8.34)

Here $\mathbf{n_c}$ is normal to the black circle at angle θ, while $\mathbf{n_e}$ is normal to the red ellipse. In polar coordinates, we know that $\hat{\mathbf{n}}_c = \hat{\mathbf{r}}$, and we wish to compute $\hat{\mathbf{n}}_e$. Our motivation is to compute angle α which will then lead us to an expression for a. So far we know that $\cos\alpha = \hat{\mathbf{n}}_c \bullet \hat{\mathbf{n}}_e = \hat{\mathbf{r}} \bullet \hat{\mathbf{n}}_e$.

Define

$$f(r,\theta) = r - R_2 - a\cos 2\theta$$

and the ellipse (8.8.33) is then given by $f(r,\theta) = 0$. We know that the normal to a 2D surface is given by its gradient, so in polar coordinates we have

$$\mathbf{n_e} = \nabla f(r,\theta) = (\partial_r f)\hat{\mathbf{r}} + (1/r)(\partial_\theta f)\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} + 2(a/r)\sin 2\theta\ \hat{\boldsymbol{\theta}}$$

$$|\mathbf{n_e}| = (1 + [2(a/r)\sin 2\theta]^2)^{1/2}$$

$$1/|\mathbf{n_e}| = (1 + [2(a/r)\sin 2\theta]^2)^{-1/2} \approx 1 - (1/2)\,[2(a/r)\sin 2\theta]^2 \qquad (8.8.35)$$

$$\cos\alpha = \hat{\mathbf{n}}_c \bullet \hat{\mathbf{n}}_e = \hat{\mathbf{r}} \bullet \hat{\mathbf{n}}_e = \hat{\mathbf{r}} \bullet \mathbf{n_e} / |\mathbf{n_e}| = \hat{\mathbf{r}} \bullet [\,\hat{\mathbf{r}} + 2(a/r)\sin 2\theta\ \hat{\boldsymbol{\theta}}] / |\mathbf{n_e}| = 1/|\mathbf{n_e}|$$

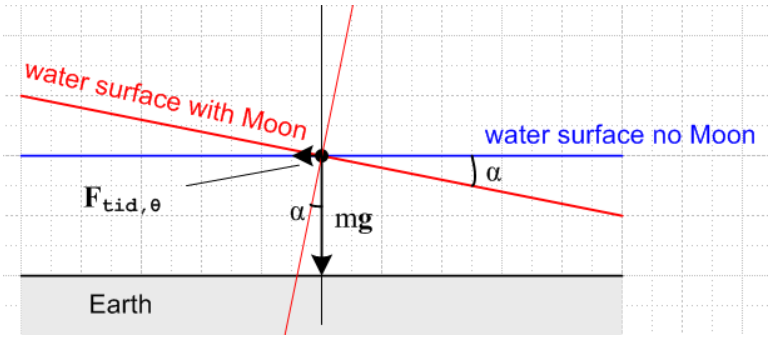
$$\approx 1 - (1/2)\,[2(a/r)\sin 2\theta]^2$$

$$\approx 1 - \alpha^2/2\ . \qquad (8.8.36)$$

Therefore using $r = R_2$,

$$\alpha = 2(a/R_2)\sin 2\theta\ . \qquad (8.8.37)$$

Having found a geometric value for α, we now seek another expression for α based on physics. Consider,



(8.8.38)

Since the affected water surface is assumed stable, the total force on the particle of water at the dot must be normal to the red surface. Therefore

$$|F_{tid,\theta}| / mg = \tan\alpha \approx \alpha \qquad \text{where } g = GM_2/R_2^2 \tag{8.8.39}$$

From (8.8.32) we then write

$$\alpha \approx |F_{tid,\theta}| / mg = (3/2)\,(mM_1G\,R_2/d_0^3)\sin 2\theta \;/\; (mGM_2/R_2^2)$$

$$= (3/2)\,(M_1/M_2)\,(R_2/d_0)^3 \sin 2\theta \ . \tag{8.8.40}$$

Comparing this α to that of (8.8.37) one finds,

$$2(a/R_2)\sin 2\theta = (3/2)\,(M_1/M_2)\,(R_2/d_0)^3 \sin 2\theta$$

so

$$a = (3/4)\,R_2\,(M_1/M_2)\,(R_2/d_0)^3 \ . \tag{8.8.41}$$

Thus the shape "ansatz" (8.8.33) was a good one. This result for a appears in Butikov as (9) and (11). Taylor obtains the same result in his (9.18) (h=2a) by treating the water surface as an equipotential surface (see Footnote at the end of this section).

The variation between low and high tides is H = 2a and we may compute this from (8.8.41) for both the Moon-Earth and Sun-Earth systems (in km),

```
Hme := (3/2)*RE*(MM/ME)*(RE/rME)^3;
```

Hme := .0005348920260 *km*

```
Hse := (3/2)*RE*(MS/ME)*(RE/rES)^3;
```

Hse := .0002458339634 *km*

Therefore,

$$H_{lunar_tide} = 53.49 \text{ cm} \sim 1.8 \text{ feet} \ .$$
$$H_{solar_tide} = 24.58 \text{ cm} \sim 0.8 \text{ feet} \ . \tag{8.8.42}$$

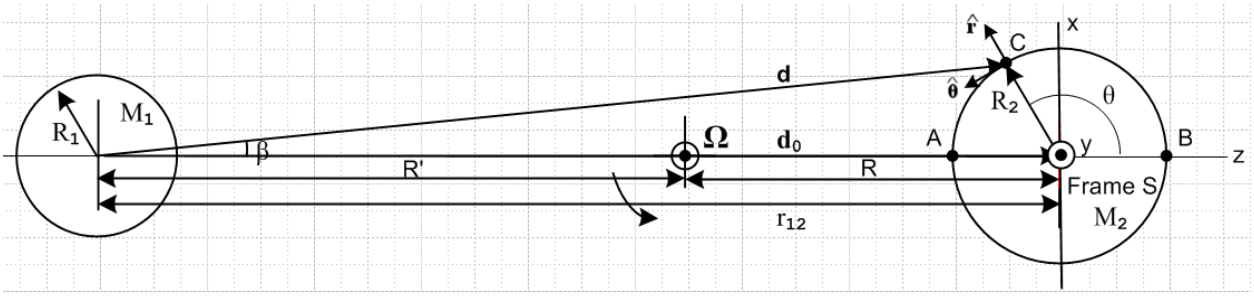
A non-inlander will recognize these as reasonable ballpark values for ocean tides, lending credence to the model at hand.

Comment: The red "ellipse" shown in (8.8.34) is the cross section in the plane of paper of a red ellipsoid formed by rotating the ellipse about the z axis. This ellipsoid then specifies the water level at all places on the Earth.

Tidal patterns for an arbitrary rotation axis of the Earth

We now turn the rotation of the Earth back on (we turned if off earlier). For the real Earth, there are many complications that arise. There are land masses. Lake water has nowhere to go. There is friction between the water and the land which slows down the Earth's rotation slightly over time. There is weather and there are ocean tidal currents which do not flow infinitely fast. We shall not attempt to analyze this general situation.

Instead, we imagine an idealized Earth covered with water and the Earth turns under the water with no "friction", and an Observer just stands in the water and measures the tide height as a function of time. What does that Observer see? It of course depends on where the Earth's axis of rotation is located relative to our picture. Consider,



(8.8.43)

We have now redefined the Earth frame on the right to be Frame S (formerly it was Frame S') and we have drawn new x,y,z axes for this new Frame S so the z axis points away from mass M_1. In Frame S then the angle θ is the usual spherical-coordinates polar angle.

We now assume that the Earth rotates about some axis $\hat{\mathbf{z}}'$ (new Frame S') which is obtained by rotating the $\hat{\mathbf{z}}$ axis by angles θ_1 and φ_1 as follows (see (E.2.2) for the matrix),

$$\hat{\mathbf{z}}' = R_z(\varphi_1)\, R_y(\theta_1)\, \hat{\mathbf{z}} \equiv R_1\, \hat{\mathbf{z}} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \hat{\mathbf{z}}$$

$$= \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta_1\cos\varphi_1 \\ \sin\theta_1\sin\varphi_1 \\ \cos\theta_1 \end{pmatrix}. \qquad (8.8.44)$$

This rotation is sufficient to put the $\hat{\mathbf{z}}'$ in any desired direction $(\theta_1.\varphi_1)$. The rotation R_1 moves all vectors $\mathbf{r}$ to new vectors $\mathbf{r}' = R_1\mathbf{r}$. Thus we may write

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_1\,\mathbf{r}$$

and

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\varphi \\ -\sin\varphi & \cos\varphi & 0 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_1^{-1}\,\mathbf{r}' = R_1^{T}\,\mathbf{r}' \qquad (8.8.45)$$

where the second matrix is the transpose of the first since rotations are real orthogonal $R^{-1} = R^{T}$. If we define spherical coordinates (r,θ,φ) for (x,y,z) and $(r'.\theta',\varphi')$ for (x',y',z') (of course $r' = r$) then the above may be written, cancelling the r factors,

$$\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & \cos\theta_1\sin\varphi_1 & -\sin\theta_1 \\ -\sin\varphi_1 & \cos\varphi_1 & 0 \\ \sin\theta_1\cos\varphi_1 & \sin\theta_1\sin\varphi_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \sin\theta'\cos\varphi' \\ \sin\theta'\sin\varphi' \\ \cos\theta' \end{pmatrix} \qquad (8.8.46)$$

which is three scalar equations. The third equation is this

$$\cos\theta = \sin\theta_1\cos\varphi_1\sin\theta'\cos\varphi' + \sin\theta_1\sin\varphi_1\sin\theta'\sin\varphi' + \cos\theta_1\cos\theta' \,. \qquad (8.8.47)$$

If the Earth turns at rate ω so $\varphi' = \omega t$, we then have

$$\cos\theta = \sin\theta_1\cos\varphi_1\sin\theta'\cos\omega t + \sin\theta_1\sin\varphi_1\sin\theta'\sin\omega t + \cos\theta_1\cos\theta'$$

$$\cos 2\theta = 2\cos^2\theta - 1 \,. \qquad (8.8.48)$$

Recall the equation of the water surface from (8.8.33),

$$r(\theta) = R_2 + a\cos 2\theta \,. \qquad a > 0 \qquad \textit{(8.8.33)}$$

This implies a tide height of

$$h(\theta) = a\cos 2\theta \,. \qquad (8.8.49)$$

Therefore on our idealized Earth which rotates about an axis (θ_1,φ_1) relative to Fig (8.8.43) we obtain the following tide height during the day

$$h(t) = a\cos 2\theta(t) = a\,[\,2\cos^2\theta(t) - 1\,]$$

$$= a\,[\,2\,(\sin\theta_1\cos\varphi_1\sin\theta'\cos\omega t + \sin\theta_1\sin\varphi_1\sin\theta'\sin\omega t + \cos\theta_1\cos\theta')^2 - 1\,] \,. \qquad (8.8.50)$$

Here θ' indicates the line of latitude at which our Observer is positioned.

Example 1: ($\hat{\mathbf{z}}' = \hat{\mathbf{y}}$)

Suppose the Earth's rotation axis were in the $\hat{\mathbf{y}}$ direction in Fig (8.8.43) (pointing out of the plane of paper). In that case one has $\theta_1 = \pi/2$ and $\varphi_1 = \pi/2$, since

$$\hat{\mathbf{z}}' = R_z(\pi/2)\, R_y(\pi/2)\, \hat{\mathbf{z}} = \begin{pmatrix} \sin\theta_1\cos\varphi_1 \\ \sin\theta_1\sin\varphi_1 \\ \cos\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{\mathbf{y}} \ . \tag{8.8.51}$$

Then from (8.8.50),

$$h(t) = a\,[\ 2(\sin\theta_1\cos\varphi_1\sin\theta'\cos\omega t + \sin\theta_1\sin\varphi_1\sin\theta'\sin\omega t + \cos\theta_1\cos\theta')^2 - 1\ ]$$

$$= a\,[\ 2(\sin\theta'\sin\omega t)^2 - 1\ ]$$

$$= a\,[\ 2\sin^2\theta'\sin^2\omega t - 1\ ] \ . \tag{8.8.52}$$

If the Observer were at the Earth's equator $\theta' = \pi/2$ (which is in the plane of paper of Fig (8.8.43)), one would have

$$h(t) = a\,[\ 2\sin^2\omega t - 1\ ] = a\sin(2\omega t) = a\cos(2\omega t - \pi/2) \ . \tag{8.8.53}$$

Comparison with (8.8.49) shows that $2\theta = 2\omega t - \pi/2$ so $\theta = \omega t - \pi/4$. The Observer sees a full amplitude swing of $\pm a$ in the tide. As this Observer moves toward the pole so θ' decrease, the amplitude of the tide decreases as (8.8.52) shows. At the pole, where $\theta' = 0$, one finds $h(t) = -a$ (a constant) all the time, which seems reasonable since $\theta = \pi/2$ all the time and so $h(\theta) = a\cos 2\theta = a\cos\pi = -a$.

Here is a Maple rendition of this Example where we set $a = 1$ and $\omega = 1$ so one day lasts $T = 2\pi$. In this code we refer to θ' as $\theta 2$:

```
costheta := sin(theta1)*cos(phi1)*cos(theta2)*cos(omega*t) +
sin(theta1)*sin(phi1)*sin(theta2)*sin(omega*t) + cos(theta1)*cos(theta2);
```

$$costheta := \sin(\theta 1)\cos(\phi 1)\cos(\theta 2)\cos(\omega t) + \sin(\theta 1)\sin(\phi 1)\sin(\theta 2)\sin(\omega t) + \cos(\theta 1)\cos(\theta 2)$$

```
theta1 := evalf((Pi/180)*90): phi1 := evalf((Pi/180)*90):    a := 1:    omega := 1:
costheta;
```

$$-.2051033807\ 10^{-9}\cos(\theta 2)\cos(t) + 1.\sin(\theta 2)\sin(t) - .2051033807\ 10^{-9}\cos(\theta 2)$$

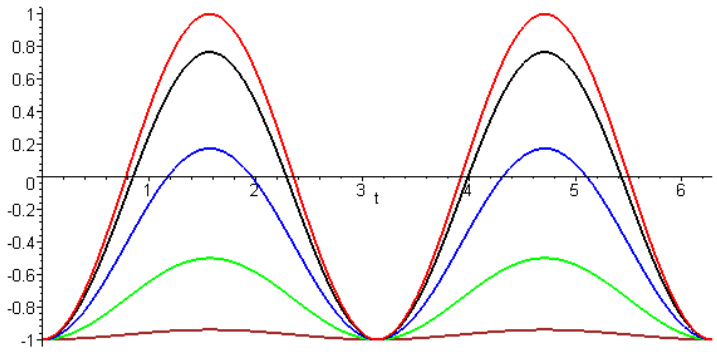
```
h := a * (2*costheta^2-1);
```

$$h := 2\,(-.2051033807\ 10^{-9}\cos(\theta 2)\cos(t) + 1.\sin(\theta 2)\sin(t) - .2051033807\ 10^{-9}\cos(\theta 2))^2 - 1$$

```
angle_set := evalf((Pi/180)*[90,70,50,30,10]);
```

$$angle_set := [1.570796327,\ 1.221730477,\ .8726646262,\ .5235987757,\ .1745329252]$$

```
plot([seq(h(t),theta2 = angle_set)],t=0..2*Pi,color
=[red,black,blue,green,brown],thickness=2);
```



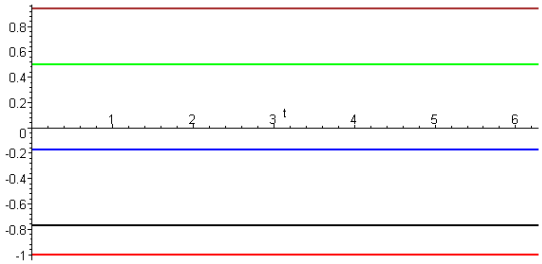
(8.8.54)

The top trace is for θ' = 90° (equator) which has the full tide amplitude, and then as one approaches the pole in steps of 20° this amplitude decreases ending up with h(t) ≈ -a for θ' = 10°. On *this* Earth there are always two equal high tides per day.

Example 2: (tipping $\hat{\mathbf{z}}'$ toward the $\hat{\mathbf{x}}$ direction)

Suppose however that the Earth's rotation axis points in the $\hat{\mathbf{z}}$ direction of (8.8.43) so $\theta_1 = 0$ and $\varphi_1 = 0$. In this case we expect to have no tides at all since for any θ' latitude line θ = θ' is constant. This is borne out in the above Maple code Maple where we again plot h(t),

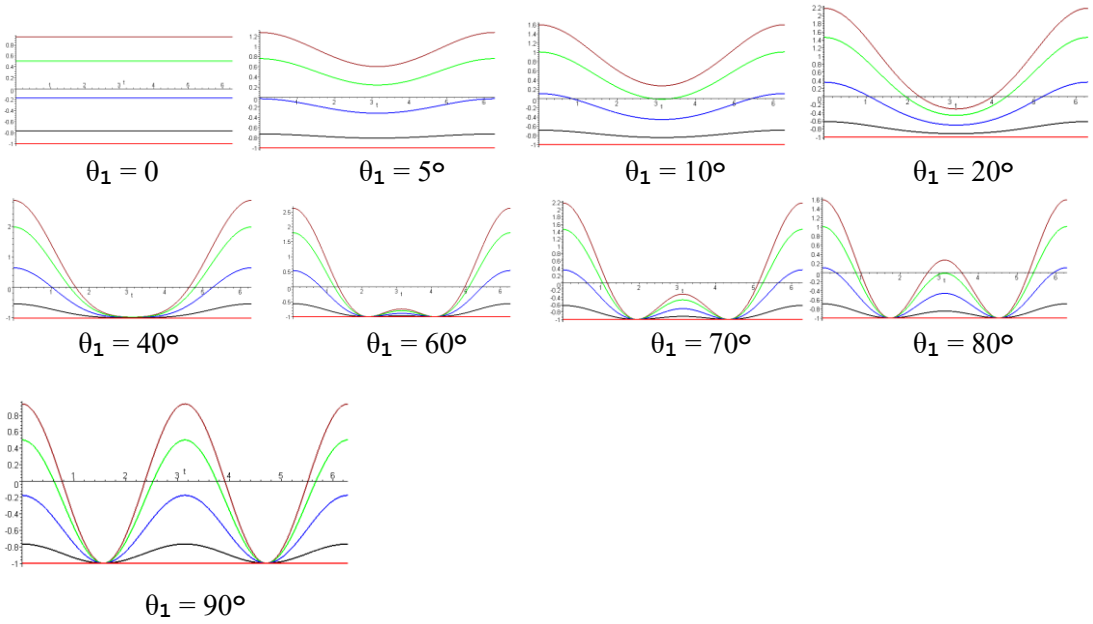
```
theta1 := evalf((Pi/180)*0): phi1 := evalf((Pi/180)*0):
```



(8.8.55)

At the equator the Observer is stuck low tide all the time (bottom trace). Conversely, an Observer at the pole is stuck at high tide all the time (top trace is θ' = 10°).

As we start increasing θ_1 away from 0, the nature of the tidal traces changes. Here is a set of trace sets for various θ_1 always with $\varphi_1 = 0$:

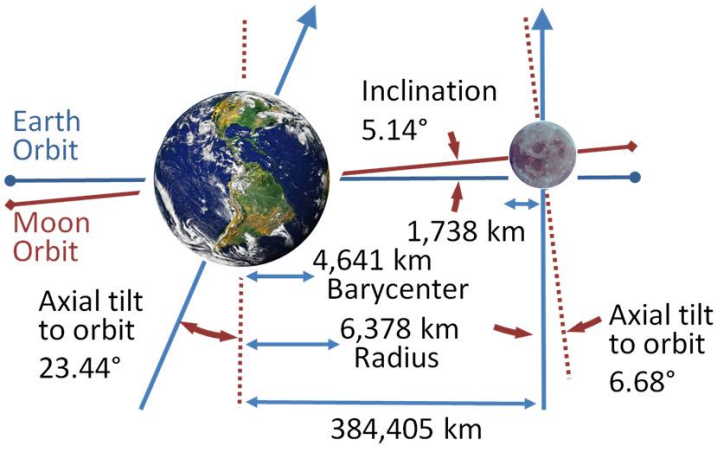


(8.8.56)

When $\theta_1 = 0°$ there are no tides at all since the rotation is in the $\hat{\mathbf{z}}$ direction in Fig (8.8.43). As we gradually tip the Earth's rotation axis toward the $\hat{\mathbf{x}}$ direction, things change. Up to about $\theta_1 = 40°$ there is one tide per day, but beyond this point there are two unequal tides per day and they become equal when θ_1 reaches 90°. The point here is that many tidal patterns are possible depending on the Earth's direction of rotation.

Tidal patterns for the rotating Earth (but still a water world in which water flows instantly)

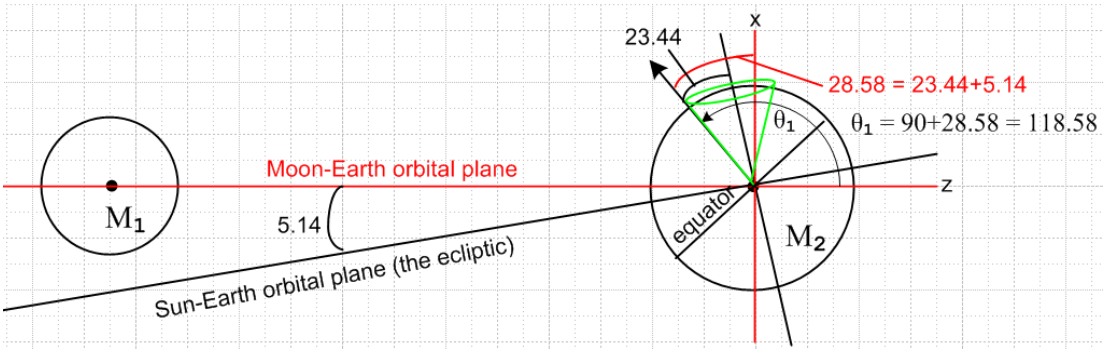
Relative to Fig (8.8.43) the actual axis $\hat{\mathbf{z}}'$ of the Earth's rotation varies over time, as suggested by this picture from wiki,



(8.8.57)

https://upload.wikimedia.org/wikipedia/commons/4/43/Earth-Moon.PNG

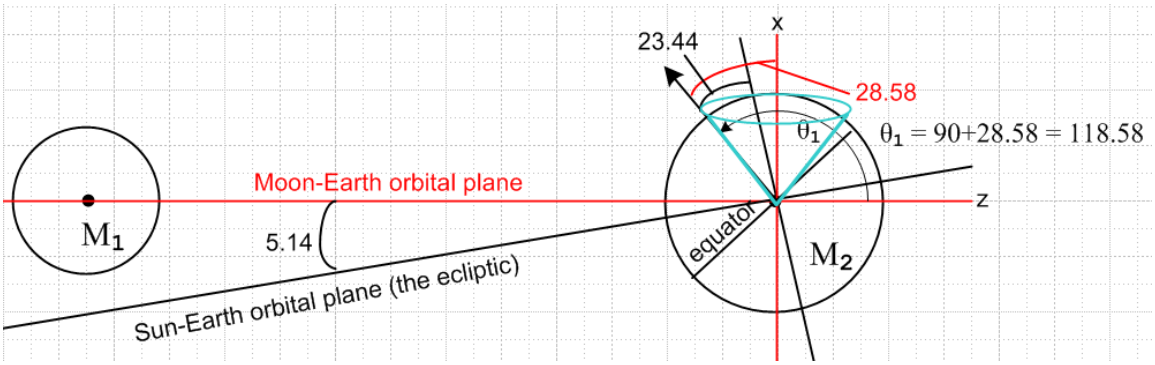
We transcribe the situation depicted above into a drawing more compatible with Fig (8.8.43),



(8.8.58)

Here the black arrow is $\hat{\mathbf{z}}'$ (Earth's rotation axis) and it is located at $\theta_1 = 118.6°$ and is in the plane of paper so $\varphi_1 = 0$. The intersection of the two orbital planes is called the line of nodes and for the time indicated in the picture, that line is perpendicular to the plane of paper. This situation of maximum tilt 28.58° occurs once every 18.6 years, a time called the "major lunar standstill". At a time 9.3 years later than the above drawing, the Earth's rotation axis in effect moves to the right edge of the green cone and then the 28.58° = 23.44+5.14 gets replaced by 18.30° = 23.44-5.14 which is the "minor lunar standstill". The half-angle of the green cone is 23.44°.

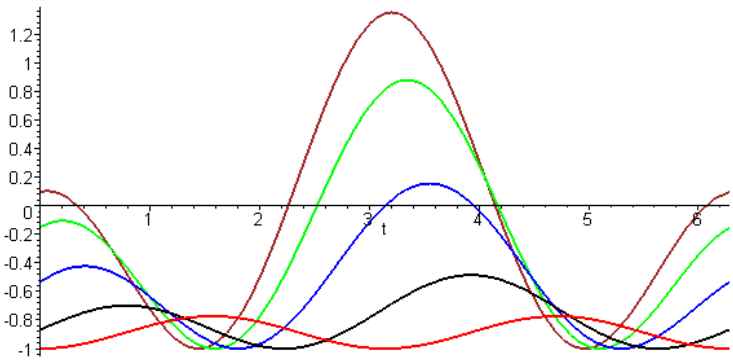
Near the time depicted in the picture, the Earth's rotation axis in effect moves around a different cone once per month as indicated in blue in this picture (blue cone half-angle = 28.58°)



(8.8.59)

Thus, a half month later than the configuration shown, one will have $\theta_1 = 90 - 28.58 = 61.42°$. At times in between, θ_1 lies in the range (61.4°, 118.6°) and φ_1 takes small values with $|\varphi_1| \leq 28.58°$.

Here we repeat our plot set above for $\theta_1 = 100°$ and $\varphi_1 = 20°$ just as an example:



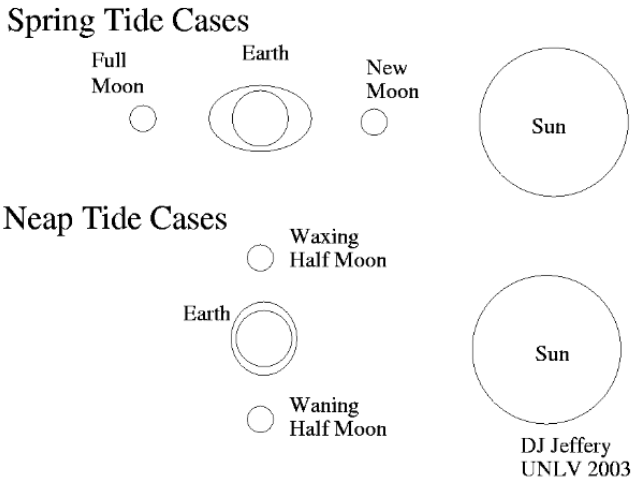
(8.8.60)

At the equator (red) there are two small equal tides per day. At other latitudes there are still two tides per day, but they are unequal.

As the Earth rotates, it is true that at any latitude θ' there is an outward-pointing centrifugal force of equal magnitude all around the Earth, but we expect this not to affect the tides.

The high tides are nominally 12 hours apart. In fact the Moon moves with a 27.3 day period in the same direction the Earth rotates, so when 12 hours has passed, the Moon has moved ahead 12/27.3 = 0.44 hours = 26.4 minutes, so one has to wait another 26 minutes for the next lunar high tide, so the time between high tides is about 12 hours 26 minutes. This causes the time of high tide to move relative to a wall clock in any location, which is why we have tide tables and tide clocks.

Roughly the solar tides have half the influence of the lunar ones as indicated in (8.8.42). They add and cancel depending on the position of the Sun and Moon. This nice picture of D.J. Jeffery shows the extremal situations (the word spring does not mean the season Spring)



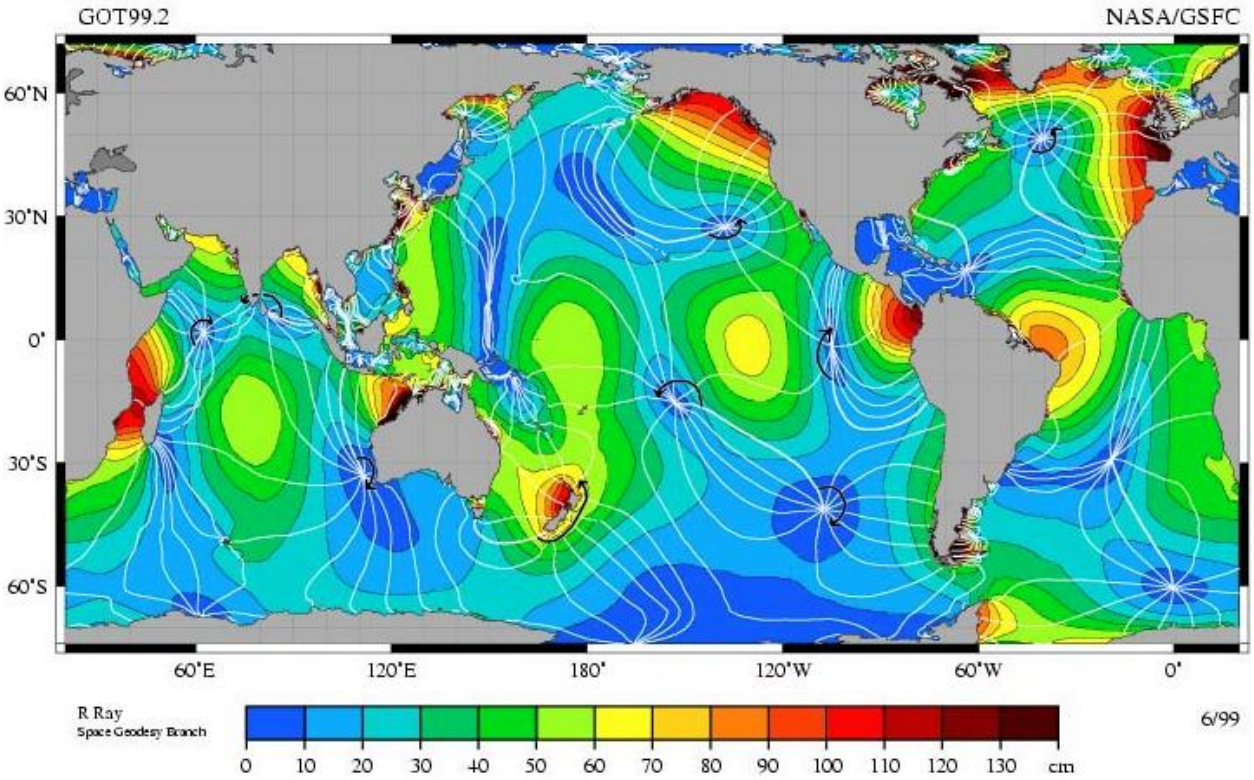
(8.8.61)

So the maximal spring tides are about 2 weeks apart and the same is true for the intervening minimal neap tides.

A good discussion of the above tidal model is given in Taylor's textbook p 330-336. A more detailed discussion is presented in the excellent (and downloadable) paper by Butikov. Both sources are very readable.

Tides on the real Earth

We leave the reader with this perhaps disappointing picture of *actual* tides on the Earth.



http://www.nauticed.org/sailing-blog/how-the-tides-work/ (8.8.62)

(We are unable to locate the original source of this graphic.) Presumably these are measured long-term averages of tidal high-low differences and one sees how they generally range from 0 cm to 140 cm. A wave resonance at the Bay of Fundy can cause a 1700 cm (50 foot) high/low tide difference there.

Recall our toy model calculation that

$$H_{\texttt{lunar_tide}} = 53.49 \text{ cm} \sim 1.8 \text{ feet} \ .$$
$$H_{\texttt{solar_tide}} = 24.58 \text{ cm} \sim 0.8 \text{ feet} \ . \qquad (8.8.42)$$

At least the values in the map are in the same ballpark as the toy model! The sailing author at the above link notes that tides are almost non-existent in the Caribbean and Mediterranean seas, which are dark blue in the figure.

As Butikov points out, the simple model in which the Earth rotates under a static ellipsoid of water in a frictionless manner is very far from reality. One must consider the dynamical aspects of the problem which involve the massive water currents which attempt to maintain the tidal ellipsoid and the interaction of such currents with land masses and the ocean bottom and with themselves (water has some viscosity).

The currents of course are subject to Coriolis forces. Water flow velocity is not infinite and is affected by ocean depth. One result is that there is a delay between the moon being at local meridian (transit) and the occurrence of high tide (the "local lunitidal interval"). This delay is extremely variable, ranging from a few minutes to ~20 hours. The white lines in (8.8.2) are loci on which a high tide occurs at the same time (equal tidal phase) and the white line nodes have no tides at all (known as amphidromes) for a particular tidal "component" like the main one called M_2. Tides flow around these points. The problem is one of wave dynamics and forced oscillation on a rotating object and is well beyond the scope of our document (but is treated by Butikov and in his references).

Footnote: Equation of the Water Surface by the Potential Method

There are many ways to set up the potential method for determining the idealized tidal shape of Fig (8.8.34), see Taylor and Butikov for alternatives to our method. In the following presentation we omit some detailed steps which involve small-ε approximations.

We seek a potential V(x,y) which solves this equation (no minus sign in $\nabla V = \mathbf{F}$)

$$\nabla V(x,y) = \mathbf{F}_{g2} + \mathbf{F}_{tid} = -(GM_2 m/r^2)\,\hat{\mathbf{r}} + (mM_1 G/d_0^3)(2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}) \qquad (8.8.63)$$

where we use (8.8.24) for $\mathbf{F}_{tid}$. The exact solution for V is the following (by inspection) :

$$V(x,y) = (GM_2 m/r) + (GM_1 m/d_0^3)(x^2 - y^2/2)\,. \qquad //\ \nabla(1/r) = -(1/r^2)\hat{\mathbf{r}},\quad r > 0 \qquad (8.8.64)$$

Of interest are surfaces on which V is a constant (since we expect the water surface to be such a surface) so we let k be a constant and write

$$(M_2/r) + (M_1/d_0^3)(x^2 - y^2/2) = k \qquad \dim(k) = M/L \qquad (8.8.65)$$

which is the same as

$$(M_2/k)^2 = (x^2+y^2)(1 - 2\varepsilon x^2 + \varepsilon y^2) = r^2(1 - 2\varepsilon r^2\cos^2\theta + \varepsilon r^2\sin^2\theta)$$

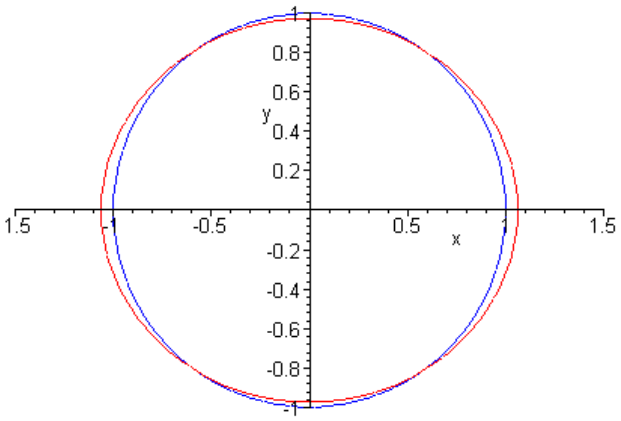
$$\varepsilon \equiv (M_1/kd_0^3)\,. \qquad \dim(\varepsilon) = L^{-2} \qquad (8.8.66)$$

This is a quartic equation whose shape is very similar to the "ellipse" of (8.8.33). For example, for $M_2/k = 1$ and $\varepsilon = .05$ we may compare the quartic (red) to the unit circle (blue) :

```
f := (x^2+y^2)*(1-2*e*x^2+e*y^2);
```

$$f := (x^2+y^2)(1-2ex^2+ey^2)$$

```
e := 0.05:
s1 := implicitplot(f=1,x=-1.5..1.5,y=-2..2, scaling = constrained,
view=[-1.5..1.5,-1..1],color=red):
s2 := implicitplot(x^2+y^2=1,x=-1.5..1.5,y=-2..2,color=blue):
display(s1,s2);
```



(8.8.67)

To zeroth order in ε we can identify $(M_2/k) \approx R_2$, the radius of the Earth. We can use this within quantities already of order ε, but more accuracy is needed for the standalone (M_2/k). Equation (8.8.66) is a quadratic in r^2 and we solve it for $r^2(\theta)$ and then $r(\theta)$ to get, always for small ε,

$$r(\theta) \approx (M_2/k)[\, 1 - (1/8)\varepsilon(\sin^2\theta - 2\cos^2\theta)(M_2/k)^2] \ . \tag{8.8.68}$$

We then demand that $\langle r(\theta)\rangle = R_2$ (averaged over θ), just as was done in (8.8.33), so

$$R_2 = \langle r(\theta)\rangle = (M_2/k)[\, 1 - (1/8)\varepsilon(\tfrac{1}{2} - 2\,\tfrac{1}{2})(M_2/k)^2] = (M_2/k)[\, 1 + (1/16)(M_2/k)^2\varepsilon\,] \ . \tag{8.8.69}$$

This is then solved for (M_2/k) to order ε with the result

$$(M_2/k) \approx R_2[1 - (1/16)\varepsilon R_2^2] \tag{8.8.70}$$

which shows the first order correction from just using R_2. The locus of the quartic (8.8.66) is now,

$$R_2^2[1-(1/16)\varepsilon R_2^2]^2 = (x^2+y^2)\,[1 - \varepsilon(2x^2 - y^2)]$$

$$\varepsilon \approx (M_1/M_2)R_2/d_0^3 . \tag{8.8.71}$$

We then examine the quartic at its right edge (high tide) and its top edge (low tide) to find

$$|\Delta|_{high} = (15/16)(M_1/M_2)R_2^4/d_0^3$$
$$|\Delta|_{low} = (9/16)(M_1/M_2)R_2^4/d_0^3$$

$$H = |\Delta|_{high} + |\Delta|_{low} = (M_1/M_2)R_2^4/d_0^3\ \{\,(15/16) + (9/16)\,\} \tag{8.8.72}$$

$$= (3/2)(M_1/M_2)R_2^4/d_0^3 \ .$$

But H = 2a in terms of (8.8.33) so we get

$$a = (3/4)(M_1/M_2)R_2^4/d_0^3 \qquad (8.8.73)$$

in agreement with (8.8.41).

Comment: Requiring $\langle r(\theta)\rangle = R_2$ (averaged over θ) both here and implicitly in (8.8.33) is a reasonable approximation, but in the true 3D problem one should really average over θ,φ so that the water volume displaced by low tides equals that piled up by high tides.

9. Notation comparison with Marion (1970) and Thornton & Marion (2003)

In this Section we compare our notation for rotating-frame kinematics and non-inertial-frame physics to that of Marion (1970) and Thornton and Marion (T&M 2003).

Their notation is close to our "swap" notation, so before making any comparisons, we restate various of our equations in swap notation. A swap notation equation has an s subscript on the equation number and is obtained from the corresponding non-swap equation by prime ↔ noprime (**b** and **ω** do not change) :

$$\mathbf{r}' = \mathbf{b} + \mathbf{r} \qquad (6.1)_s$$

$$\mathbf{v}' = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \qquad (6.6a)_s$$

$$\mathbf{F}' = m\mathbf{a}' = m\ddot{\mathbf{b}}_{S'} + m\mathbf{a} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad (8.1.3)_s$$

$$m\mathbf{a} = \mathbf{F}_{eff} = \mathbf{F}' - m\ddot{\mathbf{b}}_{S'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad (8.1.5)_s$$

In the above swap notation equations, Frame S' is fixed (f) and Frame S is the rotating frame (r) :

We now compare these equations with those of the Marion series authors:

$$\mathbf{r}' = \mathbf{b} + \mathbf{r} \qquad (6.1)_s$$
$$\mathbf{r}' = \mathbf{R} + \mathbf{r} \qquad \text{// Marion p 341 (11.1)}$$
$$\text{// T\&M p 388 (10.1)}$$

$$\mathbf{v}' = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \qquad (6.6a)_s$$
$$\mathbf{v}_f = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{V} \qquad \text{// Marion p 344 (11.12)}$$
$$\text{// T\&M p 391 (10.17)}$$

$$\mathbf{F}' = m\mathbf{a}' = m\ddot{\mathbf{b}}_{S'} + m\mathbf{a} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad (8.1.3)_s$$
$$\mathbf{F} = m\mathbf{a}_f = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{// Marion p 344 (11.17)}$$
$$\text{// T\&M p 392 (10.23)}$$

$$m\mathbf{a} = \mathbf{F}_{eff} = \mathbf{F}' - m\ddot{\mathbf{b}}_{S'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad (8.1.5)_s$$
$$m\mathbf{a}_r = \mathbf{F}_{eff} = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{// Marion p 344 (11.19)}$$
$$\text{// T\&M p 392 (10.25)}$$

Based on these comparisons, we make the following translation table:

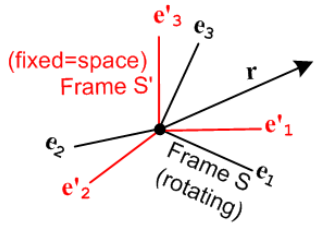
(9.1)

our swap notation	Marion authors	
S	r	name of the rotating frame (r = rotating)
S'	f	name of the fixed frame (f = fixed)
$\partial_{\mathtt{S}}$	$(d/dt)_{\mathtt{rotating}}$	time derivative in the rotating frame
$\partial_{\mathtt{S'}}$	$(d/dt)_{\mathtt{fixed}}$	time derivative in the fixed frame
$\mathbf{r}$	$\mathbf{r}$	position in rotating frame
$\mathbf{v}$	$\mathbf{v}_{\mathtt{r}}$	velocity in rotating frame
$\mathbf{a}$	$\mathbf{a}_{\mathtt{r}}$	acceleration in rotating frame
$\mathbf{r'}$	$\mathbf{r'}$	position in fixed frame
$\mathbf{v'}$	$\mathbf{v}_{\mathtt{f}}$	velocity in fixed frame
$\mathbf{a'}$	$\mathbf{a}_{\mathtt{f}}$	acceleration in fixed frame
$\mathbf{F'}$	$\mathbf{F}$	force in fixed frame = true force in rotating frame
$\mathbf{F}_{\mathtt{eff}}$	$\mathbf{F}_{\mathtt{eff}}$	total effective force in the rotating frame
$\mathbf{F}_{\mathtt{fict}}$		total fictitious force in the rotating frame
$\mathbf{b}$	$\mathbf{R}$	location of the rotating frame origin (measured in the fixed frame)
$\dot{\mathbf{b}}_{\mathtt{S'}}$	$\dot{\mathbf{R}}_{\mathtt{f}}$,$\mathbf{V}$	velocity of the rotating frame origin (measured in the fixed frame)
$\ddot{\mathbf{b}}_{\mathtt{S'}}$	$\ddot{\mathbf{R}}_{\mathtt{f}}$	acceleration of the rotating frame origin (measured in the fixed frame)

10. Notation comparison with Goldstein (1950) and Goldstein, Poole and Safko (2001)

In this Section we compare our notation for rotating-frame kinematics and non-inertial-frame physics to that of Goldstein on (1950) and Goldstein, Poole and Safko (GFS 2001).

These texts are also close to our "swap" notation. However, in neither text is the reader informed of the location of the origin of the inertial frame or of the rotating frame. In Goldstein we are told that **r** is a "vector from the origin of the terrestrial system" and that "terrestrial measurements are usually made with respect to a coordinate system fixed *in* the earth, which therefore rotates with a constant angular velocity ω relative to *the inertial system*". Having studied their rotational equations, it is our conclusion that the two frames of reference must have their origins co-sited at the center of the Earth,



(10.1)

One would likely align $\mathbf{e_3}$ with $\mathbf{e'_3}$ and call it $\hat{\mathbf{z}}$. Most of the follow-on discussion is about the Coriolis force which involves velocity and not **r**, so the location of the origin is not important.

In terms of our Fig 1, this means that **b** = 0 and so **r'** = **r** and we are simultaneously Special Case #1 and Special Case #2 since the rotation axis passes through both origins. For this situation, in our velocity table (1.8.4) the left and right sides are exactly the same. Similarly acceleration table (1.8.6) is identical to table (1.8.5). For example, with **b** = 0 we have,

$$\mathbf{v}_{S'} = (\partial \mathbf{r}/\partial t)_{S'} = (\partial \mathbf{r'}/\partial t)_{S'} = v'_{S'} = \mathbf{v'}$$

$$\mathbf{a}_{S'} = (\partial^2 \mathbf{r}/\partial t^2)_{S'} = (\partial^2 \mathbf{r'}/\partial t^2)_{S'} = a'_{S'} = \mathbf{a'} \,. \qquad (10.2)$$

Of course $\ddot{\mathbf{b}}_{S'} = 0$ when **b** = 0. We now gather up some of our equations in swap notation and state them for the case that **b** = 0, making use of (10.2) above:

$$\mathbf{r'} = \mathbf{r} \qquad // \ \mathbf{b} = 0 \qquad (6.1)_s$$

$$\mathbf{v'} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} \qquad (6.5)_s$$

$$\mathbf{a'} = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (7.4)_s$$

$$m\mathbf{a} = \mathbf{F_{eff}} = \mathbf{F'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \,. \qquad (8.1.5)_s$$

In the above swap notation equations, Frame S' is fixed (space) and Frame S is the rotating frame (r) :

We start our comparison with equation $(6.5)_s$:

$$\mathbf{v'} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} \qquad (6.5)_s$$

$$\mathbf{v_s} = \mathbf{v_r} + \boldsymbol{\omega} \times \mathbf{r} \,. \qquad // \text{ Goldstein p 135 (4-104)}$$

// GPS p 175 (4.88)

The next comparison is (we add $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ to their equations),

$$\mathbf{a}' = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad (7.4)_{\mathbf{s}}$$

$$\mathbf{a_s} = \mathbf{a_r} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v_r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad .$$

// Goldstein p 135 (4-105)
// GPS p 175 (4.89)

And finally,

$$m\mathbf{a} = \mathbf{F_{eff}} = \mathbf{F}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad (8.1.5)_{\mathbf{s}}$$

$$m\mathbf{a_r} = \mathbf{F_{eff}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v_r} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad .$$

// Goldstein p 135 (4-106,7)
// GPS p 175 (4.90,1)

Based on these comparisons, we construct the following translation table:

our swap notation	Goldstein authors	$\mathbf{b} = 0$ so $\mathbf{r} = \mathbf{r}'$	(10.3)
S	r	name of the rotating frame (r = rotating)	
S'	s	name of the fixed frame (s = space)	
$\partial_{\mathbf{S}}$	$(d/dt)_{\mathbf{r}}$	time derivative in the rotating frame	
$\partial_{\mathbf{S}'}$	$(d/dt)_{\mathbf{s}}$	time derivative in the fixed frame	
$\mathbf{r}$	$\mathbf{r}$	position in rotating frame	
$\mathbf{v}$	$\mathbf{v_r}$	velocity in rotating frame	
$\mathbf{a}$	$\mathbf{a_r}$	acceleration in rotating frame	
$\mathbf{r}'$	$\mathbf{r}$	position in fixed frame	
$\mathbf{v}'$	$\mathbf{v_s}$	velocity in fixed frame (s = space)	
$\mathbf{a}'$	$\mathbf{a_s}$	acceleration in fixed frame	
$\mathbf{F}'$	$\mathbf{F}$	force in fixed frame = true force in rotating frame	
$\mathbf{F_{eff}}$	$\mathbf{F_{eff}}$	total effective force in the rotating frame	
$\mathbf{F_{fict}}$		total fictitious force in the rotating frame	
$\mathbf{b}$	0	location of the rotating frame origin (measured in the fixed frame)	
$\dot{\mathbf{b}}_{\mathbf{S}'}$	0	velocity of the rotating frame origin (measured in the fixed frame)	
$\ddot{\mathbf{b}}_{\mathbf{S}'}$	0	acceleration of the rotating frame origin (measured in the fixed frame)	

Since $\mathbf{b} = 0$ is assumed, the Goldstein and GPS texts only treat a special case of the "rotating frames of reference" scenario we depict in Fig 1 in which $\mathbf{b}(t)$ is a general dynamic vector.

11. Angular Momentum and Fictitious Torques; the Reynolds Transport Theorem

11.1 Introduction

Nomenclature is an issue for this subject, and here is a vague partial table of usage:

	"physics"	continuum mechanics	
r x **p**	angular momentum **L**	moment of momentum (moment of **p**)	
r x **F**	torque **N** or $\boldsymbol{\tau}$	moment of force (moment of **F**), moment (**M** or **m**)	
$r_\perp = r\sin\theta$	moment arm	moment arm	
$\Sigma_i \mathbf{r}_i \times \mathbf{F}_i$ with $\Sigma_i \mathbf{F}_i = 0$	a sum of torques	a couple, a torque, a pure moment	(11.1.1)

The moment arm $r_\perp$ is the component of **r** which is perpendicular to **F** (or **p**) as in the drawings below.

We shall use the "physics" terminology, so :

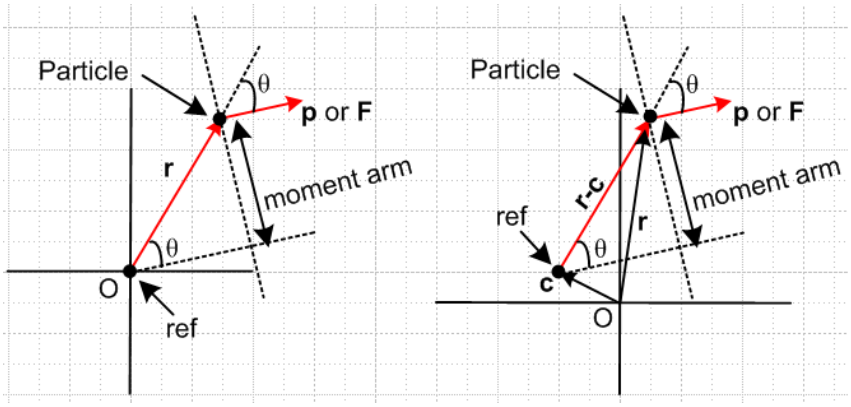
A Particle located at position **r** and having linear momentum **p** is said to have angular momentum **L** = **r** x **p** about the origin, in reference to the origin, or with respect to the origin. (11.1.2)

A Particle located at position **r** which is acted upon by a force **F** is said to experience a torque **N** = **r** x **F** about the origin, in reference to the origin, or with respect to the origin. (11.1.3)

In other words, the tail of vector **r** is at the origin of the coordinate system and **L** and **N** as above are both shown with respect to that point, as in the picture on the left below.

Note that a Particle could refer to an actual point particle, or to a small piece of a rigid body, or to a small chunk of a fluid or an elastic solid.

Sometimes we want to use **L** and **N** with respect to some *other* reference point, call it **c,** which is not the origin of the coordinate system. This is shown on the right where the origin of the picture on the left has been translated slightly down and to the right,



(11.1.4)

The angular momentum and torque with respect to point **c** are given by

$$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{p} \qquad \mathbf{L}^{(0)} = \mathbf{r} \times \mathbf{p}$$
$$\mathbf{N}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{F} \qquad \mathbf{N}^{(0)} = \mathbf{r} \times \mathbf{F} \quad . \qquad (11.1.5)$$

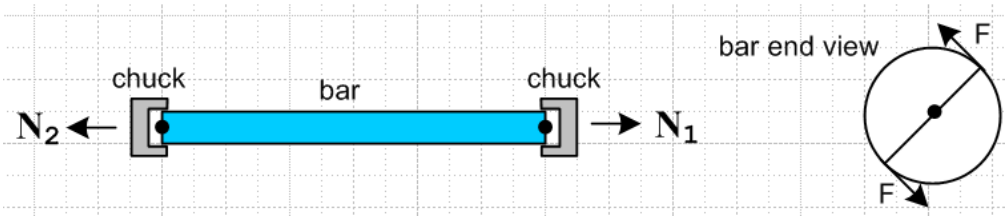
We therefore introduce a label "**c**" to indicate the point of reference for an **L** or an **N**. If the reference point is the origin, then we write things as shown on the right above.

In the work below, a certain amount of complexity is introduced by allowing $\mathbf{c} \neq 0$, but the intention is to "do the general case". This then brings up the significance of the last line in (11.1.1) above, and suggests the benefits of dealing with "a couple" when possible. Consider:

Theorem: If the sum of a set of forces acting on an object is 0, then the sum of the torques associated with those forces (acting on that same object) is independent of the point **c** chosen as the reference point for all the torques. (11.1.6)

proof: $\mathbf{N}^{(c)} = \Sigma_i(\mathbf{r}_i - \mathbf{c}) \times \mathbf{F}_i = \Sigma_i \mathbf{r}_i \times \mathbf{F}_i - \mathbf{c} \times (\Sigma_i \mathbf{F}_i) = \Sigma_i \mathbf{r}_i \times \mathbf{F}_i - 0 = \mathbf{N}^{(0)}$.

Example: Imagine a cylindrical steel bar in a state of torsional strain due to equal and opposite torques applied to the ends of the bar through small grabber chucks, one at each end (no gravity). So $\mathbf{N}_1 = -\mathbf{N}_2 = \mathbf{N}$ and each torque is twisting the bar counterclockwise as seen looking at each end. The effect of a chuck on the bar can be represented as a continuous sum of tangential forces acting on the thin band of surface of the bar under the chuck, which forces add up to zero (think pairwise). Therefore, since the total force of a chuck on the bar is zero, the torque of a chuck on the bar is independent of reference point. Since the same is true for each end, one can pick some arbitrary point **c** (such as $\mathbf{c} = 0$) and reference both torques to that point, and then add them "legally" to conclude that the total torque on the bar is 0. The bar thus shows no angular acceleration.

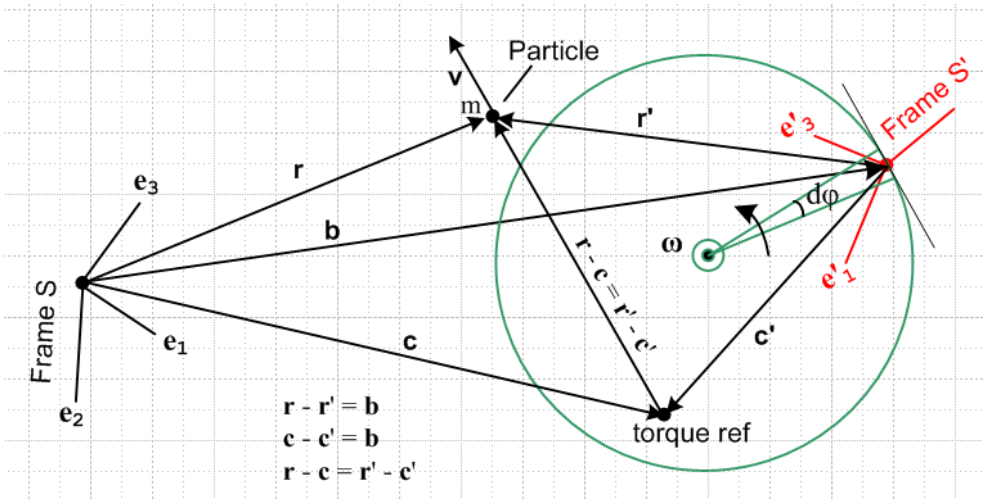


(11.1.7)

In this example, each chuck represents a couple or pure moment acting on the bar.

11.2 Expression of $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ in terms of Frame S' objects

We replicate Fig (1.9.1) which now includes a single torque reference point which is **c** is Frame S and **c'** in Frame S'.



(11.2.1)

The following three equations are obvious from the drawing,

$$\mathbf{r} - \mathbf{r}' = \mathbf{b} \qquad (11.2.2)$$

$$\mathbf{c} - \mathbf{c}' = \mathbf{b} \qquad (11.2.3)$$

$$\mathbf{r} - \mathbf{c} = \mathbf{r}' - \mathbf{c}' \; . \qquad (11.2.4)$$

For arbitrary reference point **c** one has from (1.9.4-7),

$$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times m\mathbf{v} \qquad \textit{(1.9.4)} \qquad (11.2.5)$$

$$\dot{\mathbf{L}}^{(c)} = (\mathbf{r}-\mathbf{c}) \times m\mathbf{a} - \dot{\mathbf{c}} \times m\mathbf{v} \qquad \textit{(1.9.5)} \qquad (11.2.6)$$

$$\mathbf{L}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}' \qquad \textit{(1.9.6)} \qquad (11.2.7)$$

$$\dot{\mathbf{L}}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{a}' - \dot{\mathbf{c}}' \times m\mathbf{v}' \quad . \qquad \textit{(1.9.7)} \qquad (11.2.8)$$

The reference point **c** might be moving. We know from Sections 6 and 7 how **r**,**v**,**a** and **r'**,**v'**,**a'** are related,

$$\mathbf{r} = \mathbf{r}' + \mathbf{b} \qquad \textit{(6.1)} \qquad (11.2.9)$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad \text{or} \qquad \mathbf{p} = \mathbf{p}' + m(\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) \qquad \textit{(6.6a)} \qquad (11.2.10)$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \quad . \qquad \textit{(7.6a)} \qquad (11.2.11)$$

The middle equation above can be written,

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \,. \qquad (11.2.12)$$

The torque reference point **c** in Fig (11.2.1) is just like the particle point **r**, so the above for **c** becomes

$$\dot{\mathbf{c}} = \dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S \,. \qquad (11.2.13)$$

Part of the goal of Section 5 is to express $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ in terms of Frame S' quantities. In the following all algebra is shown to provide an easily traceable path since there won't be any result verifications:

$$\begin{aligned}
\mathbf{L}^{(c)} &= (\mathbf{r}-\mathbf{c}) \times \mathbf{p} = (\mathbf{r}'-\mathbf{c}') \times [\, \mathbf{p}' + m(\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S)] && \text{// (11.2.4), (11.2.10)} \\
&= (\mathbf{r}'-\mathbf{c}') \times \mathbf{p}' + m(\mathbf{r}'-\mathbf{c}') \times (\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) \\
&= \mathbf{L}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [\, (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S] \,. && \text{// (11.2.7)}
\end{aligned}$$

Now take the ∂_S time derivative of $\mathbf{L}^{(c)}$ to get,

$$\begin{aligned}
\dot{\mathbf{L}}^{(c)} &= m(\mathbf{r}-\mathbf{c}) \times \mathbf{a} - m\dot{\mathbf{c}} \times \mathbf{v} && \text{// (11.2.6)} \\
&= m(\mathbf{r}'-\mathbf{c}') [\, \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] - m\dot{\mathbf{c}} \times \mathbf{v} && \text{// (11.2.4), (11.2.11)} \\
&= [m(\mathbf{r}'-\mathbf{c}') \times \mathbf{a}' - m\dot{\mathbf{c}}' \times \mathbf{v}'] + m(\mathbf{r}'-\mathbf{c}') \times [\, \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] - m\dot{\mathbf{c}} \times \mathbf{v} + m\dot{\mathbf{c}}' \times \mathbf{v}' \\
&= \dot{\mathbf{L}}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [\, \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] && \text{// (11.2.8)} \\
&\quad - m\dot{\mathbf{c}} \times \mathbf{v} + m\dot{\mathbf{c}}' \times \mathbf{v}' \\
&= \dot{\mathbf{L}}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [\, \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \\
&\quad - m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\, \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + m\,\dot{\mathbf{c}}' \times \mathbf{v}' \,. && \text{// (11.2.13), (11.2.10)}
\end{aligned}$$

These results may now be summarized:

$$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [\, (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S] \qquad (11.2.14)$$

$$\begin{aligned}
\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} &+ m(\mathbf{r}'-\mathbf{c}') \times [\, \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \\
&- m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\, \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + m\,\dot{\mathbf{c}}' \times \mathbf{v}' \qquad (11.2.15)
\end{aligned}$$

These equations express $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ entirely in terms of Frame S' objects, for the general case where the angular momentum reference point **c** is arbitrarily selected. The second equation can be rewritten

$$\dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{F}_{fict} + m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v}' \qquad (11.2.16)$$

$$\mathbf{F}'_{fict} = -m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \quad . \qquad (8.1.8)$$

Comment: Just as with the **v** and **a** equations of Section 6, the above equations involving angular momentum and its time derivative are valid even if both Frame S and Frame S' are non-inertial. This is so because the derivations above are based on the G Rule and those **v** and **a** equations, both of which do not require that either Frame be inertial.

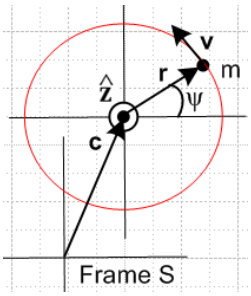
11.3 Fictitious Torques and Newton's Rotational Law in a non-inertial frame

We now assume that Frame S is an inertial frame and we compare Newton's (2nd) Law for linear motion with that for *circular* motion, both in Frame S,

$$\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a} = m\ddot{\mathbf{r}} \qquad // \dot{\mathbf{p}} \text{ and } \ddot{\mathbf{r}} \text{ "natural" in Frame S} \qquad (11.3.1)$$

$$\mathbf{N}^{(c)} = \dot{\mathbf{L}}^{(c)} = I^{(c)}\,\boldsymbol{\alpha} = I^{(c)}\,\ddot{\psi}\hat{\mathbf{z}} = (mr^2)\ddot{\psi}\hat{\mathbf{z}} \quad . \qquad // \dot{\mathbf{L}}^{(c)} \text{ and } \ddot{\psi} \text{ "natural" in Frame S} \qquad (11.3.2)$$

Here is a drawing showing the various parameters ($I^{(c)}$ is the moment of inertia about point **c**).



(11.3.3)

Although the general motion of the Particle in Fig (11.2.1) could be construed as an instantaneous circular motion about some obscure rotation axis, the above equations are not really helpful in the general case, and we work instead with torque in terms of angular momentum.

$\mathbf{L}^{(c)}$ is the angular momentum of our Particle in Frame S relative to reference point **c**, while $\mathbf{N}^{(c)}$ is some externally applied torque about that same reference point acting on the Particle.

In Frame S' we want to find some *effective* Newton's Rotational Law,

$$\mathbf{N}'^{(c')}_{eff} = \dot{\mathbf{L}}'^{(c')} \qquad (11.3.4)$$

$$\mathbf{N}'^{(c')}_{eff} = \mathbf{N}^{(c)} + \mathbf{N}'^{(c')}_{fict} \qquad // \text{ defines } \mathbf{N}'^{(c')}_{fict} \qquad (11.3.5)$$

where $\mathbf{N}'^{(c')}_{fict}$ is the fictitious torque that mysteriously appears in non-inertial Frame S', so then

$$\mathbf{N}^{(c)} + \mathbf{N'}^{(c')}_{fict} = \dot{\mathbf{L}}'^{(c')} \ . \qquad (11.3.6)$$

We can then apply this bogus Newton's Rotational Law in non-inertial Frame S'. This is in complete analogy with the use of fictitious *forces* as reviewed in Section 8.

Solve (11.3.6) for $\mathbf{N'}^{(c')}_{fict}$ and then replace $\mathbf{N}^{(c)}$ by $\dot{\mathbf{L}}^{(c)}$ in (11.3.2) to get

$$\mathbf{N'}^{(c')}_{fict} = \dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)} \ . \qquad (11.3.7)$$

Replacing the right side of the above using (11.2.16) gives

$$\mathbf{N'}^{(c')}_{fict} = (\mathbf{r'-c'}) \times \mathbf{F'}_{fict} + m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v}' \qquad (11.3.8)$$

where

$$\mathbf{F'}_{fict} = \underbrace{- m\ddot{\mathbf{b}}_S}_{\text{frame}} \underbrace{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')}_{\text{centrifugal}} \underbrace{- 2m\, \boldsymbol{\omega} \times \mathbf{v}'}_{\text{Coriolis}} \underbrace{- m\dot{\boldsymbol{\omega}} \times \mathbf{r}'}_{\text{Euler}} \ . \qquad \textit{(8.1.8)} \qquad (11.3.9)$$

Writing this out in full,

$$\mathbf{N'}^{(c')}_{fict} = - (\mathbf{r'-c'}) \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\, \boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}'] + m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v}' \ . \qquad (11.3.10)$$

In the special case that $\mathbf{b} = 0$ and $\mathbf{c} = 0$ we know from (11.2.3) that $\mathbf{c}' = 0$ as well. In this case, (11.3.8) simplifies to become

$$\mathbf{N'}^{(c')}_{fict} = \mathbf{r}' \times \mathbf{F'}_{fict} \ . \qquad (11.3.11)$$

11.4 Application: Fictitious Torques in Fluid Dynamics

The object of interest is a blob of fluid contained in a moving volume V_m. The boundary S_m of this volume V_m moves and changes shape such that every point on S_m moves at a velocity which matches the local fluid flow velocity $\mathbf{v}(\mathbf{r},t)$. As a result, no particles of fluid either enter or leave the blob volume V_m as it moves. One implication is that the mass M of the blob V_m remains constant. This blob, of some fixed mass M, is our "object of mechanical interest" (later called "the system").

At time t, we imagine that V_m sheds a snake skin V_c which then remains frozen in time. Then $V_m(t)$ aligns with V_c at time t and probably at no other time. Whereas V_m moves, V_c is fixed. V_m is called a "material volume" since it flows with the material, while V_c is called a "control volume". In general, all volume integrals except those being differentiated in time are expressed as integrals over Vc.

Here is Newton's Rotational Law $\mathbf{N} = \dot{\mathbf{L}}$ (11.3.2) for the this blob of mass M in inertial Frame S,

$$\mathbf{N}^{(0)} = \int_{S_c} \mathbf{r} \times \mathbf{t}\ dS + \int_{V_c} \mathbf{r} \times \rho\mathbf{B}\ dV = (d/dt)_S \left[\int_{V_m} \mathbf{r} \times \rho\mathbf{v}\ dV \right] . \quad \text{// Lai (7.9.1)} \qquad (11.4.1)$$

This equation appears in Lai et al. p 430 as equation (7.9.1). Here **t** is a possible external "surface traction" (force per area) acting on the blob's surface S_c, **B** is some possible "body force" per unit mass acting on the blob's interior (perhaps gravity **g**), and ρ is the mass density of the fluid. Notice that $\int_{V_m} \mathbf{r} \times \rho\mathbf{v}\ dV$ is an integral of $d\mathbf{L} = \mathbf{r} \times d\mathbf{p}$ over the blob, where $d\mathbf{p} = dm\ \mathbf{v}$ and $dm = \rho dV$, so this integral $\mathbf{L}^{(0)}_{blob}$ is referenced to the point $\mathbf{c} = 0$.

Frame S' is some rotating frame whose origin is aligned with that of Frame S, so $\mathbf{b} = 0$, and the angular momentum reference point in Frame S' is chosen as $\mathbf{c}' = 0$ (recall (11.2.3) $\mathbf{c} - \mathbf{c}' = \mathbf{b}$).

The integrals for $\mathbf{N}^{(0)}$ can be evaluated in any frame one likes, and evaluating them in rotating Frame S' gives

$$\mathbf{N}^{(0)} = \int_{S_c'} \mathbf{r}' \times \mathbf{t}\ dS' + \int_{V_c'} \mathbf{r}' \times \rho\mathbf{B}\ dV' . \qquad (11.4.2)$$

Here the integration point $\mathbf{r}'$ runs over surface S_c' in the first integral, and over volume V_c' in the second. The body force **B** and the surface traction **t** are unchanged but are now expressed in terms of $\mathbf{r}'$.

What does equation (11.4.1) look like in non-inertial Frame S' ? According to (11.3.4) and (11.3.5) it is this:

$$\mathbf{N}^{(0)} + \mathbf{N}'^{(0)}_{fict} = \dot{\mathbf{L}}'^{(0)} = (d/dt)_{S'} \mathbf{L}'^{(0)} . \qquad \textit{(11.3.4),(11.3.5)}$$

Using (11.4.2) this may be written as,

$$\left(\int_{S_c'} \mathbf{r}' \times \mathbf{t}\ dS' + \int_{V_c'} \mathbf{r}' \times \rho\mathbf{B}\ dV' \right) + \mathbf{N}'^{(0)}_{fict} = (d/dt)_{S'} \left[\int_{V_m'} \mathbf{r}' \times \rho\mathbf{v}'\ dV' \right] . \qquad (11.4.3)$$

Here everything is computed in rotating Frame S', but **t** and **B** are the same as in Frame S since they are not affected by the fact that the frame S' is rotating. Also, $\rho' = \rho$ since this is mass per volume and the differential volume element is not affected by a rotation. The control volume V_c' and its boundary S_c' are fixed in Frame S'. On the right, $V_{m'}$ aligns with $V_{c'}$ at time instant t, and

$$\int_{V_{m'}} \mathbf{r}' \times \rho'\mathbf{v}'\ dV' = \mathbf{L}'^{(0)}_{blob} . \qquad (11.4.4)$$

The question remains: what is the fictitious torque $\mathbf{N}'^{(0)}_{fict}$ appearing in (11.4.3)? For a particle of mass dm' at location $\mathbf{r}'$ in the blob the contribution is, from (11.3.11), (terms reordered on second line)

$$d\mathbf{N}'^{(0)}_{fict} = \mathbf{r}' \times d\mathbf{F}'_{fict} = \mathbf{r}' \times [-\ddot{\mathbf{b}}_S - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2\,\boldsymbol{\omega} \times \mathbf{v}' - \dot{\boldsymbol{\omega}} \times \mathbf{r}']\ dm'$$

$$= \mathbf{r}' \times [-\ddot{\mathbf{b}}_S - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2\,\boldsymbol{\omega} \times \mathbf{v}']\ dm' . \qquad (11.4.5)$$

When this is integrated over the blob, one finds

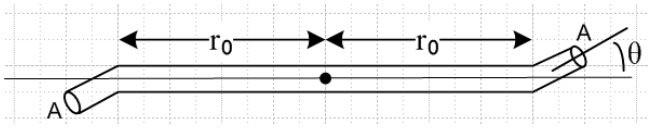
$$\mathbf{N'}^{(0)}_{fict} = \int_{V_c'} \mathbf{r'} \times [-\ddot{\mathbf{b}}_S - \dot{\boldsymbol{\omega}} \times \mathbf{r'} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) - 2\,\boldsymbol{\omega} \times \mathbf{v'}]\, dm' \qquad \text{// Lai (7.9.9)} \qquad (11.4.6)$$

$$= -(\int_{V_c'} \mathbf{r'}\, dm') \times \ddot{\mathbf{b}}_S - \int_{V_c'} \mathbf{r'} \times (\dot{\boldsymbol{\omega}} \times \mathbf{r'}]\, dm' - \int_{V_c'} \mathbf{r'} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'})]dm' - 2 \int_{V_c'} \mathbf{r'} \times (\boldsymbol{\omega} \times \mathbf{v'})dm' .$$

frame Euler centrifugal Coriolis

This expression appears in Lai et al. as p 431 (7.9.9).

Equation (11.4.3) with (11.4.6) is applied on Lai et al. pp 431-432 to a conventional rotating sprinkler. The sprinkler is at translational rest and its horizontal watering tube of length $2r_0$ rotates at ω. The control volume $V_{c'}$ is the interior of this rotating watering tube (angled ends are very short), so $V_{c'}$ is fixed in rotating Frame S'. The solution to the problem is shown to be $\omega = -(Q/A)\sin\theta/r_0$ where A is the area of the orifice on each end of the watering tube, Q is the volume of water flow delivered to the sprinkler per unit time, and θ is the angle of each tube-end opening viewed from above.



(11.4.7)

11.5 Application: Fictitious Forces in Fluid Dynamics

A similar equation applies for the fictitious *forces* (rather than torques) in this same fluid dynamics example. Perhaps this should have appeared in Section 8, but the groundwork has been laid here. The corresponding equations are

$$\mathbf{F} = d\mathbf{p}/dt$$

or

$$\mathbf{F} = (\int_{S_c} \mathbf{t}\, dS + \int_{V_c} \rho \mathbf{B}\, dV) = (d/dt)_S \left[\int_{V_m} \mathbf{r} \times \rho \mathbf{v}\, dV \right] \qquad (11.5.1)$$

and

$$\mathbf{F} + \mathbf{F'}_{fict} = (d/dt)_{S'} [\, \mathbf{p'}]$$

or

$$(\int_{S_c} \mathbf{t}\, dS + \int_{V_c} \rho \mathbf{B}\, dV) + \mathbf{F'}_{fict} = (d/dt)_{S'} \left[\int_{V_{m'}} \rho \mathbf{v'}\, dV' \right] . \qquad (11.5.2)$$

Again using (11.3.9) integrated over the blob with $dm' = \rho dV'$,

$$\mathbf{F'}_{fict} = -\ddot{\mathbf{b}}_S \int_{V'} dm' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_{V_c'} \mathbf{r'}\, dm') - 2\,\boldsymbol{\omega} \times \int_{V_c'} \mathbf{v'}\, dm' - \dot{\boldsymbol{\omega}} \times \int_{V_c'} \mathbf{r'}\, dm' .$$

(11.5.3)

Setting $M \equiv \int_{V'} dm'$ = total mass of the blob, and combining the last two equations while reordering terms, one gets

$$(d/dt)_{S'} [\int_{Vm'} \rho \mathbf{v}' \, dV'] = (\int_{Sc} \mathbf{t} \, dS + \int_{Vc} \rho \mathbf{B} \, dV)$$
$$- [M\ddot{\mathbf{b}}_S + 2\,\boldsymbol{\omega} \times \int_{Vc'} \mathbf{v}' \, dm' + \dot{\boldsymbol{\omega}} \times \int_{Vc'} \mathbf{r}' \, dm' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \int_{Vc'} \mathbf{r}' \, dm') \; .$$

// Lai (7.7.14) (11.5.4)

This appears as equation (7.7.14) on p 429 of Lai et al. (they use $\mathbf{a}_0 = \ddot{\mathbf{b}}_S$ and m = M).

Here is a translation table relating our notation to that of Lai et al. (figure on page 428)

Lai	us	
F_1	S	fixed frame
F_2	S'	rotating frame (moving frame)
$\mathbf{r}$	$\mathbf{r}$	position in fixed frame
$\mathbf{x}$	$\mathbf{r}'$	position in rotating frame
$(d\mathbf{r}/dt)_{F1} = \mathbf{v}_{F1}$	$(d\mathbf{r}/dt)_S = \mathbf{v}_S = \mathbf{v}$	velocity in fixed frame
$(d\mathbf{x}/dt)_{F2} = \mathbf{v}_{F2}$	$(d\mathbf{r}'/dt)_{S'} = \mathbf{v}'_{S'} = \mathbf{v}'$	velocity in rotating frame
$\mathbf{R}_0$	$\mathbf{b}$	vector linking frame origins
$\mathbf{r} = \mathbf{R}_0 + \mathbf{x}$	$\mathbf{r} = \mathbf{b} + \mathbf{r}'$	
$(D/Dt)_{F1} = (D/Dt)$	$(d/dt)_S$	derivative in fixed frame
$(\mathbf{a}_0)_{F1} = \mathbf{a}_0$	$\ddot{\mathbf{b}}_S$	(11.5.5)

In Lai et al. equations (7.9.1) through (7.9.8) symbol **v** means $\mathbf{v}_{F1}$ which is our **v**.
In Lai et al. equation (7.9.9) showing the fictitious torque, symbol **v** means $\mathbf{v}_{F2}$ which is our **v**' .

11.6 Comments on the Reynolds Transport Theorem

Although a bit off our path, it seems useful to tie this topic in with the previous section.

1. The operation $\partial/\partial t$ differs from the operation d/dt when applied to an "Eulerian" function of space and time, in which case d/dt is called a material derivative and is written D/Dt in fluid dynamics notation,

$$df(\mathbf{r},t)/dt = \partial f/\partial t + \nabla f \bullet d\mathbf{r}/dt = \partial f/\partial t + \mathbf{v} \bullet (\nabla f) \equiv Df(\mathbf{r},t)/Dt \quad . \qquad (11.6.1)$$

In an Eulerian function, the position coordinate **r** is the current position of a Particle of fluid as one would expect. (In a Lagrangian function, the position argument is the position at which a Particle started out at some earlier time t_0.) In general, any property of a fluid f(**r**,t) (such as temperature or density or velocity) varies with **r**, so the term $\mathbf{v} \bullet (\nabla f)$ does not in general vanish.

2. The right side of equation (11.4.1) or (11.5.1) shows the total time derivative of an integral over a material volume V_m, which integral represents a mechanical property of our blob object of interest. It is always possible to replace such a time derivative with a set of control volume and control surface integrals using a rather elegant theorem known as the **Reynolds Transport Theorem** (1903) ,

$$(d/dt)\,[\,\int_{Vm}\ T\ dV\,]\ =\int_{Vc}(\partial T/\partial t)\ dV\ +\int_{Sc}T(\mathbf{v}\bullet\mathbf{n})\ dS\ \ =\int_{Vc}[(dT/dt)+\ T\ \text{div}\ \mathbf{v}]\ dV \qquad (11.6.2)$$

where T = T(**r**,t) is any reasonable function. For example, T could be a scalar like ρ, or a component of a vector like **r** x ρ**v** in (11.5.1), or a component of any tensor $T_{ijk\ldots}$. This theorem appears as (7.4.1) and (7.4.2) in Lai et al. page 418 and a proof is given on the next page. [**n** is a unit normal to the surface]

Notice that each term has the units of T times volume/sec.

If T = ρ, the left expression is dM/dt = 0 and the far right integral being 0 for any V_c requires that (dρ/dt) + ρ div **v** = 0 which is a form of the continuity equation $\partial\rho/\partial t + \text{div}(\rho\mathbf{v})$ in which ρ is mass density and **J** = ρ**v** is the mass-current density.

3. Although we write $V_c = V_m(t)$ at time t, it is understood that V_c is independent of time -- it is that shed snake skin referred to above. Therefore, one regards $(\partial V_c/\partial t) = (dV_c/dt) = 0$, and then in (11.6.2) we can write

$$\int_{Vc}(\partial T(\mathbf{r},t)/\partial t)\ dV\ =(\partial/\partial t)\,[\int_{Vc}T(\mathbf{r},t)\ dV\,]\ =\ (d/dt)\,[\int_{Vc}T(\mathbf{r},t)\ dV\,] \qquad (11.6.3)$$

where in the last step we use the fact that the integral is a function only of time, since **r** is integrated out. So The Reynolds Transport Theorem (11.6.2) can be written this way

$$(d/dt)\,[\int_{Vm}TdV]\ =(d/dt)\,[\int_{Vc}TdV\,]\ +\int_{Sc}T(\mathbf{v}\bullet\mathbf{n})\ dS\ =\int_{Vc}[(dT/dt)+\ T\ \text{div}\ \mathbf{v}]\ dV\ . \qquad (11.6.4)$$

4. Writing T = ρb where b is some extensive blob property per unit mass, (11.6.4) becomes

$$(d/dt)\,[\int_{Vm}b\ \rho dV]\ =(d/dt)\,[\int_{Vc}b\ \rho dV\,]\ +\int_{Sc}b\rho(\mathbf{v}\bullet\mathbf{n})\ dS$$

$$=\int_{Vc}[(d(\rho b)/dt)+\ (\rho b)\ \text{div}\ \mathbf{v}]\ dV\ . \qquad (11.6.5)$$

In this case, one can regard $\int_{Vm}b\ \rho dV\ =\int_{Vm}b\ dm$ as the "total amount of b" in the moving fluid blob. This moving blob which recall maintains all its particles is sometimes called "the system", and the total amount of b in the system might be called B_{sys}. Then (11.6.5) can be written as

$$(dB_{sys}/dt)\ =(dB_{CV}/dt)\ +\int_{CS}b\rho(\mathbf{v}\bullet\mathbf{n})\ dS\ =\int_{CV}[(d(\rho b)/dt)+\ (\rho b)\ \text{div}\ \mathbf{v}]\ dV \qquad (11.6.6)$$

where CV (or C.V.) is a traditional notation for V_c, the control volume, and CS is S_c, the control surface. For example, here is a typical web appearance of the Reynolds Transport Theorem in the form of the left equality in (11.6.6) and (11.6.5),

$$\frac{dB_{sys}}{dt} = \frac{\partial B_{CV}}{\partial t} + \int_{CS} \rho b \vec{V} \cdot \vec{n} dA$$
$$= \frac{\partial}{\partial t} \int_{CV} \rho b d\forall + \int_{CS} \rho b \vec{V} \cdot \vec{n} dA \qquad (11.6.7)$$

which points out another common notation: V with a horizontal bar ($\forall$) refers to volume, to distinguish it from V without a slash which refers to velocity. We solved this problem by using lower case **v** for velocity. When the bar is short, one gets $\forall$ which looks a bit like an upside down A, and in fact the logic "for all" symbol $\forall$ is sometimes used.

5. Conceivably, the vague similarity between the left equation in (11.6.6) and the G Rule (2.1) might be the reason some people refer to the G Rule as a transport theorem. This does seem far fetched.

6. Applying the left equality of (11.6.4) to T = **r** x ρ**v** gives, in Frame S,

$$(d/dt)_S \left[\int_{V_m} \mathbf{r} \times \rho \mathbf{v} \, dV \right] = (d/dt)_S \left[\int_{V_c} \mathbf{r} \times \rho \mathbf{v} \, dV \right] + \int_{S_c} \mathbf{r} \times \rho \mathbf{v} \, (\mathbf{v} \bullet \mathbf{n}) \, dS \quad , \qquad (11.6.8)$$

so that (11.4.1) may be written

$$\mathbf{N}^{(0)} = \left(\int_{S_c} \mathbf{r} \times \mathbf{t} \, dS + \int_{V_c} \mathbf{r} \times \rho \mathbf{B} \, dV \right)$$
$$= (d/dt) \left[\int_{V_c} \mathbf{r} \times \rho \mathbf{v} \, dV \right] + \int_{S_c} (\mathbf{r} \times \rho \mathbf{v}) \, (\mathbf{v} \bullet \mathbf{n}) \, dS \quad . \qquad \text{// Lai (7.9.8)} \qquad (11.6.9)$$

This says that the total torque on a fluid blob equals the rate of change of the angular momentum contained in the frozen control volume V_c plus the rate of outflow of angular momentum from that volume. This equation appears as (7.9.8) in Lai et al. p 431.

When working in a rotating frame, we have to add to the left side of (11.6.9) the fictitious torques stated in (11.4.6) and which Lai et al. states as (7.9.9). Here then are a few quotes from Lai et al. (p 430-431) which use the continuum mechanics terminology as shown in the table at the start of this Section :

7.9 THE PRINCIPLE OF MOMENT OF MOMENTUM

The *global principle of moment of momentum* states that the total moment about a fixed point of surface and body forces on a fixed part of material is equal to the time rate of change of total moment of momentum of the part about the same point. That is,

$$\frac{D}{Dt} \int_{V_m} \mathbf{x} \times \rho \mathbf{v} dV = \int_{S_c} (\mathbf{x} \times \mathbf{t}) dS + \int_{V_c} (\mathbf{x} \times \rho \mathbf{B}) dV, \qquad (7.9.1)$$

On the other hand, if we use the Reynolds transport theorem, Eq. (7.4.1), for the left-hand side of Eq. (7.9.1), we obtain

$$\int_{S_c} (\mathbf{x} \times \mathbf{t}) dS + \int_{V_c} (\mathbf{x} \times \rho \mathbf{B}) dV = \int_{V_c} \frac{\partial}{\partial t} (\mathbf{x} \times \rho \mathbf{v}) dV + \int_{S_c} (\mathbf{x} \times \rho \mathbf{v})(\mathbf{v} \cdot \mathbf{n}) dS. \tag{7.9.8}$$

That is, the total moment about a fixed point due to surface and body forces acting on the material instantaneously inside a control volume = total rate of change of moment of momentum inside the control volume + total net rate of outflow of moment of momentum across the control surface.

If the control volume is fixed in a moving frame, then the following terms should be added to the left side of Eq. (7.9.8):

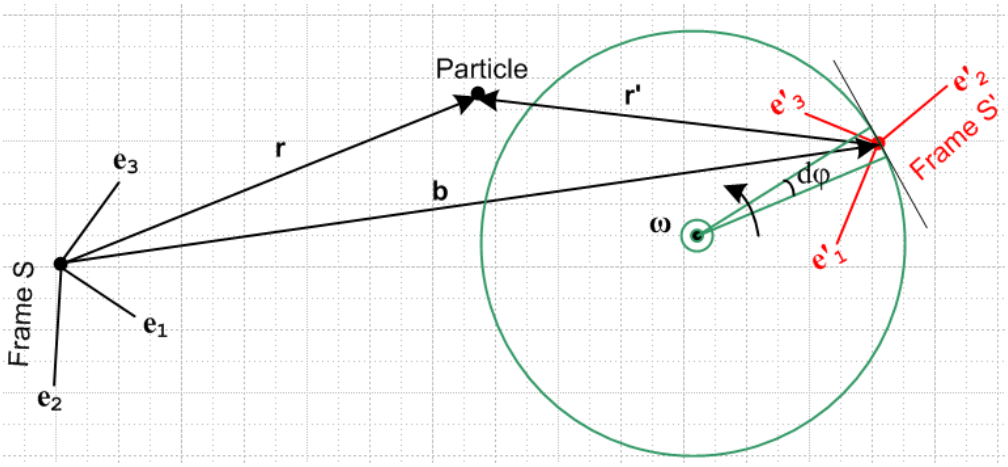
$$-\left(\int \mathbf{x} dm\right) \times \mathbf{a}_o - \int \mathbf{x} \times (\dot{\boldsymbol{\omega}} \times \mathbf{x}) dm - \int \mathbf{x} \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x})] dm - 2 \int \mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{v}) dm, \tag{7.9.9}$$

The equation (7.9.8) and its italicized interpretation express the familiar law of conservation of angular momentum. In Chapter 7 Lai et al. have similar sections for conservation of mass, energy, and linear momentum, and a final section on the inequality of entropy, each of these being a "principle". Each section uses the Reynolds Transport Theorem to replace its D/Dt [$\int_{\mathtt{vm}}$ ...] object, and each section ends up with an equation like (7.9.8) with an italicized interpretation.

12. Summary of the Forward Problem Solution

12.1 Summary of the Forward Problem equations (non-swap notation)

We now summarize the results of Sections 6, 7, 8 and 11. The first set of equations below is valid regardless of whether either of these frames is inertial (they could both be non-inertial). The second set of equations involving fictitious forces and torques assumes that Frame S is inertial (and Frame S' is not).



(12.1.1)

Definitions and Equations

$\mathbf{r}, \mathbf{v}, \mathbf{a}$	position, natural velocity and natural acceleration in Frame S		(12.1.2)
$\mathbf{r}', \mathbf{v}', \mathbf{a}'$	position, natural velocity and natural acceleration in Frame S'		
$\boldsymbol{\omega}$	angular velocity of Frame S' relative to Frame S		
$\mathbf{b}$	vector directed from origin of Frame S to origin of Frame S'		
$\mathbf{r} = \mathbf{b} + \mathbf{r}'$		*(6.1)*	(a)
$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S$		*(6.6a)*	(b)
$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}$		*(6.6c)*	(c)
$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S$		*(7.6a)*	(d)
S S' Euler Coriolis centripetal frame			
$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \ddot{\mathbf{b}}_{S'}$		*(7.6b)*	(e)
$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [(\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]$		*(11.2.14)*	(f)
$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S]$ $- m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + m\,\dot{\mathbf{c}}' \times \mathbf{v}'$		*(11.2.15)*	(g)

Fictitious Forces (Section 8)

For using fictitious forces we have (for these equations Frame S is inertial)

$\mathbf{F} = m\mathbf{a}$ // true Newton's Law in inertial Frame S *(8.1.2)*

$\mathbf{F'}_{\mathtt{eff}} = m\mathbf{a'}$ // fake Newton's Law in rotating Frame S' *(8.1.4)*

$$\mathbf{F'}_{\mathtt{eff}} = \mathbf{F} + \mathbf{F'}_{\mathtt{fict}} \ . \qquad \textit{(8.1.7)} \qquad (12.1.3)$$

For the general case, the fictitious forces can be expressed as

$$\mathbf{F'}_{\mathtt{fict}} = \underbrace{-m\ddot{\mathbf{b}}_S}_{\text{frame}} \underbrace{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'})}_{\text{centrifugal}} \underbrace{- 2m\,\boldsymbol{\omega} \times \mathbf{v'}}_{\text{Coriolis}} \underbrace{- m\dot{\boldsymbol{\omega}} \times \mathbf{r'}}_{\text{Euler}} \ . \qquad \textit{(8.1.8)} \qquad (12.1.4)$$

For Special Case # 1 problems ($\boldsymbol{\omega}$ axis passes through Frame S origin), we have

$$\mathbf{F'}_{\mathtt{fict}} = \underbrace{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{centrifugal}} \underbrace{- 2m\,\boldsymbol{\omega} \times \mathbf{v'}}_{\text{Coriolis}} \underbrace{- m\dot{\boldsymbol{\omega}} \times \mathbf{r}}_{\text{Euler}} \ . \quad \text{Special Case \#1} \qquad \textit{(8.4.2)} \qquad (12.1.5)$$

Fictitious Torques (Section 11) // Frame S is inertial

$$\mathbf{N'}^{(\mathtt{c'})}_{\mathtt{fict}} = (\mathbf{r'}-\mathbf{c'}) \times \mathbf{F'}_{\mathtt{fict}} + m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c'} + \dot{\mathbf{b}}_S) \times (\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v'} \qquad \textit{(11.3.8)}$$

or

$$\mathbf{N'}^{(\mathtt{c'})}_{\mathtt{fict}} = -(\mathbf{r'}-\mathbf{c'}) \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2m\,\boldsymbol{\omega} \times \mathbf{v'} + m\dot{\boldsymbol{\omega}} \times \mathbf{r'}] \qquad (12.1.6)$$

$$+ m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c'} + \dot{\mathbf{b}}_S) \times (\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v'} \qquad \textit{(11.3.10)}$$

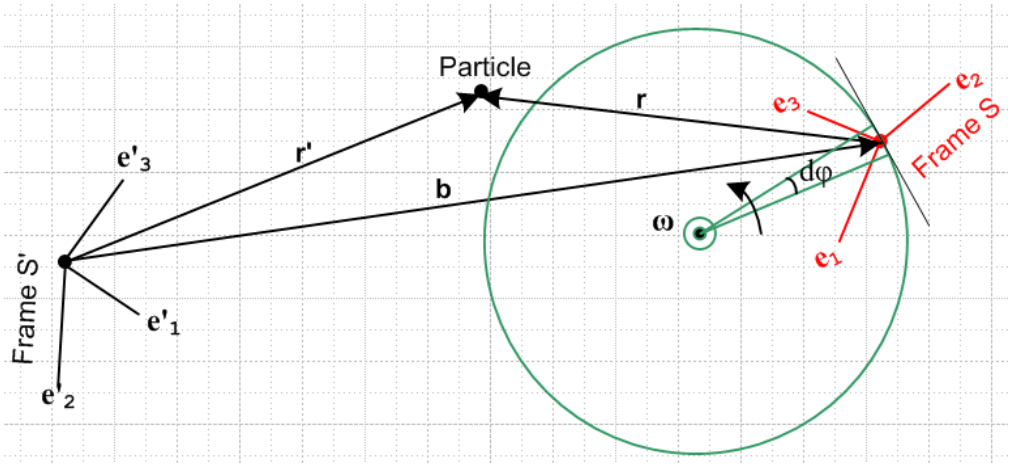
12.2 Summary of the Forward Problem equations (swap notation)

As a reminder, we quote from above:

Swap Notation. The meaning of "swap notation" is that, in Fig 1, the vectors **ω** and **b** stay put, but all other vectors undergo **V**↔**V**'. This latter group includes basis vectors $\mathbf{e}_i \leftrightarrow \mathbf{e}'_i$, **r** ↔ **r**' , **v** ↔ **v**', **a** ↔ **a**' and of course Frame S ↔ Frame S'. This is nothing more than a change of the way things are labeled. If a non-swap notation equation has the number (x.x.x), then the corresponding equation in swap notation will be given the number $(x.x.x)_S$. *(8.5.10)*

We are now going to translate everything in Section 12.1 into swap notation. Fig (12.2.1) is the same as Fig (12.1.1) but the primes are swapped as just discussed.

The first set of equations below is valid regardless of whether either of these frames is inertial (they could both be non-inertial). The second set of equations involving fictitious forces assumes that Frame S' is inertial (and therefore Frame S is not).



(12.2.1)

Definitions and Equations

r', **v'**, **a'**	position, natural velocity and natural acceleration in Frame S'		(12.2.2)
r, **v**, **a**	position, natural velocity and natural acceleration in Frame S		
ω	angular velocity of Frame S relative to Frame S'		
b	vector directed from origin of Frame S' to origin of Frame S		
$\mathbf{r}' = \mathbf{b} + \mathbf{r}$		$(6.1)_S$	(a)
$\mathbf{v}' = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_S$,		$(6.6a)_S$	(b)
$\mathbf{v}' = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S$		$(6.6c)_S$	(c)
$\mathbf{a}' = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \ddot{\mathbf{b}}_S$,		$(7.6a)_S$	(d)
S' S Euler Coriolis centripetal frame			

$$\mathbf{a'} = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_S + \ddot{\mathbf{b}}_S \qquad (7.6b)_S \quad \text{(e)}$$

$$\mathbf{L'}^{(c')} = \mathbf{L}^{(c)} + m(\mathbf{r}-\mathbf{c}) \times [\,(\boldsymbol{\omega} \times \mathbf{r}) + \dot{\mathbf{b}}_{S'}] \qquad (11.2.14)_S \quad \text{(f)}$$

$$\dot{\mathbf{L}}'^{(c')} = \dot{\mathbf{L}}^{(c)} + m(\mathbf{r}-\mathbf{c}) \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \ddot{\mathbf{b}}_{S'}]$$
$$- m(\dot{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c} + \dot{\mathbf{b}}_{S'}) \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}) + m\,\dot{\mathbf{c}} \times \mathbf{v} \qquad (11.2.15)_S \quad \text{(g)}$$

Fictitious Forces (Section 8.1)

For using fictitious forces we have (Frame S' is inertial)

$\mathbf{F'} = m\mathbf{a'}$ // true Newton's Law in inertial Frame S' $(8.1.2)_S$

$\mathbf{F}_{eff} = m\mathbf{a}$ // fake Newton's Law in rotating Frame S $(8.1.4)_S$

$$\mathbf{F}_{eff} = \mathbf{F'} + \mathbf{F}_{fict} \qquad (8.1.7)_S \qquad (12.2.3)$$

For the general case, the fictitious forces can be expressed as

$$\mathbf{F}_{fict} = \underbrace{-m\ddot{\mathbf{b}}_{S'}}_{\text{frame}} \underbrace{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\text{centrifugal}} \underbrace{- 2m\,\boldsymbol{\omega} \times \mathbf{v}}_{\text{Coriolis}} \underbrace{- m\dot{\boldsymbol{\omega}} \times \mathbf{r}}_{\text{Euler}} \qquad (8.1.8)_S \qquad (12.2.4)$$

For Special Case # 1 problems ($\boldsymbol{\omega}$ axis passes through Frame S' origin), we have

$$\mathbf{F}_{fict} = \underbrace{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'})}_{\text{centrifugal}} \underbrace{- 2m\,\boldsymbol{\omega} \times \mathbf{v}}_{\text{Coriolis}} \underbrace{- m\dot{\boldsymbol{\omega}} \times \mathbf{r'}}_{\text{Euler}} \quad \text{Special Case \#1} \qquad (8.4.1)_S \qquad (12.2.5)$$

Fictitious Torques (Section 11.3)

$$\mathbf{N}^{(c)}_{fict} = (\mathbf{r}-\mathbf{c}) \times \mathbf{F}_{fict} + m(\dot{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c} + \dot{\mathbf{b}}_{S'}) \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}) - m\dot{\mathbf{c}} \times \mathbf{v} \qquad (11.3.8)_S$$

or

$$\mathbf{N}^{(c)}_{fict} = -(\mathbf{r}-\mathbf{c}) \times [\,m\ddot{\mathbf{b}}_{S'} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r}] \qquad (12.2.6)$$
$$+ m(\dot{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c} + \dot{\mathbf{b}}_{S'}) \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}) - m\dot{\mathbf{c}} \times \mathbf{v} \qquad (11.3.10)_S$$

13. The Inverse Problem

Consider these two problems which concern the exact same physical situation (non-swap notation):

Forward Problem: given: $\mathbf{r}'$, $\mathbf{v}'$, $\mathbf{a}'$, $\mathbf{L}'^{(c')}$, $\dot{\mathbf{L}}'^{(c')}$

find: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\mathbf{L}^{(c)}$, $\dot{\mathbf{L}}^{(c)}$ // summarized in Section 12.1

Inverse Problem: given: $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, $\mathbf{L}^{(c)}$, $\dot{\mathbf{L}}^{(c)}$

find: $\mathbf{r}'$, $\mathbf{v}'$, $\mathbf{a}'$, $\mathbf{L}'^{(c')}$, $\dot{\mathbf{L}}'^{(c')}$ // to be summarized in Section 13.3 below

Recall that equations involving just the above quantities are valid even if both Frames S and S' are non-inertial. On the other hand, equations involving $\mathbf{F}'_{fict}$ and $\mathbf{N}'^{(c)}{}_{fict}$ require that Frame S be inertial. If Frame S is inertial for the Forward Problem, it is also inertial for the Inverse Problem and in that case we know that there is no fictitious force or torque in Frame S, so these items are not included on the list of quantities shown above.

We shall first compute the inverse equations by brute force, then at the end show how they can be obtained by a set of simple swap rules.

13.1 Brute Force Method

Looking at the Section 12.1 summary, equation (6.1) is easily inverted

$$\mathbf{r} = \mathbf{b} + \mathbf{r}' \;\Rightarrow$$

$$\mathbf{r}' = \mathbf{r} - \mathbf{b} \tag{13.1.1}$$

Similarly for (6.6a), where the third line below uses identity (6.2b),

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \tag{6.6a}$$

$$\mathbf{v}' = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}' - \dot{\mathbf{b}}_S \tag{13.1.2}$$

$$\mathbf{v}' = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} - \dot{\mathbf{b}}_{S'} \;. \tag{13.1.3}$$

Equation (7.6a) requires a bit more effort to invert. We first solve (7.6a*)* for $\mathbf{a}'$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \tag{7.6a}$$

$$\mathbf{a}' = \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v}' - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S \;. \tag{13.1.4}$$

Replace $\mathbf{v}'$ using (13.1.2),

$$\mathbf{a}' = \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times [\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}' - \dot{\mathbf{b}}_S] - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S$$

$$= \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v} + 2\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2\,\boldsymbol{\omega} \times \dot{\mathbf{b}}_S - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S$$

$$= \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2\,\boldsymbol{\omega} \times \dot{\mathbf{b}}_S - \ddot{\mathbf{b}}_S \ . \qquad (13.1.5)$$

With $\mathbf{r}' = \mathbf{r} - \mathbf{b}$ we can regard the RHS of (13.1.5) as being expressed entirely in terms of Frame S objects.

Now, by first shuffling terms in (7.11),

$$\ddot{\mathbf{b}}_{S'} = \ddot{\mathbf{b}}_S - \dot{\boldsymbol{\omega}} \times \mathbf{b} - 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_S + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad \Rightarrow \qquad (7.11)$$

$$2\boldsymbol{\omega} \times \dot{\mathbf{b}}_S - \ddot{\mathbf{b}}_S = -\dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) - \ddot{\mathbf{b}}_{S'}$$

we can replace the last two terms in (13.1.5) to get

$$\mathbf{a}' = \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) - \ddot{\mathbf{b}}_{S'}$$

$$= \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r} - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \ddot{\mathbf{b}}_{S'} \qquad (13.1.6)$$

Inversion of the $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ can also be done by this brute force method, but we spare the reader and instead quote the results later after establishing the "swap rules method".

Summary of the inverse problem results obtained by brute force:

$$\mathbf{r}' = \mathbf{r} - \mathbf{b} \qquad (13.1.1)$$

$$\mathbf{v}' = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}' - \dot{\mathbf{b}}_S \qquad (13.1.2)$$

$$\mathbf{v}' = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} - \dot{\mathbf{b}}_{S'} \qquad (13.1.3)$$

$$\mathbf{a}' = \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2\,\boldsymbol{\omega} \times \dot{\mathbf{b}}_S - \ddot{\mathbf{b}}_S \qquad (13.1.5)$$

$$\mathbf{a}' = \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r} - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \ddot{\mathbf{b}}_{S'} \qquad (13.1.6)$$

13.2 Swap Rules Method

Without any justification yet, let us postulate that we can obtain our inverse problem equations directly from the forward problem equations (and vice versa) using this set of Swap Rules :

forward problem equations ← swap rules → inverse problem equations

$$\mathbf{r} \leftrightarrow \mathbf{r}' \qquad \mathbf{b} \leftrightarrow -\mathbf{b} \qquad \boldsymbol{\omega} \leftrightarrow -\boldsymbol{\omega} \qquad \mathbf{L}^{(c)} \leftrightarrow \mathbf{L}'^{(c')} \tag{13.2.1}$$

$$\mathbf{v} \leftrightarrow \mathbf{v}' \qquad \dot{\mathbf{b}}_S \leftrightarrow -\dot{\mathbf{b}}_{S'} \qquad \dot{\mathbf{L}}^{(c)} \leftrightarrow \dot{\mathbf{L}}'^{(c')}$$

$$\mathbf{a} \leftrightarrow \mathbf{a}' \qquad \ddot{\mathbf{b}}_S \leftrightarrow -\ddot{\mathbf{b}}_{S'} \qquad \text{// Swap Rules}$$

This set of rules is *different* from our rules for going between swap and no-swap notation, which are these

$$\mathbf{r} \leftrightarrow \mathbf{r}' \qquad \mathbf{b} \leftrightarrow \mathbf{b} \qquad \boldsymbol{\omega} \leftrightarrow \boldsymbol{\omega} \qquad \mathbf{L}^{(c)} \leftrightarrow \mathbf{L}'^{(c')} \qquad S \leftrightarrow S' \tag{13.2.2}$$

$$\mathbf{v} \leftrightarrow \mathbf{v}' \qquad \dot{\mathbf{b}}_S \leftrightarrow \dot{\mathbf{b}}_{S'} \qquad \dot{\mathbf{L}}^{(c)} \leftrightarrow \dot{\mathbf{L}}'^{(c')}$$

$$\mathbf{a} \leftrightarrow \mathbf{a}' \qquad \ddot{\mathbf{b}}_S \leftrightarrow \ddot{\mathbf{b}}_{S'} \qquad \text{// rules for going between swap and no-swap notation.}$$

The big difference is that with the Swap Rules we are negating the vectors **b** (and its derivatives) and **ω**. Later in Section 13.5 we will justify the set of rules (13.2.1). Whereas the change from non-swap notation to swap notation is just a cosmetic relabeling of a problem's variables, application of the Swap Rules changes a problem which is the Forward Problem into a different problem which is the Inverse Problem defined at the start of Section 13.1.

Since the inverse problem equations computed by brute force are sitting just above, let's apply the Swap Rules (13.2.1) to them and see what we get:

$$\mathbf{r} = \mathbf{r}' + \mathbf{b} \qquad (13.1.1)_{\texttt{swapped}}$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \qquad (13.1.2)_{\texttt{swapped}}$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad (13.1.3)_{\texttt{swapped}}$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \ddot{\mathbf{b}}_{S'} \qquad (13.1.5)_{\texttt{swapped}}$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \quad . \qquad (13.1.6)_{\texttt{swapped}}$$

And now we directly quote from the summary from Section 12.1 above

$$\mathbf{r} = \mathbf{b} + \mathbf{r}' \qquad (6.1)$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \qquad (6.6c)$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad (6.6a)$$

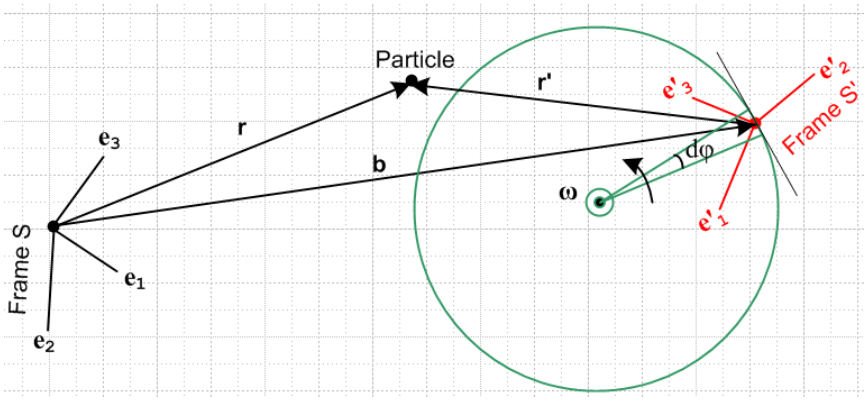
$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \ddot{\mathbf{b}}_{S'} \qquad (7.6b)$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \qquad (7.6a)$$

Since the last two sets of equations are identical, we have demonstrated that the Swap Rules presented in (13.2.1) do indeed convert either set of equations into the other.

13.3 Summary of the Inverse Problem Equations (non-swap notation)

As noted above, we obtain the Inverse Problem equations by applying the Swap Rules to the Forward Problem equations. But the Swap Rules can be thought of as having two steps: (1) do the swap one usually does to go from swap to non-swap notation; (2) then take **b**→ **-b** (including derivatives) and **ω**→ **-ω** (including derivatives**)**. We have step (1) already carried out in Section 12.2, so to get the equations below we need only carry out step (2) on the Section 12.2 equations. Here is a repeat of Fig (12.1.1) which of course applies to both the Forward Problem and the Inverse Problem :



(13.3.1)

r', **v'**, **a'**	position, natural velocity and natural acceleration in Frame S'	(13.3.2)
r, **v**, **a**	position, natural velocity and natural acceleration in Frame S	
ω	angular velocity of Frame S' relative to Frame S	
b	vector directed from origin of Frame S to origin of Frame S'	

$$\mathbf{r}' = -\mathbf{b} + \mathbf{r} \qquad \text{(a)}$$

$$\mathbf{v}' = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} - \dot{\mathbf{b}}_{S'} \qquad \text{// all these equations are from (12.2.2) with } \boldsymbol{\omega} \text{ and } \mathbf{b} \text{ negated} \qquad \text{(b)}$$

$$\mathbf{v}' = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r}' - \dot{\mathbf{b}}_{S} \qquad \text{(c)}$$

$$\mathbf{a}' = \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r} - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \ddot{\mathbf{b}}_{S'} \qquad \text{(d)}$$

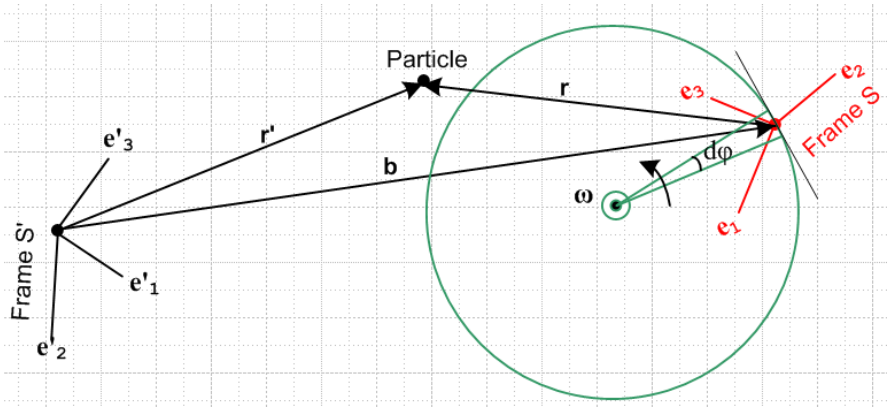
$$\mathbf{a}' = \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2\,\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S} - \ddot{\mathbf{b}}_{S} \qquad \text{(e)}$$

$$\mathbf{L}'^{(c')} = \mathbf{L}^{(c)} + m(\mathbf{r}-\mathbf{c}) \times [-(\boldsymbol{\omega} \times \mathbf{r}) - \dot{\mathbf{b}}_{S'}] \qquad \text{(f)}$$

$$\dot{\mathbf{L}}'^{(c')} = \dot{\mathbf{L}}^{(c)} + m(\mathbf{r}-\mathbf{c}) \times [-\dot{\boldsymbol{\omega}} \times \mathbf{r} - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \ddot{\mathbf{b}}_{S'}]$$
$$- m(\dot{\mathbf{c}} - \boldsymbol{\omega} \times \mathbf{c} - \dot{\mathbf{b}}_{S'}) \times (\mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} - \dot{\mathbf{b}}_{S'}) + m\,\dot{\mathbf{c}} \times \mathbf{v} \qquad \text{(g)}$$

13.4 Summary of the Inverse Problem Equations (swap notation)

The results of this section are those of Section 13.3 but with S ↔ S' and **v'↔v** for all vectors except **b** and **ω**. Again, this is just a change of labeling. We repeat Fig (12.2.1) which shows the swap notation case :



(13.4.1)

r, **v**, **a**	position, natural velocity and natural acceleration in Frame S	(13.3.2)
r', **v'**, **a'**	position, natural velocity and natural acceleration in Frame S'	
ω	angular velocity of Frame S relative to Frame S'	
b	vector directed from origin of Frame S' to origin of Frame S	

$$\mathbf{r} = -\mathbf{b} + \mathbf{r}' \qquad \text{(a)}$$

$$\mathbf{v} = \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{r}' - \dot{\mathbf{b}}_S \qquad \text{(b)}$$

$$\mathbf{v} = \mathbf{v}' - \boldsymbol{\omega} \times \mathbf{r} - \dot{\mathbf{b}}_{S'} \qquad \text{(c)}$$

$$\mathbf{a} = \mathbf{a}' - \dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S \qquad \text{(d)}$$

$$\mathbf{a} = \mathbf{a}' - \dot{\boldsymbol{\omega}} \times \mathbf{r} - 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} - \ddot{\mathbf{b}}_{S'} \qquad \text{(e)}$$

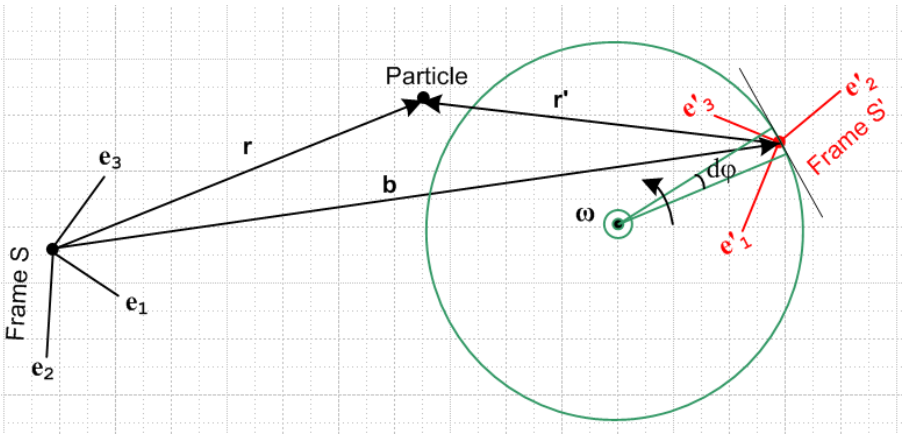
$$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [-(\boldsymbol{\omega} \times \mathbf{r}') - \dot{\mathbf{b}}_S] \qquad \text{(f)}$$

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [-\dot{\boldsymbol{\omega}} \times \mathbf{r}' - 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - \ddot{\mathbf{b}}_S]$$
$$- m(\dot{\mathbf{c}}' - \boldsymbol{\omega} \times \mathbf{c}' - \dot{\mathbf{b}}_S) \times (\mathbf{v}' - \boldsymbol{\omega} \times \mathbf{r}' - \dot{\mathbf{b}}_S) + m\,\dot{\mathbf{c}}' \times \mathbf{v}' \qquad \text{(g)}$$

13.5 Why the Swap Rules Work

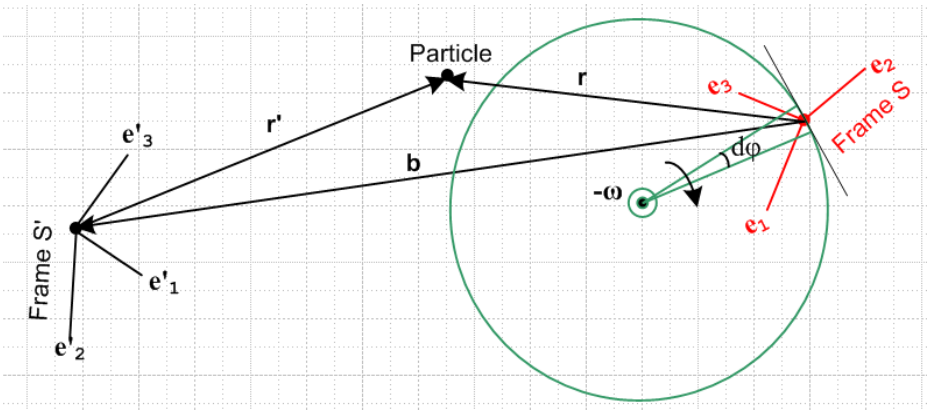
Preview: We are going to show here that if one starts with an initial picture of the physical situation between two frames of references and a Particle, and if one applies the Swap Rules (13.2.1) to that picture, one ends up with a final picture which is exactly the same as the initial picture. Therefore, if the initial physical picture is described by a set of equations, then applying the Swap Rules to those equations gives new equations which also apply to the initial picture (since it is the same as the final picture). Thus, if the initial set of equations is valid, so is the final set of equations obtained via these Swap Rules. The equations obtained by application of the Swap Rules provide the "answers" for our Inverse Problem defined at the start of Section 13.

Start with Fig (12.1.1),



(13.5.1)

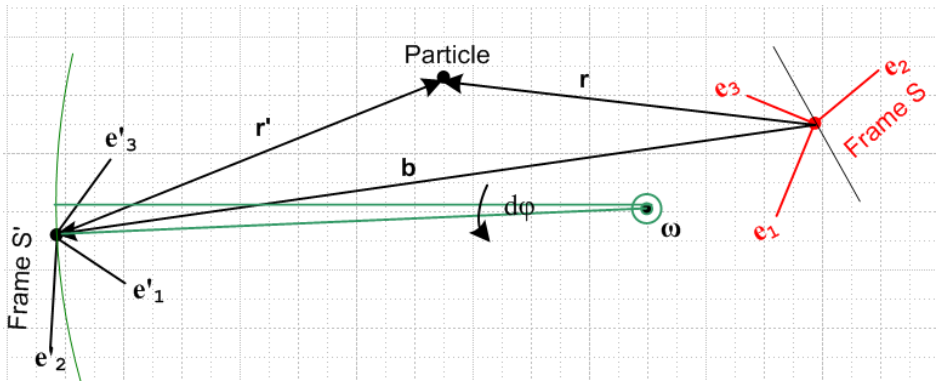
Apply the Swap Rules of (13.2.1) to get



(13.5.2)

Rather than take **b** → **-b** as a label, we have flipped the arrow on the **b** vector to achieve the same result.

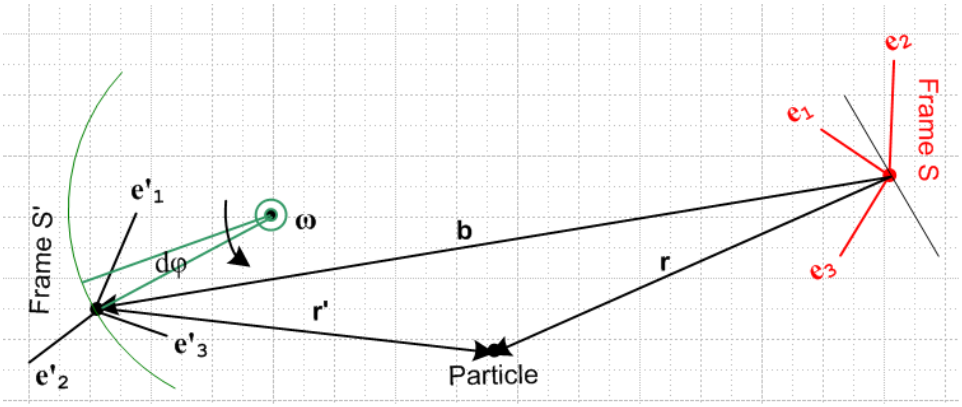
Now observe the above scenario from a camera platform which is rotating counterclockwise at rate **ω** with its rotation axis the same as that shown in the figure. Viewed from this camera's rotating frame, Frame S is at rest, and Frame S' is rotating counterclockwise. That camera-viewed picture is then,



(13.5.3)

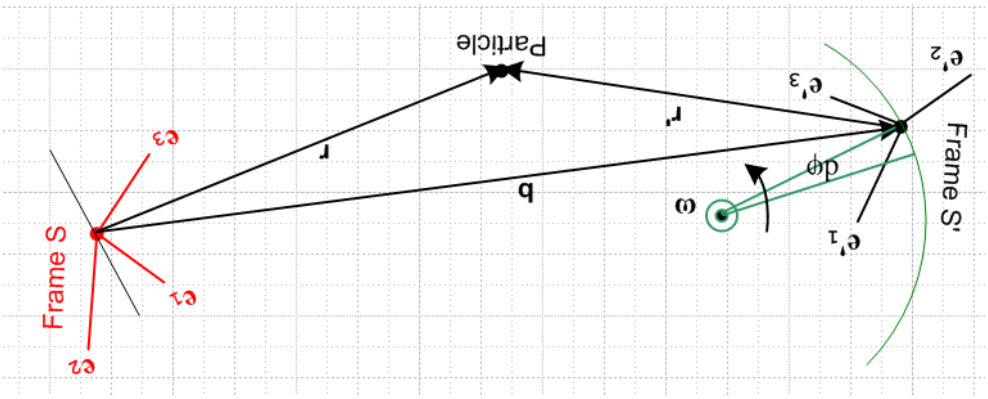
This is the picture one sees if one observes things from Frame S. Our equations of interest remain valid despite the camera's rotation because these equations are based on the G Rule which is a function only of the *relative* relationship between frames, and making the camera rotate above does not change this relationship.

Moreover, these equations are the same regardless of where one puts the **ω** axis, regardless of where one places the Particle, and regardless of where and how one orients the two Reference frames. So let's move these things around a bit in the above picture to get this new drawing which has the same equations,



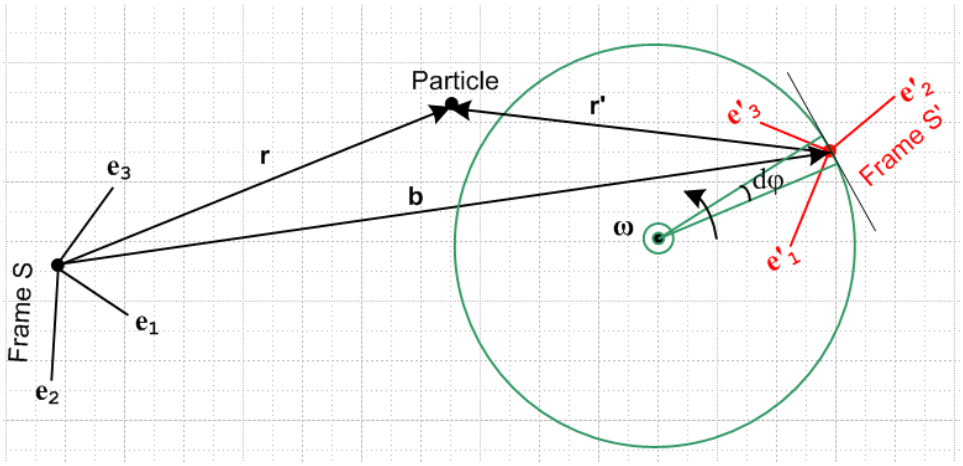
(13.5.4)

Now we rotate this picture about an axis near the center and perpendicular to the plane of paper. Such a rotation again does not change the equations associated with the picture. We then have



(13.5.5)

But this is the same as the picture we started with above (apart from colors and text orientation),



(13.5.1)

Since the Swap Rules, along with various equation-invariant reorientations, produce a final picture which is the same as the initial picture, when those Swap Rules are applied to a valid set of equations which apply to the initial picture, the resulting equations are also valid for the initial picture since this is the same as the final picture.

14. Rotating Frames in Curvilinear Coordinates

The solution equations for our Forward Problem are summarized in Section 12.1 and 12.2 above, and those for the Inverse Problem are summarized in Section 13.3 and 13.4. All equations are stated in bolded vector notation. Such equations may be projected onto (dotted with) any complete set of basis vectors, such as the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ used in spherical coordinates. Every orthogonal curvilinear coordinate system has such a set of orthonormal unit basis vectors which we shall call $\hat{\boldsymbol{e}}_i$, orthonormality meaning $\hat{\boldsymbol{e}}_i \bullet \hat{\boldsymbol{e}}_j = \delta_{i,j}$. In general, curvilinear basis vectors like $\hat{\boldsymbol{e}}_i = \hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ are different at different points in space, so one can think of them as $\hat{\boldsymbol{e}}_i(\mathbf{r})$. It is appropriate then to use them as basis vectors for a vector *field* $\mathbf{V}(\mathbf{r})$ or for a vector associated with a discrete Particle located at position $\mathbf{r}$ such as the velocity or acceleration of that Particle.

We might want to use one curvilinear system of coordinates ξ_i with basis unit vectors $\hat{\boldsymbol{e}}_i$ for Frame S, and an entirely different system ξ'_i with basis unit vectors $\hat{\boldsymbol{e}}'_i$ for Frame S'. We might, for example, have ξ_i be spherical coordinates and ξ'_i be toroidal coordinates. Here then is the situation,

	Cartesian coords and basis vectors		Curvilinear coords and basis vectors		
Frame S	r_i	$\mathbf{e}_i$	ξ_i	$\hat{\boldsymbol{e}}_i$	
Frame S'	$(r')_i$	$\mathbf{e}'_i$	$(\xi')_i$	$\hat{\boldsymbol{e}}'_i$	(14.1)

$$\mathbf{e}_i \bullet \mathbf{e}_j = \mathbf{e}'_i \bullet \mathbf{e}'_j = \hat{\boldsymbol{e}}_i \bullet \hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}'_i \bullet \hat{\boldsymbol{e}}'_j = \delta_{i,j} \quad \text{// orthonormality of all bases} \tag{14.2}$$

There must exist some matrix $R(\boldsymbol{\xi})$ such that $\hat{\boldsymbol{e}}_i(\mathbf{r}) = R(\boldsymbol{\xi})\mathbf{e}_i$ for any given curvilinear system. Recall now equations (1.1.1) and (1.1.2),

$$\mathbf{e}_n = R\,\mathbf{e}'_n \quad n = 1,2,3 \quad \text{or} \quad (\mathbf{e}_n)_i = R_{ij}(\mathbf{e}'_n)_j \tag{1.1.1}$$

$$\mathbf{e}_n = (R^{-1})_{nm}\,\mathbf{e}'_m \quad \text{or} \quad \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \;. \tag{1.1.2}$$

In these equations replacing $\mathbf{e}_n \rightarrow \hat{\boldsymbol{e}}_n$ and $\mathbf{e}'_n \rightarrow \mathbf{e}_n$ and $R \rightarrow R(\xi)$ gives,

$$\hat{\boldsymbol{e}}_n = R(\xi)\,\mathbf{e}_n \quad n = 1,2,3 \quad \text{or} \quad (\hat{\boldsymbol{e}}_n)_i = R(\xi)_{ij}(\mathbf{e}_n)_j$$

$$\hat{\boldsymbol{e}}_n = (R(\xi)^{-1})_{nm}\,\mathbf{e}_m \quad \text{or} \quad \mathbf{e}_n = R(\xi)_{nm}\,\hat{\boldsymbol{e}}_m \tag{14.3}$$

which we summarize in the first line below. The second line is for then some other curvilinear coordinate system in Frame S'.

$$\hat{\boldsymbol{e}}_n = R(\xi)\,\mathbf{e}_n \Rightarrow \mathbf{e}_n = R(\xi)_{nm}\hat{\boldsymbol{e}}_m \quad \text{or} \quad \hat{\boldsymbol{e}}_n = [R(\xi)]^{-1}{}_{nm}\,\mathbf{e}_m$$

$$\hat{\boldsymbol{e}}'_n = R'(\xi')\,\mathbf{e}'_n \Rightarrow \mathbf{e}'_n = R'(\xi')_{nm}\hat{\boldsymbol{e}}'_m \quad \text{or} \quad \hat{\boldsymbol{e}}'_n = [R'(\xi')]^{-1}{}_{nm}\,\mathbf{e}'_m \;. \tag{14.4}$$

Example of an R(ξ) matrix. In spherical coordinates with ordering 1,2,3 = r,θ,φ, where θ is the polar angle and φ the azimuth, the matrix R(ξ) is given by (note that $R^{-1} = R^{T}$),

$$R(\xi) = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \qquad [R(\xi)]^{-1} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \tag{14.5}$$

as shown in (A.9), (A.11). We can then use (14.3) that $\hat{\boldsymbol{e}}_n = [R(\xi)]^{-1}_{nm}\,\mathbf{e}_m$ to write (in matrix notation),

$$\begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{14.6}$$

or

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\varphi\sin\theta\,\hat{\mathbf{x}} + \sin\varphi\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\varphi\cos\theta\,\hat{\mathbf{x}} + \sin\varphi\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \quad . \end{aligned} \tag{14.7}$$

Expansions and naming. If **V** is an arbitrary vector, we then have these four expansions of interest :

$$\begin{aligned} &\mathbf{V} = V_i\mathbf{e}_i && V_i = \mathbf{V}\bullet\mathbf{e}_i \\ &\mathbf{V} = (V)'_i\,\mathbf{e}'_i && (V)'_i = \mathbf{V}\bullet\mathbf{e}'_i \\ &\mathbf{V} = (V)_i\,\hat{\boldsymbol{e}}_i && (V)_i = \mathbf{V}\bullet\hat{\boldsymbol{e}}_i \\ &\mathbf{V} = (V)'_i\,\hat{\boldsymbol{e}}'_i && (V)'_i = \mathbf{V}\bullet\mathbf{e}_i \end{aligned} \tag{14.8}$$

where we use italics to denote curvilinear vector components. It is common practice, once a curvilinear system is selected, to make these replacements so the italics are no longer needed,

$$(V)_i \rightarrow V_{\xi_i} \qquad (V)'_i \rightarrow V_{\xi_i'} \tag{14.9}$$

In cylindrical coordinates r,θ,z and r',θ',z' this would mean, for example,

$$\begin{aligned} &(V)_1 \rightarrow V_r && (V)'_1 \rightarrow V_{r'} \\ &(V)_2 \rightarrow V_\theta && (V)'_1 \rightarrow V_{\theta'} \\ &(V)_3 \rightarrow V_z && (V)'_2 \rightarrow V_{z'} \quad . \end{aligned} \tag{14.10}$$

Equation Example. Consider now this equation taken from the Section 12.1 summary,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r} + \dot{\mathbf{b}}_{S'} \quad . \qquad \textit{(6.6c)} \tag{14.11}$$

We can view such an equation in any of our four bases as just discussed above,

$$
\begin{aligned}
(v)_i &= (v')_i + \varepsilon_{ijk}(\omega)_j(r)_k + (\dot{b}_{S'})_i && \text{components in basis } \mathbf{e}_i \\
(v)'_i &= (v')'_i + \varepsilon_{ijk}(\omega)'_j(r)'_k + (\dot{b}_{S'})'_i && \text{components in basis } \mathbf{e}'_i \\
(v)_i &= (v')_i + \varepsilon_{ijk}(\omega)_j(r)_k + (\dot{b}_{S'})_i && \text{components in basis } \hat{\boldsymbol{e}}_i \\
(v)'_i &= (v')'_i + \varepsilon_{ijk}(\omega)'_j(r)'_k + (\dot{b}_{S'})'_i \quad . && \text{components in basis } \hat{\boldsymbol{e}}'_i
\end{aligned}
\tag{14.12}
$$

For example, in r,θ,z cylindrical coordinates if we have $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, then $(\omega)_j = \delta_{j3}\omega$, so in the third line above we get

$$\varepsilon_{ijk}(\omega)_j(r)_k = \varepsilon_{ijk}\,\omega\,\delta_{j3}\,(r)_k = \omega\,\varepsilon_{i3k}(r)_k = -\,\omega\,\varepsilon_{ik3}(r)_k$$

so that line becomes

$$(v)_i = (v')_i - \omega\,\varepsilon_{ik3}(r)_k + (\dot{b}_{S'})_i \qquad \text{components in basis } \hat{\boldsymbol{e}}_i$$

or

$$
\begin{aligned}
(v)_1 &= (v')_1 - \omega\,\varepsilon_{123}(r)_2 + (\dot{b}_{S'})_1 \\
(v)_2 &= (v')_2 - \omega\,\varepsilon_{213}(r)_1 + (\dot{b}_{S'})_2 \\
(v)_3 &= (v')_3 - \omega\,\varepsilon_{3k3}(r)_k + (\dot{b}_{S'})_3 = (v')_3 + (\dot{b}_{S'})_3 \quad .
\end{aligned}
\tag{14.13}
$$

This then translates into (since $\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}} = r_r\hat{\mathbf{r}} + r_z\hat{\mathbf{z}}$ and $r_\theta = 0$)

$$
\begin{aligned}
v_r &= v'_r - \omega\, r_\theta + (\dot{b}_{S'})_r = v'_r + (\dot{b}_{S'})_r \\
v_\theta &= v'_\theta + \omega\, r_r + (\dot{b}_{S'})_\theta = v'_\theta + \omega\, r + (\dot{b}_{S'})_\theta \\
v_z &= v'_z + (\dot{b}_{S'})_z \quad .
\end{aligned}
\tag{14.14}
$$

For any Special Case #1 problem (see Section 4.4) one has $\dot{\mathbf{b}}_{S'} = 0$ and the above equations become extremely simple

$$
\begin{aligned}
v_r &= v'_r \\
v_\theta &= v'_\theta + \omega\, r \\
v_z &= v'_z \quad .
\end{aligned}
\tag{14.15}
$$

15. Ant on Turntable Problems

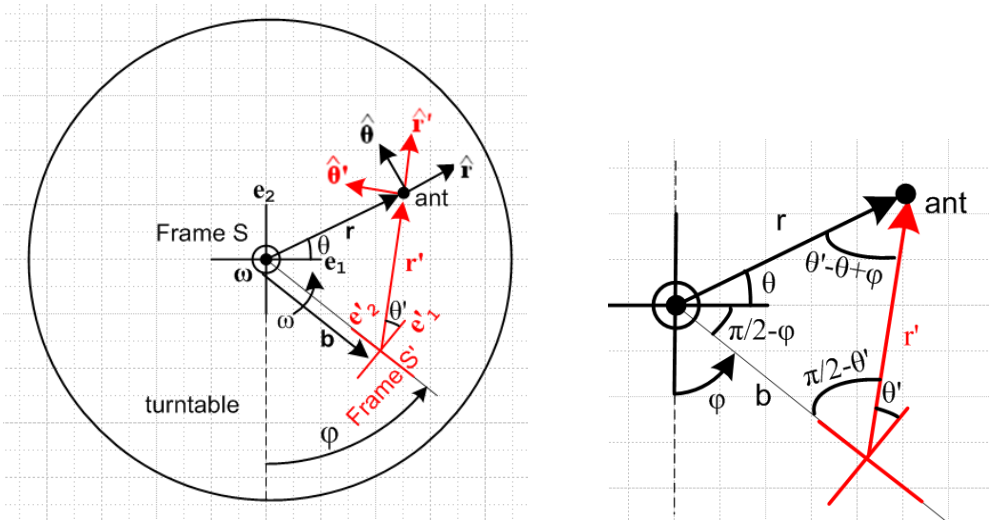
The main purpose of the following four "ant problem" examples is to exercise the results summarized in Sections 12 and 13 above and to demonstrate the use of non-Cartesian coordinates as outlined in Section 14.

Problems 1 and 2 are "forward" problems, while Problems 3 and 4 are "inverse" problems. Problem 4 concludes the analysis of the 4-projectile problem begun in Section 8. Everything is done in "no-swap" notation where Frame S' is the rotating frame.

Some secondary purposes are to provide the reader with many examples of manipulating basis vectors, using the bulletproof vector component notation of Section 1, and applying simple matrix methods.

Kinematics common to all Ant Problems

Consider a turntable occupied by an ant as shown in this drawing. Here Frame S is a fixed frame with origin at the turntable center, while Frame S' (glued to the turntable surface) is a rotating frame.



(15.1)

In Frame S, the vector **r** has coordinates (r,θ) in standard polar coordinates.
In Frame S', the vector **r'** has coordinates (r',θ') in standard polar coordinates.

When φ = 0, red Frame S' lies directly *under* black Frame S and the axes line up. For any angle φ one has **b** = -b **e'**$_2$. Since the rotation axis goes through the origin of Frame S, the turntable problems fall into Special Case #1 of Section 4.4. Basis vectors **e**$_3$ = **e'**$_3$ (not labeled) point to the viewer as does the **ω** vector for ω>0.

The relation between the three angles θ, θ' and φ is complicated and can be indirectly obtained by writing the laws of sines and cosines for the triangle shown on the right above. The left and bottom internal triangle angles are obvious. The top one is then

$$\pi - (\theta + \pi/2 - \varphi) - (\pi/2 - \theta') = \theta'-\theta+\varphi \tag{15.2}$$

None of this angle detail will be needed below (except in a Reader Exercise).

In the first two Problems considered below, an ant executes some crawling motion on the turntable as described by certain **r'**, **v'**, and **a'** in Frame S'. Our task in each problem is to use our Section 12.1 summary results to compute **r**, **v**, and **a** as seen in Frame S and to plot some trajectories **r**(t).

In the third problem, the ant becomes a flying ant doing a straight-line fly-by at constant velocity in Frame S just over the turntable surface, a fly-by described by a certain **r**, **v**, and **a**. This is an example of the Inverse Problem discussed in Section 13 and our goal here is to compute **r'**, **v'**, and **a'** in Frame S' using the equations provided in Section 13.3.

In each frame we define Cartesian and cylindrical coordinates and unit vectors as follows:

Frame S	$r_i = x,y,z$	basis vectors	$\mathbf{e}_i = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$	
	$\xi_i = r,\theta,z$	basis vectors	$\hat{\boldsymbol{e}}_i = \hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$.	
Frame S'	$r'_i = x',y',z'$	basis vectors	$\mathbf{e}'_i = \hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}'$	
	$\xi'_i = r',\theta',z'$	basis vectors	$\hat{\boldsymbol{e}}'_i = \hat{\mathbf{r}}', \hat{\boldsymbol{\theta}}', \hat{\mathbf{z}}'$	(15.3)

We are thus providing a specific example of (14.1) concerning general curvilinear coordinates in two frames of reference.

What do we know about all the basis vectors?

In order to illustrate some of the work of Section 1.1, we provide the reader with a complete set of socket wrenches even though only a few of these tools will actually be used below. All of the following relations can be obtained by inspection from the above figure:

Within Frame S we have

$$\hat{\boldsymbol{e}}_i = R_z(\theta)\mathbf{e}_i \text{ .} \qquad \text{for example} \qquad \hat{\mathbf{r}} = R_z(\theta)\,\hat{\mathbf{x}} \qquad \text{(a)}$$

A corresponding equation applies in Frame S' ,

$$\hat{\boldsymbol{e}}'_i = R_z(\theta')\mathbf{e}'_i \text{ .} \qquad \text{for example} \qquad \hat{\mathbf{r}}' = R_z(\theta')\,\hat{\mathbf{x}}' \qquad \text{(b)}$$

The relation between the Frame S and Frame S' Cartesian unit vectors is

$$\mathbf{e}'_i = R_z(\varphi)\mathbf{e}_i \text{ .} \qquad \text{for example} \qquad \hat{\mathbf{x}}' = R_z(\varphi)\,\hat{\mathbf{x}} \qquad \text{(c)}$$

The relation between the Frame S and Frame S' cylindrical unit vectors is

$$\hat{\boldsymbol{e}}'_i = R_z(\varphi)\hat{\boldsymbol{e}}_i \text{ .} \qquad \text{for example} \qquad \hat{\mathbf{r}}' = R_z(\varphi)\,\hat{\mathbf{r}} \qquad \text{(d)}$$

Relations (d) and (a) can be combined to get

$$\hat{\boldsymbol{e}}'_i = R_z(\varphi+\theta)\mathbf{e}_i \text{ .} \qquad \text{for example} \qquad \hat{\mathbf{r}}' = R_z(\varphi+\theta)\,\hat{\mathbf{x}} \qquad \text{(e)} \qquad (15.4)$$

Writing the basis vector relations in matrix notation

Recall the following theorem of (1.1.1) and (1.1.2) : (we use dummy basis vector names $\mathbf{a}_n$ and $\mathbf{a}'_n$) ,

$$\mathbf{a}_n = R\,\mathbf{a}'_n \qquad \Leftrightarrow \qquad \mathbf{a}_n = \Sigma_m (R^{-1})_{nm}\,\mathbf{a}'_m \tag{15.5}$$

On the left, we rotate vector $\mathbf{a}'_n$ to get vector $\mathbf{a}_n$.
On the right, we express $\mathbf{a}_n$ as a linear combination of the basis vectors $\mathbf{a}'_n$.
Remember that the subscripts on the **a** and **a'** are labels, not components!

Suppose we take the k^{th} component of the equation on the right of (15.5),

$$[\mathbf{a}_n]_k = \Sigma_m (R^{-1})_{nm}\,[\mathbf{a}'_m]_k \quad . \tag{15.6a}$$

One can write this as

$$A_{nk} = \Sigma_m (R^{-1})_{nm}\; A'_{mk} \qquad \text{where} \quad A_{nk} = [\mathbf{a}_n]_k \;\text{ and }\; A'_{nk} = [\mathbf{a}'_n]_k \,. \tag{15.6b}$$

For a matrix A_{nk} one knows that n is the row index and k is the column index. Therefore, saying $A_{nk} = [\mathbf{a}_n]_k$ is the same as saying that the vector $\mathbf{a}_n$ is the n^{th} row of matrix A. Thus we can write (15.6a) in this manner,

$$\begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \end{pmatrix} = R^{-1} \begin{pmatrix} \mathbf{a}'_1 \\ \mathbf{a}'_2 \\ \mathbf{a}'_3 \end{pmatrix} \qquad \Leftrightarrow \qquad \mathbf{a}_n = R\,\mathbf{a}'_n \quad n = 1,2,3 \quad . \tag{15.7}$$

All our rotations of interest in (15.4) are z-rotations which, from (A.1), have the form

$$R_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{A.1}_z$$

Example 1: Apply (15.7) to (15.4e) which says $\hat{\boldsymbol{e}}'_i = R_z(\varphi+\theta)\mathbf{e}_i$:

$$\begin{pmatrix} \hat{\boldsymbol{e}}'_1 \\ \hat{\boldsymbol{e}}'_2 \\ \hat{\boldsymbol{e}}'_3 \end{pmatrix} = R_z(-\theta' - \varphi) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \cos(\theta'+\varphi) & \sin(\theta'+\varphi) & 0 \\ -\sin(\theta'+\varphi) & \cos(\theta'+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{\mathbf{r}}' \\ \hat{\boldsymbol{\theta}}' \\ \hat{\mathbf{z}}' \end{pmatrix} = \begin{pmatrix} \cos(\theta'+\varphi) & \sin(\theta'+\varphi) & 0 \\ -\sin(\theta'+\varphi) & \cos(\theta'+\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} .$$

Writing out the linear combinations, one gets

$$\hat{\mathbf{r}}' = \cos(\theta'+\varphi)\,\hat{\mathbf{x}} + \sin(\theta'+\varphi)\,\hat{\mathbf{y}}$$
$$\hat{\boldsymbol{\theta}}' = -\sin(\theta'+\varphi)\,\hat{\mathbf{x}} + \cos(\theta'+\varphi)\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{z}}' = \hat{\mathbf{z}} \quad . \tag{15.8}$$

Example 2: Apply (15.8) to (15.4c) which says $\mathbf{e}'_i = R_z(\varphi)\mathbf{e}_i$:

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = R_z(-\varphi)\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

or

$$\begin{pmatrix} \hat{\mathbf{x}}' \\ \hat{\mathbf{y}}' \\ \hat{\mathbf{z}}' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \quad .$$

Writing out the linear combinations, one gets

$$\hat{\mathbf{x}}' = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}$$
$$\hat{\mathbf{y}}' = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \qquad // \ \mathbf{e}'_2 = -\sin\varphi\,\mathbf{e}_1 + \cos\varphi\,\mathbf{e}_2$$
$$\hat{\mathbf{z}}' = \hat{\mathbf{z}} \quad . \tag{15.9}$$

We could reduce our 3x3 matrix work to 2x2 for the turntable examples, but other problems require the full 3x3 notation so we maintain it throughout.

Relation between Frame S and Frame S'

Assume at time t = 0 we have $\varphi = \varphi_0$ in Fig (15.1).

If the rotation follows some angular velocity profile $\omega = \omega(t)$, and since $\omega = d\varphi/dt$, one has

$$d\varphi/dt = \omega(t) \qquad \Rightarrow \qquad \varphi(t) = \varphi_0 + \int_0^t \omega(\tau)d\tau \quad . \tag{15.10}$$

For simplicity, we shall assume constant ω in which case,

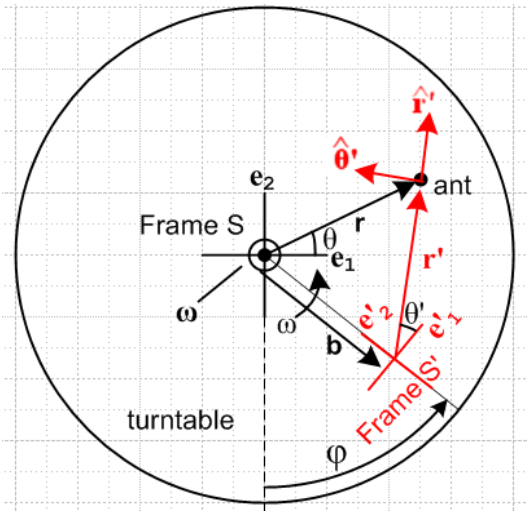
$$\varphi(t) = \varphi_0 + \omega t \quad . \tag{15.11}$$

Motion of vector **b**

From Fig (15.1) and from (15.9) one finds,

$$\mathbf{b}(t) = -b\,\mathbf{e}'_2 = -b\,[-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}] = b\sin\varphi\,\hat{\mathbf{x}} - b\cos\varphi\,\hat{\mathbf{y}} \quad . \tag{15.12}$$

15.1 Problem 1: Ant crawls at constant speed V to the Origin of Frame S'



(15.1.1)

Ant's Motion in Frame S'.

Assume the ant starts at some (r'_0,θ'_0) at t = 0 and crawls with constant speed V toward the S' origin,

$$\mathbf{v}' = -V\hat{\mathbf{r}}' \quad . \tag{15.1.2}$$

We can integrate this within Frame S' (where $\hat{\mathbf{r}}'$ is fixed) to get

$$\mathbf{r}'(t) = \mathbf{r}'_0 - Vt\,\hat{\mathbf{r}}' \qquad \text{where} \quad \mathbf{r}'_0 = r'_0\hat{\mathbf{r}}' = (r'_0,\theta'_0) \ . \tag{15.1.3}$$

The magnitude of $\mathbf{r}'(t)$ is given by

$$r' = r'_0 - Vt \tag{15.1.4}$$

since we shall only be interested in times small enough so r' > 0. The angle θ' never changes, so

$$\theta' = \theta'_0 \quad . \tag{15.1.5}$$

Finally, since V = constant, the acceleration is

$$\mathbf{a}' = 0 \ . \tag{15.1.6}$$

Thus, in line with our Forward Problem statement, these are the given quantities in Frame S' ,

$$\begin{aligned} \mathbf{r}' &= \mathbf{r}'_0 - Vt\,\hat{\mathbf{r}}' \\ \mathbf{v}' &= -V\,\hat{\mathbf{r}}' \\ \mathbf{a}' &= 0 \ . \end{aligned} \tag{15.1.7}$$

Using (15.8) for $\hat{\mathbf{r}}'$, we can write $\mathbf{v}' = -V\hat{\mathbf{r}}'$ as

$$\mathbf{v}' = -V\cos(\theta'+\varphi)\,\hat{\mathbf{x}} - V\sin(\theta'+\varphi)\,\hat{\mathbf{y}} \quad . \tag{15.1.8}$$

Our goal is to compute **r**, **v** and **a** as seen in Frame S.

Trajectory **r**(t) of the ant in Frame S

Above we found that

$$\mathbf{b}(t) = b\sin\varphi\,\hat{\mathbf{x}} - b\cos\varphi\,\hat{\mathbf{y}} \tag{15.12}$$

$$\hat{\mathbf{r}}' = \cos(\theta'+\varphi)\,\hat{\mathbf{x}} + \sin(\theta'+\varphi)\,\hat{\mathbf{y}} \quad . \tag{15.8}$$

Recall from (12.2.1a) that,

$$\mathbf{r} = \mathbf{b} + \mathbf{r}' = \mathbf{b} + r'\,\hat{\mathbf{r}}' \,. \tag{12.2.1a}$$

Therefore from (15.12) and (15.8) quoted just above we can write

$$\mathbf{r}(t) = \left(b\sin\varphi\,\hat{\mathbf{x}} - b\cos\varphi\,\hat{\mathbf{y}} \right) + r'\left(\cos(\theta'+\varphi)\,\hat{\mathbf{x}} + \sin(\theta'+\varphi)\,\hat{\mathbf{y}}\right)$$

$$= [b\sin\varphi + r'\cos(\theta'+\varphi)]\,\hat{\mathbf{x}} + [-b\cos\varphi + r'\sin(\theta'+\varphi)]\,\hat{\mathbf{y}} \quad .$$

Setting $r' = (r'_0 - Vt)$ by (15.1.4), and thinking of $\varphi = \varphi(t)$ (ie, a function of time) as in (15.10,11),

$$\mathbf{r}(t) = [\,b\sin\varphi + (r'_0 - Vt)\cos(\theta'+\varphi)]\,\hat{\mathbf{x}} + [\,-b\cos\varphi + (r'_0 - Vt)\sin(\theta'+\varphi)]\,\hat{\mathbf{y}}$$

or

$$\mathbf{r}(t) = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$$

where (15.1.9)

$$x = b\sin\varphi + (r'_0 - Vt)\cos(\theta'+\varphi)$$
$$y = -b\cos\varphi + (r'_0 - Vt)\sin(\theta'+\varphi) \quad .$$

This **r**(t) then is the trajectory of the ant in Frame S.

Velocity **v**(t) of the ant in Frame S

Since the turntable falls into our Special Case #1 of Section 4.4 (**ω** through origin of Frame S), we know that $\dot{\mathbf{b}}_{S'} = 0$ (vector **b** is soldered to the Frame S' unit vectors). From (12.1.2b) we have,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \tag{12.1.2b}$$

which then says, setting $\dot{\mathbf{b}}_{S'} = 0$,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} \quad . \tag{15.1.10}$$

The first term $\mathbf{v}'$ we will replace by (15.1.8); the second we compute,

$$\mathbf{v}' = -V\cos(\theta'+\varphi)\,\hat{\mathbf{x}} - V\sin(\theta'+\varphi)\,\hat{\mathbf{y}} \tag{15.1.8}$$

$$\boldsymbol{\omega} \times \mathbf{r} = [\omega\hat{\mathbf{z}}] \times [x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}] = \omega x\hat{\mathbf{y}} - \omega y\hat{\mathbf{x}} \quad . \tag{15.1.11}$$

Therefore (15.1.10) says

$$\mathbf{v} = [-V\cos(\theta'+\varphi)\,\hat{\mathbf{x}} - V\sin(\theta'+\varphi)\,\hat{\mathbf{y}}] + \omega x\hat{\mathbf{y}} - \omega y\hat{\mathbf{x}}$$

$$= [-V\cos(\theta'+\varphi) - \omega y]\,\hat{\mathbf{x}} + [-V\sin(\theta'+\varphi) + \omega x]\,\hat{\mathbf{y}}$$

or

$$\mathbf{v} = v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}}$$

where

$$v_x = -V\cos(\theta'+\varphi) - \omega y$$
$$v_y = -V\sin(\theta'+\varphi) + \omega x \tag{15.1.12}$$

where x,y are given in (15.1.9). We can go ahead and insert x and y from there to get

$$-\omega y = -\omega[-b\cos\varphi + (r'_0 - Vt)\sin(\theta'+\varphi)] = \omega b\cos\varphi - \omega(r'_0 - Vt)\sin(\theta'+\varphi)$$

$$\omega x = \omega[b\sin\varphi + (r'_0 - Vt)\cos(\theta'+\varphi)] = \omega b\sin\varphi + \omega(r'_0 - Vt)\cos(\theta'+\varphi)$$

so

$$v_x = -V\cos(\theta'+\varphi) + \omega b\cos\varphi - \omega(r'_0 - Vt)\sin(\theta'+\varphi)$$
$$v_y = -V\sin(\theta'+\varphi) + \omega b\sin\varphi + \omega(r'_0 - Vt)\cos(\theta'+\varphi) \quad . \tag{15.1.13}$$

This $\mathbf{v}(t)$ then is the velocity of the ant in Frame S.

Acceleration $\mathbf{a}(t)$ of the ant in Frame S

From (12.1.2e) we find that

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \ddot{\mathbf{b}}_{S'} \tag{12.1.2e}$$

but in this Special Case #1 problem we have $\dot{\mathbf{b}}_{S'} = 0$ and $\ddot{\mathbf{b}}_{S'} = 0$ so

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \,. \tag{15.1.14}$$

We shall ponder the terms one at a time.

As noted in (15.1.6), $\mathbf{a}' = 0$.

Our turntable is restricted to have $\dot{\boldsymbol{\omega}} = \dot{\omega}\,\hat{\mathbf{z}}$ so, similar to (15.1.11) above, we find

$$\dot{\boldsymbol{\omega}} \times \mathbf{r} = \dot{\omega}x\hat{\mathbf{y}} - \dot{\omega}y\hat{\mathbf{x}} \quad . \tag{15.1.15}$$

Next, we install (15.1.8) for $\mathbf{v}'$ to get

$$\boldsymbol{\omega} \times \mathbf{v}' = [\omega\hat{\mathbf{z}}] \times [-V\cos(\theta'+\varphi)\,\hat{\mathbf{x}} - V\sin(\theta'+\varphi)\,\hat{\mathbf{y}}] = -\omega V\cos(\theta'+\varphi)\,\hat{\mathbf{y}} + \omega V\sin(\theta'+\varphi)\,\hat{\mathbf{x}}$$

$$= \omega V\sin(\theta'+\varphi)\,\hat{\mathbf{x}} - \omega V\cos(\theta'+\varphi)\,\hat{\mathbf{y}} \quad . \tag{15.1.16}$$

With (15.1.11) the last term of (15.1.14) becomes

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = [\omega\hat{\mathbf{z}}] \times [\omega x\hat{\mathbf{y}} - \omega y\hat{\mathbf{x}}] = -\omega^2 x\hat{\mathbf{x}} - \omega^2 y\hat{\mathbf{y}} \quad . \qquad // = -\omega^2\,\mathbf{r}\text{, centripetal accel.} \tag{15.1.17}$$

Combining all the terms then gives

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$= 0 + (\dot{\omega}x\hat{\mathbf{y}} - \dot{\omega}y\hat{\mathbf{x}}) + 2\omega V\sin(\theta'+\varphi)\,\hat{\mathbf{x}} - 2\omega V\cos(\theta'+\varphi)\,\hat{\mathbf{y}} - \omega^2 x\hat{\mathbf{x}} - \omega^2 y\hat{\mathbf{y}}$$

$$= [-\dot{\omega}y + 2\omega V\sin(\theta'+\varphi) - \omega^2 x]\,\hat{\mathbf{x}} + [\dot{\omega}x - 2\omega V\cos(\theta'+\varphi) - \omega^2 y]\hat{\mathbf{y}}$$

or

$$\mathbf{a} = a_x\,\hat{\mathbf{x}} + a_y\,\hat{\mathbf{y}}$$

where (15.1.18)

$$a_x = -\dot{\omega}y + 2\omega V\sin(\theta'+\varphi) - \omega^2 x$$

$$a_y = \dot{\omega}x - 2\omega V\cos(\theta'+\varphi) - \omega^2 y$$

where x,y are given by (15.1.9).

This $\mathbf{a}(t)$ then is the acceleration of the ant in Frame S.

Summary of the Solution to Problem 1

$$\mathbf{r}(t) = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$$

where *(15.1.9)*

$$x = b\sin\varphi + (r'_0 - Vt)\cos(\theta'+\varphi)$$
$$y = -b\cos\varphi + (r'_0 - Vt)\sin(\theta'+\varphi)$$

$$\mathbf{v} = v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}}$$

where *(15.1.12)*

$$v_x = -V\cos(\theta'+\varphi) - \omega y$$
$$v_y = -V\sin(\theta'+\varphi) + \omega x$$

$$\mathbf{a} = a_x\,\hat{\mathbf{x}} + a_y\,\hat{\mathbf{y}}$$

where *(15.1.18)*

$$a_x = -\dot{\omega}y + 2\omega V\sin(\theta'+\varphi) - \omega^2 x$$
$$a_y = \dot{\omega}x - 2\omega V\cos(\theta'+\varphi) - \omega^2 y$$

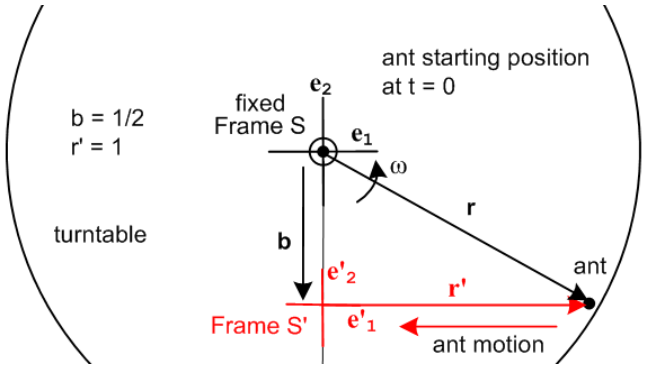
and

$$\varphi = \varphi(t) = \varphi_0 + \int_0^t \omega(\tau)d\tau = \varphi_0 + \omega t \quad \text{for constant } \omega \ . \qquad (15.10) \qquad (15.1.19)$$

The x and y in equations (15.1.19) are given by (15.1.9), and $\theta' = \theta'_0$ by (15.1.5).

Selected Plots

We set $\varphi_0 = 0$ so Frame S' starts directly below Frame S and is aligned with it, so then $\varphi = \omega t$.
We set $\theta'_0 = \theta' = 0$ so our ant approaches the Frame S' origin along the $\mathbf{e}'_1$ axis :



(15.1.20)

With these assumptions (15.1.9) becomes

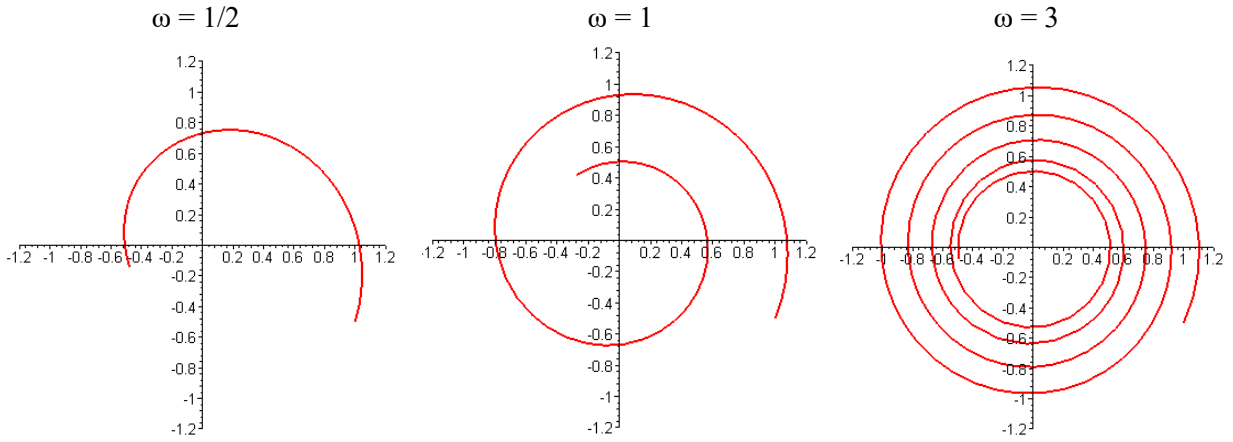
$$\mathbf{r}(t) = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$$

where (15.1.21)

$$x = b\sin(\omega t) + (r'_0 - Vt)\cos(\omega t)$$
$$y = -b\cos(\omega t) + (r'_0 - Vt)\sin(\omega t) \ .$$

Each plot is finite because the trip is over when the ant reaches the S' origin at $t_{max} = r'_0/V$.

We set b = 0.5, $r'_0 = 1$, V = 0.1. The ant therefore starts at (x,y) = $(r'_0,-b)$ = (1,-0.5). Here are trajectory plots for various values of ω :



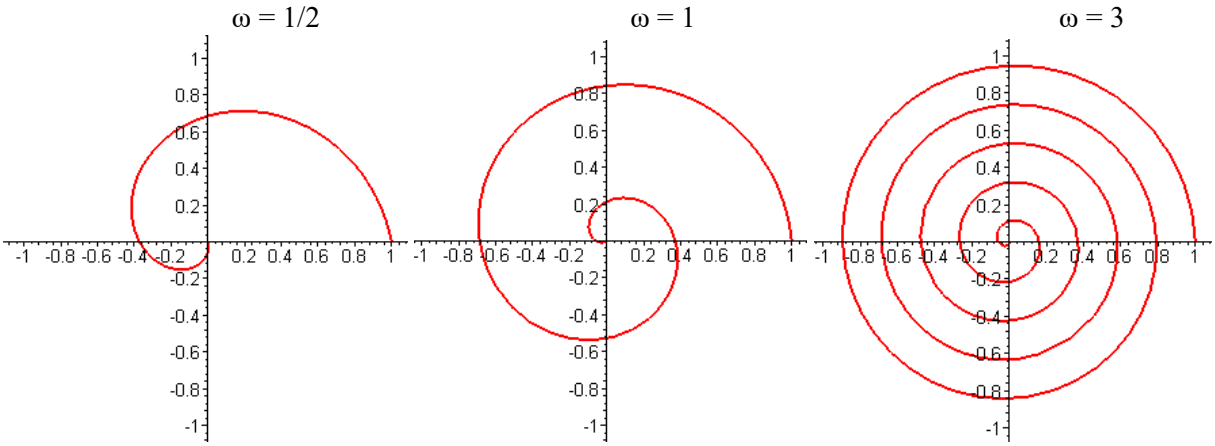
b = 0.5, $r'_0 = 1$, V = 0.1 (15.1.22)

The middle plot was generated by the following Maple code based on (15.1.21), where R ≡ r'_0,

```
b := 0.5: R := 1: V := 0.1:
tmax := R/V: omega := 1:
x :=  b*sin(omega * t) + (R-V*t)*cos(omega * t):
y := -b*cos(omega * t) + (R-V*t)*sin(omega * t):
plot([x,y,t=0..tmax],scaling = CONSTRAINED, view =[-1.2..1.2,-1.2..1.2] );
```

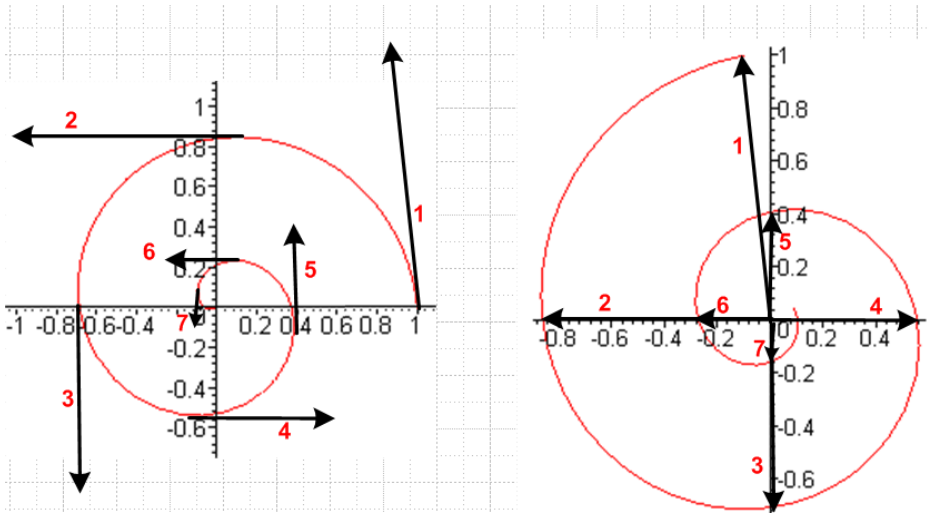
Similar code is used to make all the other plots below.

Setting b = 0, we get these more traditional plots (Frame S and Frame S' origins now coincide) :



b = 0, $r'_0 = 1$, V = 0.1 (15.1.23)

Next we plot the velocity **v** from (15.1.12) only for the middle ω = 1 case above on the right below, with the corresponding trajectory plot **r** on the left:

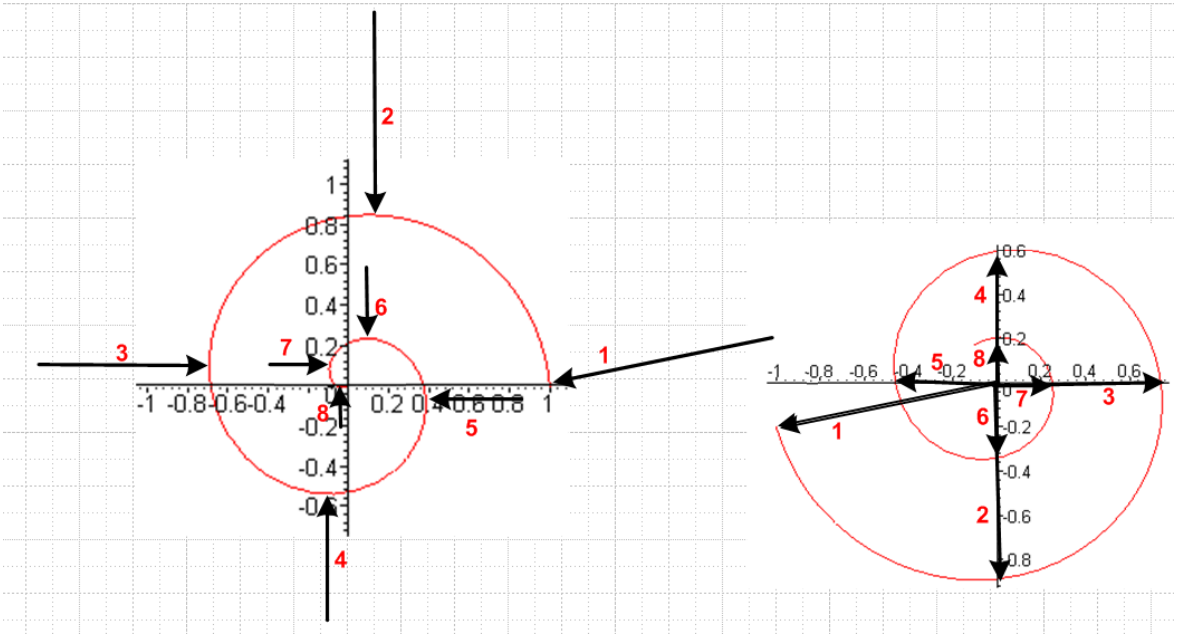


(15.1.24)

ω = 1 plot of **r**(t) ω = 1 plot of **v**(t)

For seven different (but unknown) times, we draw the velocity vector on the right and then transfer it to where we think it ought to go on the trajectory plot on the left. Things at least seem reasonable. A proper visual check would require a program to automate the above process.

Next we plot the acceleration **a** on the right below using (15.1.18), again for ω = 1, with the corresponding trajectory plot **r** on the left:



ω = 1 plot of **r**(t) ω = 1 plot of **a**(t) (15.1.25)

For the same seven (still unknown) times plus one more, we draw the acceleration vector on the right and transfer it to where we think it ought to go on the left. Again, this is just a sanity check to make sure things seem reasonable.

Reader Exercise: From (15.1.2) one has $\mathbf{v}' = -V\hat{\mathbf{r}}'$ so that

$$v'_r = \mathbf{v}' \bullet \hat{\mathbf{r}} = -V\hat{\mathbf{r}}' \bullet \hat{\mathbf{r}} = -V\cos(\theta'-\theta+\varphi)$$

according to Fig (15.1), where v'_r is the radial component of the ant velocity $\mathbf{v}'$ in polar coordinates. On the other hand, the radial component of $\mathbf{v}$ is given in (15.1.13) as,

$$v_r = \hat{\mathbf{r}} \bullet \mathbf{v} = \hat{\mathbf{r}} \bullet [v_x \hat{\mathbf{x}} + v_y \hat{\mathbf{y}}] = v_x\hat{\mathbf{r}} \bullet \hat{\mathbf{x}} + v_y\hat{\mathbf{r}} \bullet \hat{\mathbf{y}} = v_x\cos\theta + v_y\sin\theta$$

where

$$v_x = -V\cos(\theta'+\varphi) + \omega b\cos\varphi - \omega(r'_0 - Vt)\sin(\theta'+\varphi)$$
$$v_y = -V\sin(\theta'+\varphi) + \omega b\sin\varphi + \omega(r'_0 - Vt)\cos(\theta'+\varphi) \quad . \qquad (15.1.13)$$

Looking at the above expressions for v'_r and v_r, it seems unlikely that they could be equal since v_r involves terms linear in time t and is a function of ω, b, r'_0 whereas v'_r does not seem to involve these terms and parameters at all. Yet equation (14.15), which applies to any Special Case #1 problem like Problem 1, claims $v_r = v'_r$. The Exercise is to demonstrate that in fact $v_r = v'_r$.

Hints:
(1) Set $r'_0 - Vt = r'$ and show that $v_r = \omega b\cos(\varphi-\theta) - \omega r'\sin(\theta'+\varphi-\theta) - V\cos(\theta'+\varphi-\theta)$.
(2) Show that the first two terms cancel due to a Law of Sines for Fig (15.1). QED.

Results expressed in matrix notation

In the case that $b = 0$, $\dot{\omega} = 0$, $\varphi_0 = 0$, $\varphi = \omega t$ and $\theta'_0 = \theta' = 0$ (triplet of plots in (15.1.23)) we can summarize our results as follows:

$$\mathbf{r}(t) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$

where *(15.1.9)* (a)

$$x = (r'_0 - Vt)\cos(\omega t)$$
$$y = (r'_0 - Vt)\sin(\omega t)$$

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

where *(15.1.12) + (15.1.13)* (b)

$$v_x = -V\cos(\omega t) - \omega(r'_0 - Vt)\sin(\omega t)$$
$$v_y = -V\sin(\omega t) + \omega(r'_0 - Vt)\cos(\omega t)$$

$$\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}}$$

where *(15.1.18)* (c)

$$a_x = +2\omega V\sin(\omega t) - \omega^2(r'_0 - Vt)\cos(\omega t)$$
$$a_y = -2\omega V\cos(\omega t) - \omega^2(r'_0 - Vt)\sin(\omega t) \quad . \qquad (15.1.26)$$

These equations can be written in matrix notation as follows,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} r'_0 - Vt \\ 0 \\ z \end{pmatrix} = R_z(\omega t) \begin{pmatrix} r'_0 - Vt \\ 0 \\ z \end{pmatrix}$$

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -V \\ \omega(r'_0 - Vt) \\ v_z \end{pmatrix} = R_z(\omega t) \begin{pmatrix} -V \\ \omega(r'_0 - Vt) \\ v_z \end{pmatrix}$$

$$\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\omega^2(r'_0 - Vt) \\ -2\omega V \\ a_z \end{pmatrix} = R_z(\omega t) \begin{pmatrix} -\omega^2(r'_0 - Vt) \\ -2\omega V \\ a_z \end{pmatrix} . \qquad (15.1.27)$$

Using more systematic notation, we rewrite the above three matrix equations as,

$$\mathbf{r} = (r)_i \mathbf{e}_i \qquad \text{where} \qquad \begin{pmatrix} (r)_1 \\ (r)_2 \\ (r)_3 \end{pmatrix} = R_z(\omega t) \begin{pmatrix} r'_0 - Vt \\ 0 \\ (r)_3 \end{pmatrix}$$

$$\mathbf{v} = (v)_i \mathbf{e}_i \qquad \text{where} \qquad \begin{pmatrix} (v)_1 \\ (v)_2 \\ (v)_3 \end{pmatrix} = R_z(\omega t) \begin{pmatrix} -V \\ \omega(r'_0 - Vt) \\ (v)_3 \end{pmatrix}$$

$$\mathbf{a} = (a)_i \mathbf{e}_i \qquad \text{where} \qquad \begin{pmatrix} (a)_1 \\ (a)_2 \\ (a)_3 \end{pmatrix} = R_z(\omega t) \begin{pmatrix} -\omega^2(r'_0 - Vt) \\ -2\omega V \\ (a)_3 \end{pmatrix} . \qquad (15.1.28)$$

The components shown in these matrix equations describe the spiral solution path, velocity and acceleration of our Problem 1 ant in Frame S. It happens that $(r)_3 = (v)_3 = (a)_3 = 0$.

15.2 Problem 2: Ant spirals in at constant V and Ω to the Origin of Frame S'

Ant's Motion in Frame S'

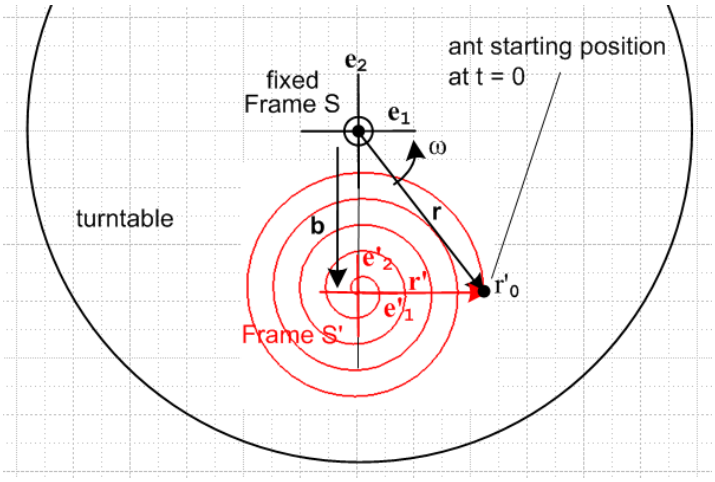
In order to challenge our formalism a bit, the ant now crawls on the turntable in a more complicated manner. The ant in Frame S' starts at $\mathbf{r'}_0 = r'_0\,\hat{\mathbf{x}}'$ and crawls in a spiral path toward the S' origin. This spiral path is the output of Problem 1 with the Problem 1 parameters set to b = 0, $\dot{\omega} = 0$, $\varphi_0 = 0$ and $\theta'_0 = 0$. The ant moves at constant radial speed V toward the S' origin while at the same time rotating CCW at constant Ω about that origin. In order to find **r'**, **v'** and **a'** for this problem, we merely adjust the results stated in (15.1.28) by taking $\omega \to \Omega$ and priming appropriate objects.

$$\mathbf{r'} = (r')'_i\mathbf{e'}_i \quad \text{where} \quad \begin{pmatrix}(r')'_1\\(r')'_2\\(r')'_3\end{pmatrix} = R_z(\Omega t)\begin{pmatrix}r'_0\text{-}Vt\\0\\(r')'_3\end{pmatrix}$$

$$\mathbf{v'} = (v')'_i\mathbf{e'}_i \quad \text{where} \quad \begin{pmatrix}(v')'_1\\(v')'_2\\(v')'_3\end{pmatrix} = R_z(\Omega t)\begin{pmatrix}-V\\\Omega(r'_0\text{-}Vt)\\(v')'_3\end{pmatrix}$$

$$\mathbf{a'} = (a')'_i\mathbf{e'}_i \quad \text{where} \quad \begin{pmatrix}(a')'_1\\(a')'_2\\(a')'_3\end{pmatrix} = R_z(\Omega t)\begin{pmatrix}-\Omega^2(r'_0\text{-}Vt)\\-2\Omega V\\(a')'_3\end{pmatrix}. \qquad (15.2.1)$$

This ant path in Frame S' has the following general appearance (depending on parameters),



(15.2.2)

We first wish to know the components of **r'**, **v'** and **a'** on the $\mathbf{e_n}$ basis vectors. This problem was addressed in (1.2.7) which we quote

$$(a')_i = (R^{-1})_{ij}(a')'_j \qquad \Leftrightarrow \qquad \begin{pmatrix}(a')_1\\(a')_2\\(a')_3\end{pmatrix} = R^{-1}\begin{pmatrix}(a')'_1\\(a')'_2\\(a')'_3\end{pmatrix} . \qquad (1.2.7)$$

where (15.2.3)

$$\mathbf{e}_n = R\,\mathbf{e}'_n \qquad n = 1,2,3 \qquad \text{or} \qquad (\mathbf{e}_n)_i = R_{ij}(\mathbf{e}'_n)_j \quad . \qquad (1.1.1)$$

In our application here, we know from (15.4c) that**,**

$$\mathbf{e}'_n = R_z(\varphi)\mathbf{e}_n \quad \Rightarrow \quad \mathbf{e}_n = R_z(-\varphi)\mathbf{e}'_n \quad \Rightarrow \quad R = R_z(-\varphi) \quad \Rightarrow \quad R^{-1} = R_z(\varphi) \ . \qquad (15.2.4)$$

Therefore, first setting $\mathbf{a}' = \mathbf{r}'$, we find from (15.2.3) and (15.2.1) that

$$\begin{pmatrix}(r')_1\\(r')_2\\(r')_3\end{pmatrix} = R_z(\varphi)\begin{pmatrix}(r')'_1\\(r')'_2\\(r')'_3\end{pmatrix} = R_z(\varphi)R_z(\Omega t)\begin{pmatrix}r'_0\text{-}Vt\\0\\(r')'_3\end{pmatrix} = R_z(\Omega t+\varphi)\begin{pmatrix}r'_0\text{-}Vt\\0\\(r')'_3\end{pmatrix}$$

$$= \begin{pmatrix}\cos(\Omega t+\varphi) & -\sin(\Omega t+\varphi) & 0\\ \sin(\Omega t+\varphi) & \cos(\Omega t+\varphi) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}r'_0\text{-}Vt\\0\\(r')'_3\end{pmatrix} \qquad \mathbf{r}' = (r')_i\mathbf{e}_i \ . \qquad (15.2.5)$$

Consider next $\mathbf{a}' = \mathbf{v}'$ and $\mathbf{a}' = \mathbf{a}'$ and use the column vectors on the right in (15.2.1) to get,

$$\begin{pmatrix}(v')_1\\(v')_2\\(v')_3\end{pmatrix} = \begin{pmatrix}\cos(\Omega t+\varphi) & -\sin(\Omega t+\varphi) & 0\\ \sin(\Omega t+\varphi) & \cos(\Omega t+\varphi) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-V\\ \Omega(r'_0\text{-}Vt)\\(v')'_3\end{pmatrix} \qquad \mathbf{v}' = (v')_i\mathbf{e}_i$$

$$\begin{pmatrix}(a')_1\\(a')_2\\(a')_3\end{pmatrix} = \begin{pmatrix}\cos(\Omega t+\varphi) & -\sin(\Omega t+\varphi) & 0\\ \sin(\Omega t+\varphi) & \cos(\Omega t+\varphi) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-\Omega^2(r'_0\text{-}Vt\ )\\ -2\Omega V\\(a')'_3\end{pmatrix} \qquad \mathbf{a}' = (a')_i\mathbf{e}_i \quad . \qquad (15.2.6)$$

Then do the matrix multiplication to obtain,

$$(r')_1 = (r'_0\text{-}Vt)\cos(\Omega t+\varphi)$$
$$(r')_2 = (r'_0\text{-}Vt)\sin(\Omega t+\varphi)$$

$$(v')_1 = -V\cos(\Omega t+\varphi) - \Omega(r'_0\text{-}Vt)\sin(\Omega t+\varphi)$$
$$(v')_2 = -V\sin(\Omega t+\varphi) + \Omega(r'_0\text{-}Vt)\cos(\Omega t+\varphi)$$

$$(a')_1 = -\Omega^2(r'_0\text{-}Vt\ )\cos(\Omega t+\varphi) + 2\Omega V\sin(\Omega t+\varphi)$$
$$(a')_2 = -\Omega^2(r'_0\text{-}Vt\ )\sin(\Omega t+\varphi) - 2\Omega V\cos(\Omega t+\varphi) \ . \qquad (15.2.7)$$

Using our less formal notation we have now shown that the trajectory of our spiraling ant in Frame S' can be expressed in terms of Frame S basis vectors as follows (later we will set $\varphi = \omega t$) ,

$$\mathbf{r'}(t) = (r')_x\,\hat{\mathbf{x}} + (r')_y\,\hat{\mathbf{y}}$$

where (a)

$$(r')_x = (r'_0 - Vt)\cos(\varphi + \Omega t)$$
$$(r')_y = (r'_0 - Vt)\sin(\varphi + \Omega t)$$

$$\mathbf{v'} = (v')_x\,\hat{\mathbf{x}} + (v')_y\,\hat{\mathbf{y}}$$

where (b)

$$(v')_x = -V\cos(\varphi + \Omega t) - \Omega\,(r'_0 - Vt)\sin(\varphi + \Omega t)$$
$$(v')_y = -V\sin(\varphi + \Omega t) + \Omega\,(r'_0 - Vt)\cos(\varphi + \Omega t)$$

$$\mathbf{a'} = (a')_x\,\hat{\mathbf{x}} + (a')_y\,\hat{\mathbf{y}}$$

where (c)

$$(a')_x = 2\Omega V\sin(\varphi + \Omega t) - \Omega^2(r'_0 - Vt)\cos(\varphi + \Omega t)$$
$$(a')_y = -2\Omega V\cos(\varphi + \Omega t) - \Omega^2(r'_0 - Vt)\sin(\varphi + \Omega t)\quad . \qquad (15.2.8)$$

Trajectory **r**(t) of the ant in Frame S

According to (15.2.8a),

$$\mathbf{r'}(t) = (r'_0 - Vt)\,[\,\cos(\varphi + \Omega t)\,\hat{\mathbf{x}} + \sin(\varphi + \Omega t)\,\hat{\mathbf{y}}\,]\quad . \qquad \textit{(15.2.8a)} \qquad (15.2.9)$$

From (12.1.2a) and then from (15.12) we have

$$\mathbf{r} = \mathbf{b} + \mathbf{r'} \qquad \textit{(12.1.2a)}$$

$$\mathbf{b}(t) = b\sin\varphi\,\hat{\mathbf{x}} - b\cos\varphi\,\hat{\mathbf{y}}\ . \qquad \textit{(15.12)}$$

Therefore, installing (15.2.9) for **r'** and just above for **b**,

$$\mathbf{r}(t) = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$$

where (15.2.10)

$$x = b\sin\varphi + (r'_0 - Vt)\cos(\varphi + \Omega t)$$
$$y = -b\cos\varphi + (r'_0 - Vt)\sin(\varphi + \Omega t)\ .$$

The $\Omega = 0$ limit of this result agrees with the $\theta' = 0$ limit of (15.1.9), the Problem 1 trajectory.

This **r**(t) then is the trajectory of the ant in Frame S.

Velocity **v**(t) of the ant in Frame S

Start with two equations used in Problem 1,

$$\mathbf{v} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r} \qquad \textit{(15.1.10)}$$
$$\boldsymbol{\omega} \times \mathbf{r} = [\omega\hat{\mathbf{z}}] \times [x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}\,] = \omega x\hat{\mathbf{y}} - \omega y\hat{\mathbf{x}}\quad . \qquad \textit{(15.1.11)}$$

Then use **v**' from (15.2.8b) in (15.1.10) just above to get

$$\mathbf{v} = v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}}$$

where (15.2.11)

$$v_x = -V\cos(\varphi + \Omega t) - \Omega\,(r'_0 - Vt)\sin(\varphi + \Omega t) - \omega y$$
$$v_y = -V\sin(\varphi + \Omega t) + \Omega\,(r'_0 - Vt)\cos(\varphi + \Omega t) + \omega x \quad .$$

Finally, insert (15.2.10) for x and y so that,

$$v_x = -V\cos(\varphi + \Omega t) - \Omega\,(r'_0 - Vt)\sin(\varphi + \Omega t) - \omega[-b\cos\varphi + (r'_0 - Vt)\sin(\varphi + \Omega t)]$$
$$v_y = -V\sin(\varphi + \Omega t) + \Omega\,(r'_0 - Vt)\cos(\varphi + \Omega t) + \omega[b\sin\varphi + (r'_0 - Vt)\cos(\varphi + \Omega t)]$$

or

$$v_x = -V\cos(\varphi + \Omega t) - (\omega + \Omega)\,(r'_0 - Vt)\sin(\varphi + \Omega t) + \omega b\cos\varphi$$
$$v_y = -V\sin(\varphi + \Omega t) + (\omega + \Omega)\,(r'_0 - Vt)\cos(\varphi + \Omega t) + \omega b\sin\varphi \quad . \qquad (15.2.12)$$

The $\Omega = 0$ limit of these last equations gives the $\theta' = 0$ limit of (15.1.13).

This **v**(t) then is the velocity of the ant in Frame S.

<u>Acceleration **a**(t) of the ant in Frame S</u>

Start again with (15.1.14),

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad \textit{(15.1.14)} \qquad (15.2.13)$$

The first term is given by (15.2.8c)

$$\mathbf{a}' = (a')_x\,\hat{\mathbf{x}} + (a')_y\,\hat{\mathbf{y}}$$

where *(15.2.8c)*

$$(a')_x = 2\Omega V\sin(\varphi + \Omega t) - \Omega^2(r'_0 - Vt)\cos(\varphi + \Omega t)$$
$$(a')_y = -2\Omega V\cos(\varphi + \Omega t) - \Omega^2(r'_0 - Vt)\sin(\varphi + \Omega t) \quad .$$

The 2nd and 4th terms we obtain by quoting these results from the previous problem,

$$\dot{\boldsymbol{\omega}} \times \mathbf{r} = \dot{\omega}x\hat{\mathbf{y}} - \dot{\omega}y\hat{\mathbf{x}} \qquad \textit{(15.1.15)}$$

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = [\omega\hat{\mathbf{z}}] \times [\omega x\hat{\mathbf{y}} - \omega y\hat{\mathbf{x}}] = -\omega^2 x\hat{\mathbf{x}} - \omega^2 y\hat{\mathbf{y}} \quad . \qquad // = -\omega^2\,\mathbf{r}\text{, centripetal accel.} \qquad \textit{(15.1.17)}$$

The third term of (15.2.13) is

$$2\boldsymbol{\omega} \times \mathbf{v}' = 2[\omega\hat{\mathbf{z}}] \times [(v')_x\,\hat{\mathbf{x}} + (v')_y\,\hat{\mathbf{y}}] = 2\omega(v')_x\,\hat{\mathbf{y}} - 2\omega\,(v')_y\,\hat{\mathbf{x}} \quad .$$

Adding these terms one can rewrite (15.2.13) as,

$$\mathbf{a} = a_x\,\hat{\mathbf{x}} + a_y\,\hat{\mathbf{y}}$$

where (15.2.14)

$$a_x = 2\Omega V\sin(\varphi + \Omega t) - \Omega^2(r'_0 - Vt)\cos(\varphi + \Omega t) - \dot{\omega}y - \omega^2 x - 2\omega\,(v')_y$$

$$a_y = -2\Omega V\cos(\varphi + \Omega t) - \Omega^2(r'_0 - Vt)\sin(\varphi + \Omega t) + \dot{\omega}x - \omega^2 y + 2\omega(v')_x$$

where

$$x = b\sin\varphi + (r'_0 - Vt)\cos(\varphi + \Omega t) \qquad \textit{(15.2.10)}$$

$$y = -b\cos\varphi + (r'_0 - Vt)\sin(\varphi + \Omega t)$$

and

$$(v')_x = -V\cos(\varphi + \Omega t) - \Omega\,(r'_0 - Vt)\sin(\varphi + \Omega t)$$

$$(v')_y = -V\sin(\varphi + \Omega t) + \Omega\,(r'_0 - Vt)\cos(\varphi + \Omega t)\ . \qquad \textit{(15.2.8b)}$$

This **a**(t) then is the acceleration of the ant in Frame S.

The result is admittedly a bit complicated, but the point is that we were able to obtain the result using our Section 12 summary equations without too much effort. [See "The Hard Way" in Section 15.4 below. We have not dealt with any differential equations in obtaining the above results.]

Trajectory Plots

We set $\varphi_0 = 0$ so Frame S' starts directly below Frame S and is aligned with it, and then $\varphi = \omega t$. Equation (15.2.10) then reads

$$\mathbf{r}(t) = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}$$

where (15.2.15)

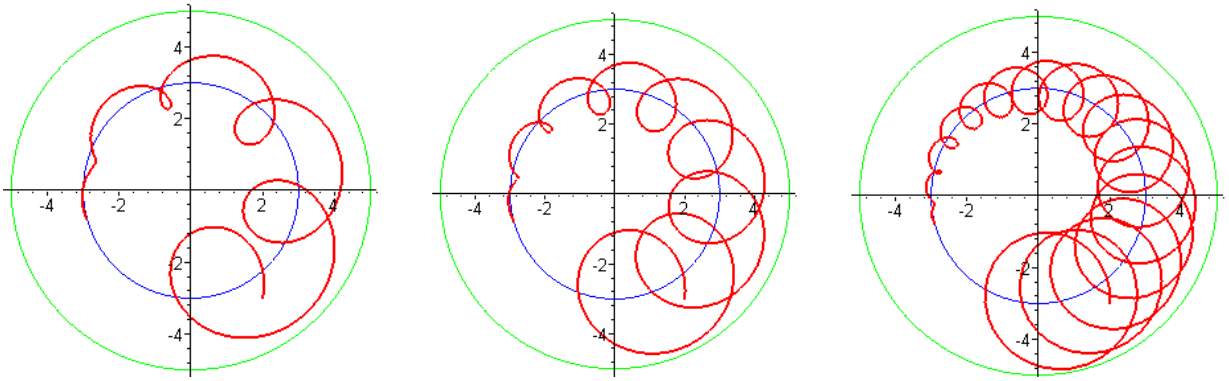
$$x = b\sin(\omega t) + (r'_0 - Vt)\cos(\omega t + \Omega t)$$

$$y = -b\cos(\omega t) + (r'_0 - Vt)\sin(\omega t + \Omega t)\ .$$

Each plot is finite because the trip is over when the ant reaches the S' origin at $t_{max} = r'_0/V$.

We set $b = 3$, $r'_0 = 2$, $V = 0.4$ and $\omega = 1$. The ant therefore starts at $(x,y) = (r'_0,-b) = (2,-3)$. Here are trajectory plots for various values of Ω :

Ω = 6 Ω = 10 Ω = 20



b = 3, r'_0 = 2, V = 0.4, ω = 1 (15.2.16)

The blue circles have radius b = 3 (origin of Frame S'), and the green circles have radius b+r'_0 = 5.

These trajectories should seem quite reasonable to the reader, knowing what that ant is up to in Frame S', shown generically in Fig (15.2.2).

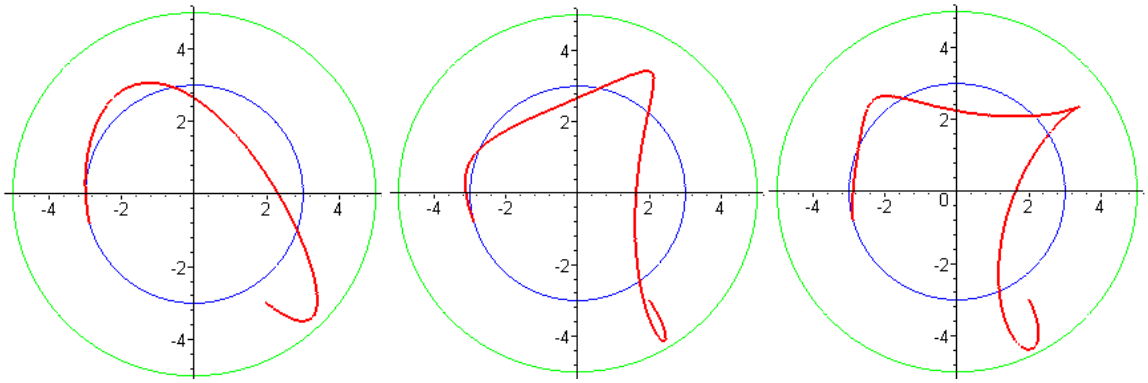
The middle plot was generated by the following Maple code based on (15.2.15), where R ≡ r'_0, :

```
b := 3: R := 2: V := 0.4:
tmax := R/V: omega := 1: Omega := 10:
x :=  b*sin(omega*t) + (R - V*t)*cos(omega*t + Omega*t):
y := -b*cos(omega*t) + (R - V*t)*sin(omega*t + Omega*t):
plot([[x,y,t=0..tmax],[b*cos(z),b*sin(z),z=0...2*Pi],[(b+R)*cos(z),(b+R)*sin(z),z=0...2*Pi]],
scaling = CONSTRAINED, thickness = [2,1,1],numpoints = 800, color = [red,blue,green] );
```

(15.2.17)

When ω and Ω have opposite sign, things can look quite different,

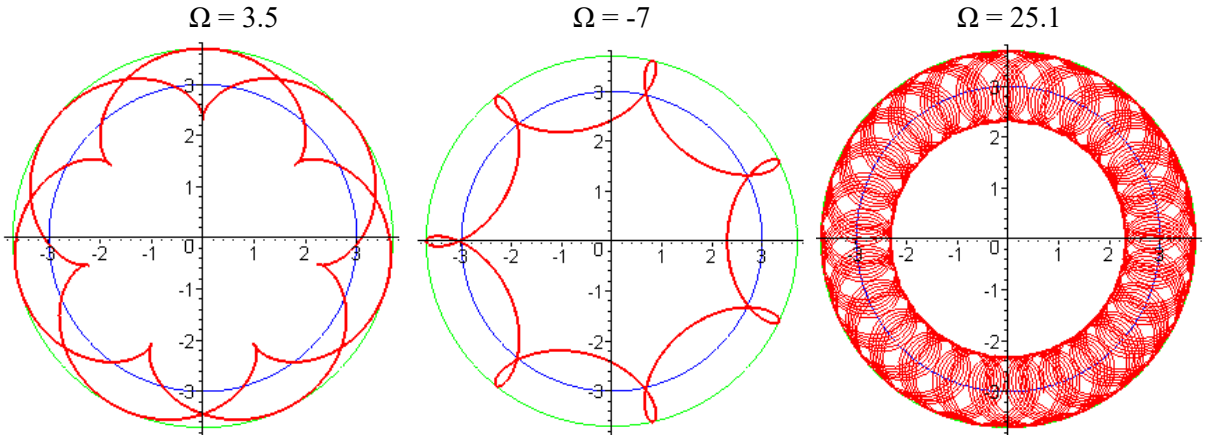
Ω = -2 Ω = -3 Ω = -3.6



(15.2.18)

Finally, here are some trajectories with V = 0 :

$r'_0 = 0.7$, $b = 3$, $V = 0$, $\omega = 1$ and :



(15.2.19)

Using the parameters of the third plot, if one were to space 10,000 ants evenly on the turntable, and have each one carry a grain of sand on its back, one would have constructed a random orbital sander. The closure of the plots occurs whenever Ω/ω is a ratio of integers, but that could take many revolutions if those integers are large.

15.3 Problem 3: Inverse Problem: Ant flies in Frame S at constant velocity V

A flying ant starting at position $\mathbf{r}_0$ flies just above the turntable surface in Frame S in a straight line at constant velocity **V** at angle θ relative to the x axis. First state **r**,**v**,**a** and then compute **r'**,**v'**,**a'** and plot the trajectory **r'** of the particle as seen in Frame S'. Use the same Frame S / Frame S' setup as in the previous problems.

In this problem continue to use both notations for the unit vectors as in (1.1.5),

$$x_1, x_2, x_3 = x, y, z \qquad \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \qquad \mathbf{e'}_1, \mathbf{e'}_2, \mathbf{e'}_3 = \hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}' \qquad \textit{(1.1.5)}$$

Ant's Motion in Frame S

The flying ant starts at location $\mathbf{r}_0$ and has velocity $\mathbf{V} = V\hat{\mathbf{n}}$ with V constant, so

$$\begin{aligned} &\mathbf{v} = V\hat{\mathbf{n}} \\ &\mathbf{r} = Vt\,\hat{\mathbf{n}} + \mathbf{r}_0 \\ &\mathbf{a} = 0 \quad . \end{aligned} \qquad (15.3.1)$$

As noted, the ant flies on a line which has angle θ relative to the $\mathbf{e}_1$ axis, so

$$\hat{\mathbf{n}} = R_z(\theta)\,\mathbf{e}_1 = R_z(\theta)\,\hat{\mathbf{x}} \quad . \qquad (15.3.2)$$

This angle θ is defined in the usual polar sense: it is counterclockwise from the positive x axis.

Trajectory **r'**(t) of the ant in Frame S'

First, we need to write $\hat{\mathbf{n}}$ in Frame S' basis vectors. Recall (15.4c) which says

$$\mathbf{e'}_i = R_z(\varphi)\,\mathbf{e}_i \qquad \text{for example} \qquad \mathbf{e'}_1 = R_z(\varphi)\,\mathbf{e}_1 \quad . \qquad \textit{(15.4c)}$$

Therefore

$$\hat{\mathbf{n}} = R_z(\theta)\,\mathbf{e}_1 = R_z(\theta)\,R_z(-\varphi)\,\mathbf{e'}_1 = R_z(\theta-\varphi)\,\mathbf{e'}_1 \quad . \qquad (15.3.3)$$

Second, how does the point $\mathbf{r}_0$ in Frame S appear in Frame S' ? Using (15.4c) just above gives

$$\mathbf{r}_0 = (r_0)_i\,\mathbf{e}_i = (r_0)_i\,R_z(-\varphi)\mathbf{e'}_i \quad . \qquad (15.3.4)$$

Third, from (15.12) (or just looking at Fig (15.1)) we know that

$$\mathbf{b} = -b\,\mathbf{e'}_2 \quad . \qquad \textit{(15.12)}$$

Now for the trajectory we start with (13.3.2a),

$$\mathbf{r}' = \mathbf{r} - \mathbf{b} \qquad (13.3.2a)$$

$$= (Vt\,\hat{\mathbf{n}} + \mathbf{r}_0) + b\,\mathbf{e}'_2 \qquad \text{// from (15.3.1) and (15.12)}$$

$$= Vt\,R_z(\theta-\varphi)\,\mathbf{e}'_1 + (r_0)_i\,R_z(-\varphi)\mathbf{e}'_i + b\,\mathbf{e}'_2 \qquad \text{// from (15.3.3) and (15.3.4)}$$

so

$$\mathbf{r}' = Vt\,R_z(\theta-\varphi)\,\mathbf{e}'_1 + R_z(-\varphi)\,[(r_0)_i\,\mathbf{e}'_i] + b\,\mathbf{e}'_2 \quad . \qquad (15.3.5)$$

We now use the following notations (these are all "natural" components in sense of Section 1.8)

$$(r_0)_1 = x_0 \qquad (r')'_1 = x'$$
$$(r_0)_2 = y_0 \qquad (r')'_2 = y' \quad .$$

In Frame S', $\mathbf{e}'_1 = (1,0,0)$, so we write (15.3.5) in matrix notation *in Frame S'* as follows :

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = Vt \begin{pmatrix} \cos(\theta-\varphi) & -\sin(\theta-\varphi) & 0 \\ \sin(\theta-\varphi) & \cos(\theta-\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

or

$$x' = Vt\cos(\theta-\varphi) + x_0\cos\varphi + y_0\sin\varphi$$
$$y' = Vt\sin(\theta-\varphi) - x_0\sin\varphi + y_0\cos\varphi + b$$
$$z' = 0 \quad .$$

The conclusion is this :

$$\mathbf{r}'(t) = x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}'$$

where (15.3.6)

$$x' = Vt\cos(\theta-\varphi) + x_0\cos\varphi + y_0\sin\varphi$$
$$y' = Vt\sin(\theta-\varphi) - x_0\sin\varphi + y_0\cos\varphi + b$$
$$\text{where } \varphi = \varphi_0 + \omega t \quad .$$

This then is the trajectory **r'** of the flying ant as seen in Frame S'.

Trajectory **r'**(t) of the ant in Frame S': Alternate Method

Since we are going to have a discrepancy with Thornton and Marion below, it seems healthy to confirm (15.3.6) by an alternate derivation that does not make use of the matrix notation used above. We start as before above (15.3.5),

$$\mathbf{r}' = \mathbf{r} - \mathbf{b} = (Vt\,\hat{\mathbf{n}} + \mathbf{r}_0) + b\,\mathbf{e}'_2 \quad . \qquad \mathbf{r}' = \text{trajectory of ant in Frame S'}$$

Using the theorem (1.1) + (1.2) we find from (15.3.3) and (15.3.4) that,

$\hat{\mathbf{n}} = R_z(\theta-\varphi)\,\mathbf{e'}_1 = R_z(\varphi-\theta)_{11}\,\mathbf{e'}_1 + R_z(\varphi-\theta)_{12}\,\mathbf{e'}_2 = \cos(\varphi-\theta)\,\mathbf{e'}_1 - \sin(\varphi-\theta)\,\mathbf{e'}_2$

$\mathbf{r}_0 = (r_0)_i\,\mathbf{e}_i = (r_0)_i\,R_z(-\varphi)\mathbf{e'}_i = x_0\,R_z(-\varphi)\mathbf{e'}_1 + y_0\,R_z(-\varphi)\mathbf{e'}_2$

$= x_0\,\{\,R_z(\varphi)_{11}\,\mathbf{e'}_1 + R_z(\varphi)_{12}\,\mathbf{e'}_2\,\} + y_0\,\{\,R_z(\varphi)_{21}\,\mathbf{e'}_1 + R_z(\varphi)_{22}\,\mathbf{e'}_2\,\}$

$= x_0\,\{\,\cos(\varphi)\,\mathbf{e'}_1 - \sin(\varphi)\,\mathbf{e'}_2\,\} + y_0\,\{\sin(\varphi)\,\mathbf{e'}_1 + \cos(\varphi)\,\mathbf{e'}_2\,\}$

$= [x_0\cos(\varphi) + y_0\sin(\varphi)]\,\mathbf{e'}_1 + [-x_0\sin(\varphi) + y_0\cos(\varphi)]\,\mathbf{e'}_2$.

Therefore,

$\mathbf{r'} = Vt\,\hat{\mathbf{n}} + \mathbf{r}_0 + b\,\mathbf{e'}_2$

$= Vt\,[\cos(\varphi-\theta)\,\mathbf{e'}_1 - \sin(\varphi-\theta)\,\mathbf{e'}_2] + [x_0\cos(\varphi) + y_0\sin(\varphi)]\,\mathbf{e'}_1 + [-x_0\sin(\varphi) + y_0\cos(\varphi)]\,\mathbf{e'}_2 + b\,\mathbf{e'}_2$

$= [Vt\cos(\varphi-\theta) + x_0\cos(\varphi) + y_0\sin(\varphi)]\,\mathbf{e'}_1 + [-Vt\sin(\varphi-\theta) - x_0\sin(\varphi) + y_0\cos(\varphi) + b]\,\mathbf{e'}_2$

$= [Vt\cos(\theta-\varphi) + x_0\cos(\varphi) + y_0\sin(\varphi)]\,\mathbf{e'}_1 + [Vt\sin(\theta-\varphi) - x_0\sin(\varphi) + y_0\cos(\varphi) + b]\,\mathbf{e'}_2$

and this does agree with (15.3.6).

Velocity $\mathbf{v'}(t)$ of the ant in Frame S'

From the Inverse Problem equations of Section 13.3 we have from (13.3.2b),

$\mathbf{v'} = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} - \dot{\mathbf{b}}_{S'}$. *(13.3.2b)*

Since this is a Special Case #1 problem, we have $\dot{\mathbf{b}}_{S'} = 0$ and then

$\mathbf{v'} = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r} = \mathbf{v} - \boldsymbol{\omega} \times (\mathbf{r'}+\mathbf{b}) = \mathbf{v} - \boldsymbol{\omega} \times \mathbf{r'} - \boldsymbol{\omega} \times \mathbf{b}$ // *(13.3.2a)* that $\mathbf{r'} = \mathbf{r} - \mathbf{b}$

$= V\,R_z(\theta-\varphi)\,\hat{\mathbf{x}}' - \boldsymbol{\omega} \times [\,x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}'] - \boldsymbol{\omega} \times [\,-b\,\hat{\mathbf{y}}']$ // (15.3.1) $\mathbf{v}$, (15.3.3) $\hat{\mathbf{n}}$ and (15.12) $\mathbf{b}$

$= V\,R_z(\theta-\varphi)\,\hat{\mathbf{x}}' - \omega x'\,\hat{\mathbf{y}}' + \omega y'\,\hat{\mathbf{x}}' - \omega b\,\hat{\mathbf{x}}'$. // $\boldsymbol{\omega} = \omega\,\hat{\mathbf{z}}'$

Using the following unofficial notations,

$(\mathbf{v'})'_1 = v'_{x'}$
$(\mathbf{v'})'_2 = v'_{y'}$

the above equation in matrix notation in Frame S' is

$$\begin{pmatrix} v'_{x'} \\ v'_{y'} \\ v'_{z'} \end{pmatrix} = V \begin{pmatrix} \cos(\theta-\varphi) & -\sin(\theta-\varphi) & 0 \\ \sin(\theta-\varphi) & \cos(\theta-\varphi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \omega y'-\omega b \\ -\omega x' \\ 0 \end{pmatrix}$$

or

$$v'_{x'} = V\cos(\theta-\varphi) + \omega(y'-b)$$
$$v'_{y'} = V\sin(\theta-\varphi) - \omega x'$$
$$v'_{z'} = 0 \quad .$$

Then using (15.3.6) for x' and y' we get,

$$\mathbf{v}'(t) = v'_{x'}\,\hat{\mathbf{x}}' + v'_{y'}\,\hat{\mathbf{y}}'$$

where (15.3.7)

$$v'_{x'} = V\cos(\theta-\varphi) + \omega[Vt\sin(\theta-\varphi) - x_0\sin\varphi + y_0\cos\varphi]$$
$$v'_{y'} = V\sin(\theta-\varphi) - \omega[Vt\cos(\theta-\varphi) + x_0\cos\varphi + y_0\sin\varphi]$$

where $\varphi = \varphi_0 + \omega t$.

This then is the velocity **v'** of the flying ant as seen in Frame S'.

Acceleration **a'**(t) of the ant in Frame S'

From the Inverse Problem equations in Section 13.3 we have from (13.3.2d),

$$\mathbf{a}' = \mathbf{a} - \dot{\boldsymbol{\omega}} \times \mathbf{r} - 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \ddot{\mathbf{b}}_{S'} \quad . \qquad \textit{(13.3.2d)}$$

Setting $\ddot{\mathbf{b}}_{S'} = 0$ for our Special Case #1 problem, and using (15.3.1) for **r**, **v** and **a** (**a** = 0) we get

$$\mathbf{a}' = -\dot{\boldsymbol{\omega}} \times [Vt\,\hat{\mathbf{n}} + \mathbf{r}_0] - 2\,\boldsymbol{\omega} \times [V\hat{\mathbf{n}}] + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [Vt\,\hat{\mathbf{n}} + \mathbf{r}_0]) \quad .$$

We shall stop here, but the calculation can be continued in a manner similar to that for **r'** and **v'**.

Trajectory Plots

Recall (15.3.6) from above

$$\mathbf{r}'(t) = x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}'$$

where *(15.3.6)*

$$x' = Vt\cos(\theta-\varphi) + x_0\cos\varphi + y_0\sin\varphi$$
$$y' = Vt\sin(\theta-\varphi) - x_0\sin\varphi + y_0\cos\varphi + b$$

where $\varphi = \varphi_0 + \omega t$.

For plotting purposes, we set $\varphi_0 = 0$ so $\varphi = \omega t$. Since b merely offsets plots vertically by amount b, we lose no interest by setting b=0, causing the Frame S and Frame S' origins to coincide. Then from (15.3.6),

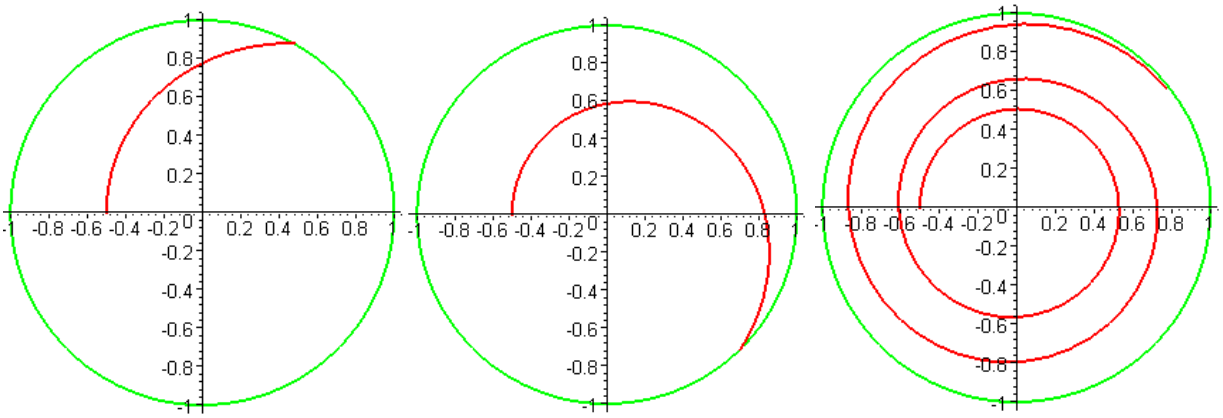
$$x' = Vt\cos(\theta-\omega t) + x_0\cos(\omega t) + y_0\sin(\omega t)$$
$$y' = Vt\sin(\theta-\omega t) - x_0\sin(\omega t) + y_0\cos(\omega t) \quad . \qquad (15.3.8)$$

In all plots below we set ω = 1. In the first three plots the flying ant starts out at $\mathbf{r}_0$ = (-1/2,0) and flies north so θ = π/2. We superpose a green circle of radius R = 1 and take note of the time T it takes for the ant to reach the circle. The Maple code used for the left plot is this (x' = xp) :

```
V := 0.85:    T := 1.03:
x0 := -0.5:   y0 := 0:
b := 0:   omega := 1:
theta := Pi/2:
xp := V*t*cos(theta - omega*t) + x0*cos(omega*t) + y0*sin(omega*t):
yp := V*t*sin(theta - omega*t) - x0*sin(omega*t) + y0*cos(omega*t) + b:
plot([[xp,yp,t=0..T],[cos(z),sin(z),z=0..2*Pi]],scaling = CONSTRAINED, thickness = 2);
```

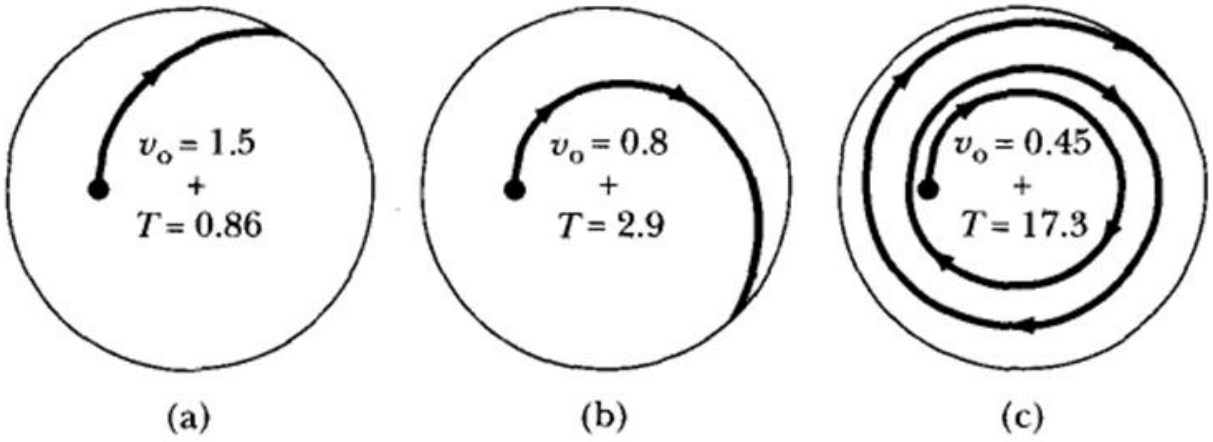
Here then are plots for three decreasing values of V. As the ant flies more slowly, the turntable turns more radians before the ant reaches a distance R = 1 from the turntable center.

V = .85 , T = 1.03 V = 0.3, T = 2.9 V = .061, T = 14



(15.3.9)

These plots may be compared to those appearing on on page 395 of Thornton and Marion,



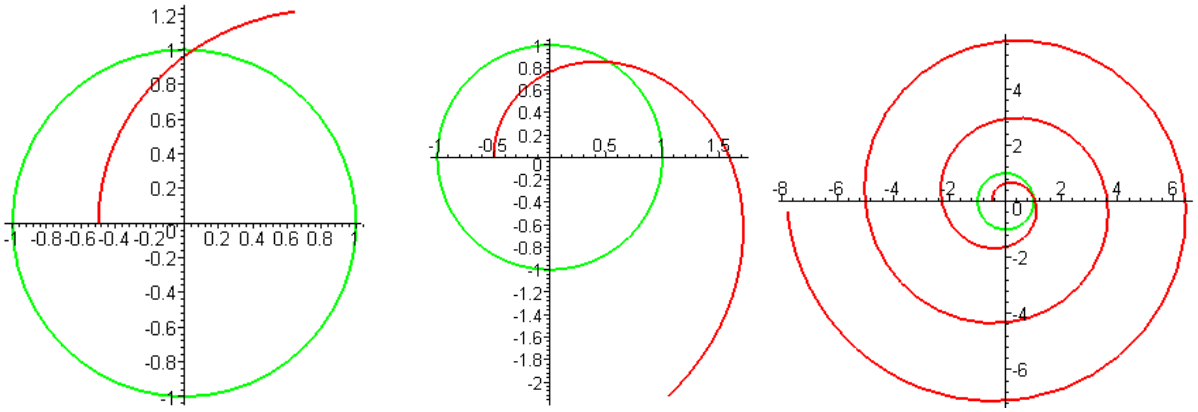
(15.3.10)

Their plotting method is to compute the fictitious force acceleration

$$\mathbf{a}' = -2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

and then to *numerically integrate* **a**' twice to get **v**' and then **r**' which is then plotted. [Again, see "The Hard Way" in Section 15.4 below.] Although our plots are close to theirs in appearance, our numbers for V and T differ significantly from theirs. Here are the three plots our code generates using the numbers specified in the T&M images above,

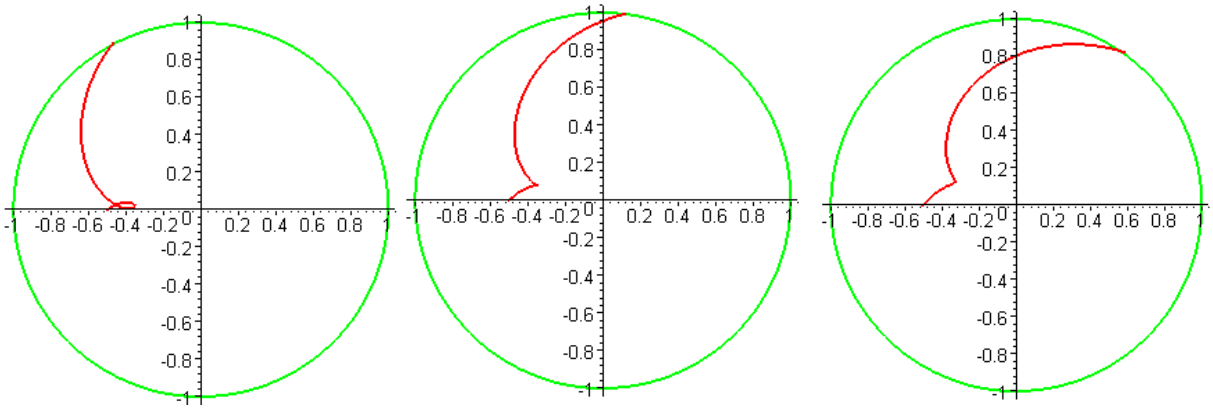
V = 1.5 , T = 0.86 V = 0.8, T = 2.9 V = 0.45, T = 17.3

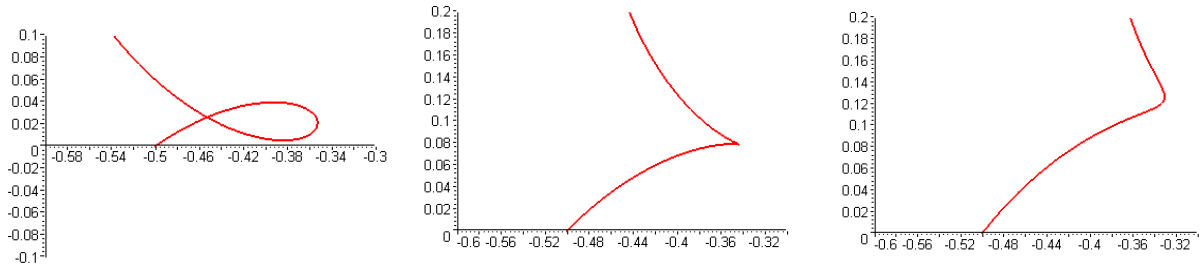


(15.3.11)

For the next three plots, we still have ω = 1 and the flying ant starts in the same place $\mathbf{r}_0$ = (-1/2,0), but now the ant flies southeast at speed V so θ = -π/4. Interestingly, we see that it is possible for the ant (as seen in Frame S') to execute a loop, a cusp, or a bump soon after taking flight. The lower set of figures show blowups of the parts of the upper paths,

V = 0.42, T = 3.1 V = 0.35, T = 3.7 V = 0.31, T = 4.2 (15.3.12)

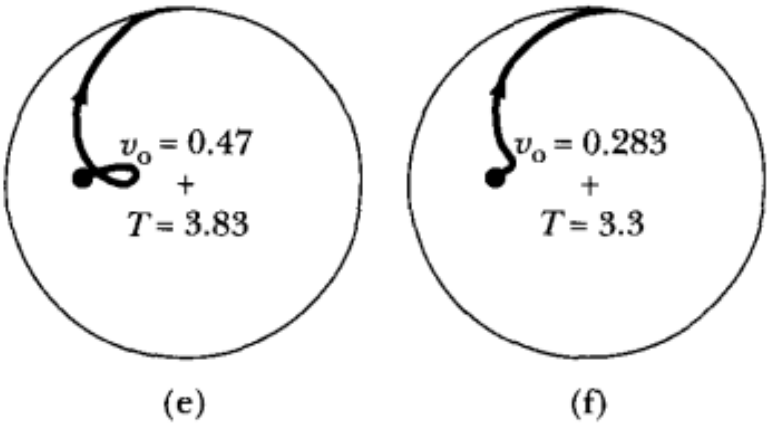




(15.3.13)

It is not easy to intuitively explain these plots. We do know that for all plots $v_t = \omega r = 1*(1/2) = 0.5$ at time t = 0. So even though the ant is flying southeast in Frame S, in Frame S' there is at t = 0 a v_t adder of 0.5 upwards (north) due to the motion of the turntable so in Frame S' the ant starts off going northeast. The ant moves to a smaller radius so v_t is reduced so the and moves more toward the east and in the left case to the south as well, but then Frame S' which started under Frame S is rotating up to the right

The left pair of plots in (15.3.12) may be compared to two other plots appearing in T&M on page 395,

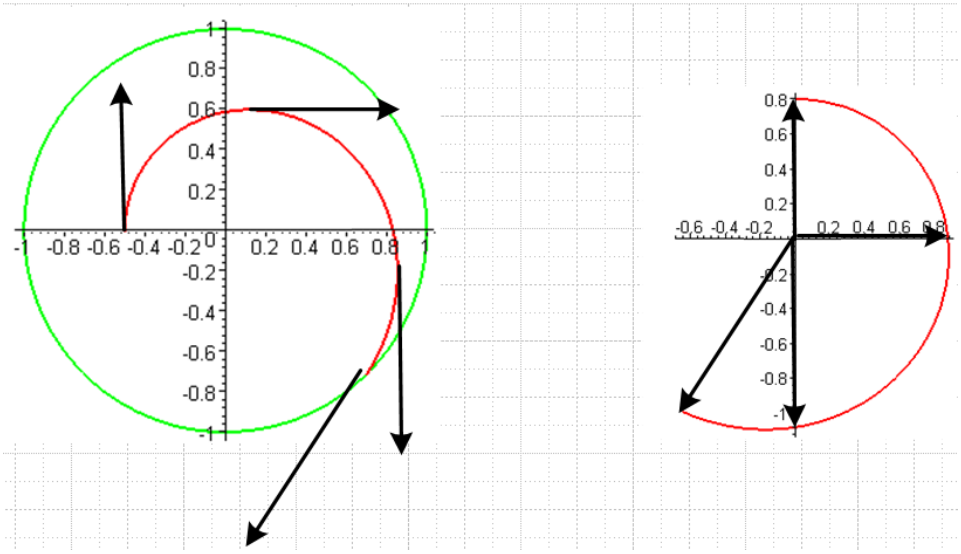


(15.3.14)

Again the plots are similar, but the numbers are different (but in the same ball park).

Using (15.3.7) we plot the *velocity* that goes with the second trajectory shown in the first triplet above, and at least things seem reasonable:

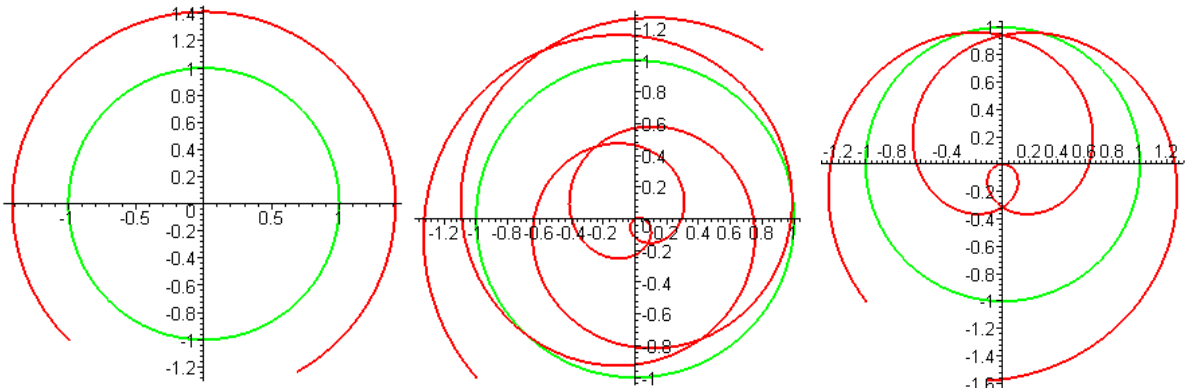
r(t) for V = 0.3, T = 2.9 **v**(t) for V = 0.3, T = 2.9



(15.3.15)

Finally, we move the starting position to (-1,-1) and have the ant fly northeast so $\theta = \pi/4$. This path takes him over the origin of Frame S (and Frame S'), so we expect to see the Frame S' trajectory touch the origin at one point along the trajectory (except in the left plot where V = 0)

V = 0, T = 5 V = 0.11, T = 25 V = 0.2, T = 15



(15.3.16)

Since the turntable is rotating counterclockwise at ω=1, these trajectories run clockwise at all times. When V = 0, the apparent motion of the static fly is circular in Frame S'. In the middle plot we see that the ant spirals in, reaches the origin, the spirals out. In the right plot he does the same thing, but more quickly.

In the 1960's some excellent frames-of-reference movies were produced. One of them involves a frictionless puck moving on a smooth table mounted to a large wooden frame which is rotated (no doubt by students). Two affable "doctors" are rotating on that frame with the table. Doctor #1 on the left launches the puck, but Doctor #2 has nothing to do since the puck just returns to Doctor #1.

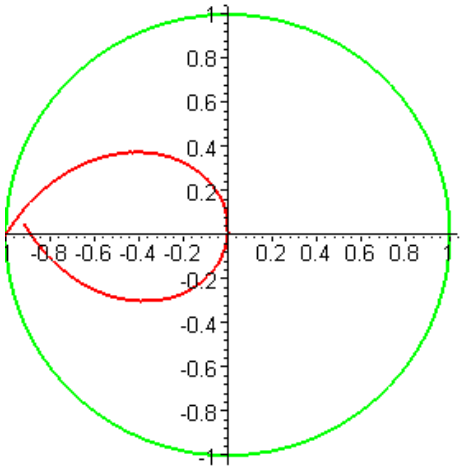




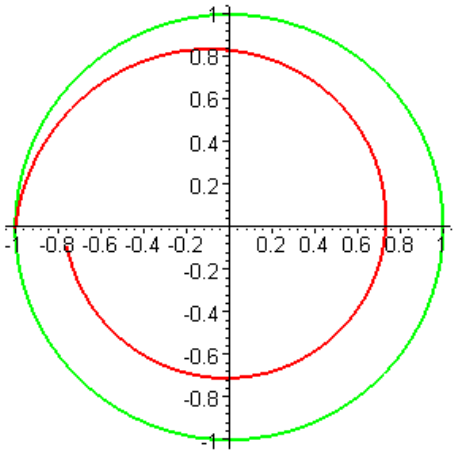
(15.3.17)

We show two possible Frame S' paths for a puck launched from $(x_0, y_0) = (-1,0)$ and $\omega = 1$.

V = 0.6, T = 3.2, θ = 0



V = 0.2, T = 5, θ = π/4



(15.3.18)

This classic movie is archived at http://www.youtube.com/watch?v=3ug23VTMies . (The next film in this series treats the Foucault Pendulum which we describe in Appendix C.)

15.4 Problem 4: The Projectile Problem of Section 8.3

It will be recalled that in Section 8.3, as a demonstration of the Coriolis force, four projectiles are fired horizontally in four directions as in Figure (8.3.1). Since each projectile is like one of our flying ants, we already have a complete solution to this problem which we shall plot below. But first, it is very enlightening to approach this problem "the hard way" and then to appreciate the power of the equations which directly relate particle properties in Frame S and Frame S'.

The Hard Way

It was noted in (8.3.4) that the projectiles (or our flying ant) experience the following fictitious forces,

$$\mathbf{F'}_{\mathtt{fict}} = m\omega^2\mathbf{r'} - 2m\,\boldsymbol{\omega} \times \mathbf{v'} \quad . \qquad \textit{(8.3.4)}$$

centrifugal Coriolis

Using bogus Newton's Law (8.1.4) that $\mathbf{F'}_{\mathtt{eff}} = m\mathbf{a'}$, the above equation can be written,

$$(d\mathbf{v'}/dt)_{\mathtt{S'}} = \omega^2\mathbf{r'} - 2\boldsymbol{\omega} \times \mathbf{v'} \quad . \qquad (15.4.1)$$

Since $\boldsymbol{\omega}$ is a constant, and working in Frame S', we differentiate once to get

$$(d^2\mathbf{v'}/dt^2)_{\mathtt{S'}} = \omega^2\,(d\mathbf{r'}/dt)_{\mathtt{S'}} - 2\boldsymbol{\omega} \times (d\mathbf{v'}/dt)_{\mathtt{S'}} \quad .$$

In the abbreviated "natural" notation of Section 1.8 this says

$$\ddot{\mathbf{v}}' + 2\boldsymbol{\omega} \times \dot{\mathbf{v}}' - \omega^2\mathbf{v'} = 0 \quad . \qquad (15.4.2)$$

Now for the rest of this section, we temporarily drop all primes just to reduce clutter. Then the above becomes

$$\ddot{\mathbf{v}} + 2\boldsymbol{\omega} \times \dot{\mathbf{v}} - \omega^2\mathbf{v} = 0 \quad . \qquad (15.4.3)$$

Expanding the vectors of interest,

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$$
$$\dot{\mathbf{v}} = \dot{v}_x\hat{\mathbf{x}} + \dot{v}_y\hat{\mathbf{y}} + \dot{v}_z\hat{\mathbf{z}}$$
$$\ddot{\mathbf{v}} = \ddot{v}_x\hat{\mathbf{x}} + \ddot{v}_y\hat{\mathbf{y}} + \ddot{v}_z\hat{\mathbf{z}}$$

$$\boldsymbol{\omega} \times \dot{\mathbf{v}} = [\omega\hat{\mathbf{z}}] \times [\dot{v}_x\hat{\mathbf{x}} + \dot{v}_y\hat{\mathbf{y}} + \dot{v}_z\hat{\mathbf{z}}] = \omega\dot{v}_x\hat{\mathbf{y}} - \omega\dot{v}_y\hat{\mathbf{x}} \quad .$$

Inserting these expansions into (15.4.3) and isolating the coefficients of the three unit vectors, we get

$$\ddot{v}_x - 2\omega\dot{v}_y - \omega^2 v_x = 0$$
$$\ddot{v}_y + 2\,\omega\dot{v}_x - \omega^2 v_y$$
$$\ddot{v}_z - \omega^2 v_z = 0 \quad . \tag{15.4.4}$$

The third equation has obvious solutions, one of which is $v_z = 0$ which is what applies to our turntable problems. That leaves the first two equations,

$$\ddot{v}_x - 2\omega\dot{v}_y - \omega^2 v_x = 0$$
$$\ddot{v}_y + 2\omega\dot{v}_x - \omega^2 v_y = 0 \quad . \tag{15.4.5}$$

This is a system of two coupled, second-order, linear ODE's with constant coefficients. It takes some amount of work to solve such an equation and we will start down that path. Apply a Laplace Transform,

$$[\, s^2 V_x(s) - s\, v_x(0) - \dot{v}_x(0)\,] - 2\omega[\, s\, V_y(s) - v_y(0)\,] - \omega^2 V_x(s) = 0$$
$$[\, s^2 V_y(s) - s\, v_y(0) - \dot{v}_y(0)\,] + 2\omega[\, s\, V_x(s) - v_x(0)\,] - \omega^2 V_y(s) = 0 \tag{15.4.6}$$

$$(s^2-\omega^2)V_x(s) - 2\omega s\, V_y(s) = s\, v_x(0) + \dot{v}_x(0) - 2\omega\, v_y(0) \qquad \text{// get } V_x \text{ and } V_y \text{ on the left}$$
$$(s^2-\omega^2)V_y(s) + 2\omega s\, V_x(s) = s\, v_y(0) + \dot{v}_y(0) + 2\omega\, v_x(0)$$

$$\begin{pmatrix} s^2-\omega^2 & -2\omega s \\ 2\omega s & s^2-\omega^2 \end{pmatrix}\begin{pmatrix} V_x(s) \\ V_y(s) \end{pmatrix} = \begin{pmatrix} s\, v_x(0) + \dot{v}_x(0) - 2\omega\, v_y(0) \\ s\, v_y(0) + \dot{v}_y(0) + 2\omega\, v_x(0) \end{pmatrix} \qquad \text{// write as matrix equation}$$

$$\begin{pmatrix} V_x(s) \\ V_y(s) \end{pmatrix} = \begin{pmatrix} s^2-\omega^2 & 2\omega s \\ -2\omega s & s^2-\omega^2 \end{pmatrix}\begin{pmatrix} s\, v_x(0) + \dot{v}_x(0) - 2\omega\, v_y(0) \\ s\, v_y(0) + \dot{v}_y(0) + 2\omega\, v_x(0) \end{pmatrix} / (s^2+\omega^2)^2 \qquad \text{// Maple assists}$$

$$V_x(s) = \{\, (s^2-\omega^2)\,[s\, v_x(0) + \dot{v}_x(0) - 2\omega\, v_y(0)] + 2\omega s\,[s\, v_y(0) + \dot{v}_y(0) - 2\omega\, v_x(0)]\,\}/(s^2+\omega^2)^2$$
$$V_y(s) = \{\, -2\omega s\,[s\, v_x(0) + \dot{v}_x(0) - 2\omega\, v_y(0)] + (s^2-\omega^2)\,[s\, v_y(0) + \dot{v}_y(0) - 2\omega\, v_x(0)]\,\}/(s^2+\omega^2)^2$$

One can then look up the inverse Laplace transforms of all these functions,

$s^3/(s^2+\omega^2)^2$	$\cos(\omega t) - (1/2)\omega t \sin(\omega t)$
$s^2/(s^2+\omega^2)^2$	$[\sin(\omega t) + \omega t\cos(\omega t)]/(2\omega)$
$s/(s^2+\omega^2)^2$	$t \sin(\omega t)/(2\omega)$
$1/(s^2+\omega^2)^2$	$[\sin(\omega t) - \omega t\cos(\omega t)]/(2\omega^3)$

(15.4.7)

and the problem is solved including the initial conditions.

But, we don't *have* to solve this coupled system of differential equations because we already know the solution, and we didn't have to even *look* at a differential equation to find it! The solution is (15.3.7) which we quote

$$\mathbf{v}'(t) = v'_{x'}\,\hat{\mathbf{x}}' + v'_{y'}\,\hat{\mathbf{y}}'$$

where *(15.3.7)*

$$v'_{x'} = V\cos(\theta-\varphi) + \omega[Vt\sin(\theta-\varphi) - x_0\sin\varphi + y_0\cos\varphi]$$
$$v'_{y'} = V\sin(\theta-\varphi) - \omega[Vt\cos(\theta-\varphi) + x_0\cos\varphi + y_0\sin\varphi]$$
where $\varphi = \varphi_0 + \omega t$.

Changing this to our temporary no-primes de-cluttered notation,

$$\mathbf{v}(t) = v_x\,\hat{\mathbf{x}} + v_y\,\hat{\mathbf{y}}$$

where

$$v_x = V\cos(\theta-\varphi) + \omega[Vt\sin(\theta-\varphi) - x_0\sin\varphi + y_0\cos\varphi]$$
$$v_y = V\sin(\theta-\varphi) - \omega[Vt\cos(\theta-\varphi) + x_0\cos\varphi + y_0\sin\varphi]$$
where $\varphi = \varphi_0 + \omega t$.

We shall now use Maple to verify that these functions solve the coupled equations (15.4.5):

$v_x = V\cos(\theta-\varphi) + \omega[Vt\sin(\theta-\varphi) - x_0\sin\varphi + y_0\cos\varphi]$

```
vx := V*cos(theta-phi) + omega*(V*t*sin(theta-phi)-x0*sin(phi)+y0*cos(phi));
```

$$vx := V\cos(-\theta+\phi) + \omega\,(-Vt\sin(-\theta+\phi) - x0\sin(\phi) + y0\cos(\phi))$$

$v_y = V\sin(\theta-\varphi) - \omega[Vt\cos(\theta-\varphi) + x_0\cos\varphi + y_0\sin\varphi]$

```
vy := V*sin(theta-phi) - omega*(V*t*cos(theta-phi)+x0*cos(phi)+y0*sin(phi));
```

$$vy := -V\sin(-\theta+\phi) - \omega\,(Vt\cos(-\theta+\phi) + x0\cos(\phi) + y0\sin(\phi))$$

$\varphi = \varphi_0 + \omega t$

```
phi := phi0 + omega*t;
```

$$\phi := \phi0 + \omega t$$

```
vxd := diff(vx,t); vyd := diff(vy,t);          # compute first derivatives
```

$$vxd := -V\sin(-\theta+\phi0+\omega t)\,\omega + \omega\,(-V\sin(-\theta+\phi0+\omega t) - Vt\cos(-\theta+\phi0+\omega t)\,\omega - x0\cos(\phi0+\omega t)\,\omega - y0\sin(\phi0+\omega t)\,\omega)$$
$$vyd := -V\cos(-\theta+\phi0+\omega t)\,\omega - \omega\,(V\cos(-\theta+\phi0+\omega t) - Vt\sin(-\theta+\phi0+\omega t)\,\omega - x0\sin(\phi0+\omega t)\,\omega + y0\cos(\phi0+\omega t)\,\omega)$$

```
vxdd := diff(vxd,t);  vydd := diff(vyd,t);    # compute second derivatives
```

$$vxdd := -V\cos(-\theta+\phi0+\omega t)\,\omega^2 + \omega\,(-2V\cos(-\theta+\phi0+\omega t)\,\omega + Vt\sin(-\theta+\phi0+\omega t)\,\omega^2 + x0\sin(\phi0+\omega t)\,\omega^2 - y0\cos(\phi0+\omega t)\,\omega^2)$$
$$vydd := V\sin(-\theta+\phi0+\omega t)\,\omega^2 - \omega\,(-2V\sin(-\theta+\phi0+\omega t)\,\omega - Vt\cos(-\theta+\phi0+\omega t)\,\omega^2 - x0\cos(\phi0+\omega t)\,\omega^2 - y0\sin(\phi0+\omega t)\,\omega^2)$$

$\ddot{v}_x - 2\omega\dot{v}_y - \omega^2 v_x = 0$

```
E1 := vxdd - 2*omega*vyd - omega^2*vx:  simplify(%);  # evaluate first coupled equation
```

$$0$$

$\ddot{v}_y + 2\omega\dot{v}_x - \omega^2 v_y = 0$

```
E2 := vydd + 2*omega*vxd - omega^2*vy:  simplify(%);  # evaluate second coupled equation
```

$$0$$

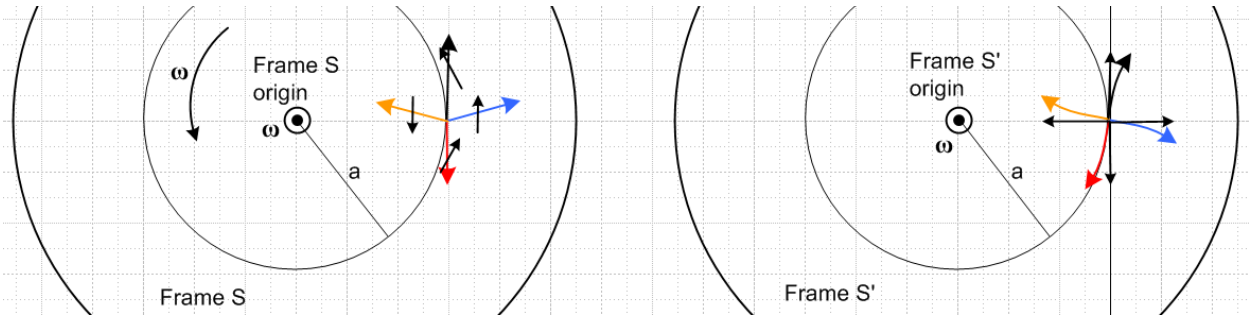
(15.4.8)

Sometimes, for a given problem in rotational motion, there is an easy way and a hard way to solve the problem. Another example of a hard way is the numeric integration mentioned after Fig (15.3.10).

At this point we restore the primes on Frame S' quantities.

The Four Projectiles

Recall Fig (8.3.1) which shows the deflecting projectiles on the right.



(8.3.1)

In Frame S' at t = 0 the four projectiles are launched in the four Frame S' axis directions and each has the same speed V' (black arrows on the right). This V' was called V in Section 8.3, but since it is a speed in Frame S' we now call it V'.

We now examine the initial speeds and launch angles θ of the four projectiles as seen in Frame S:

black (upper)	$V'\hat{\mathbf{y}} + v_t\hat{\mathbf{y}}$	$\theta = \pi/2$	$V = V' + v_t$
red (lower)	$-V'\hat{\mathbf{y}} + v_t\hat{\mathbf{y}}$	$\theta = -(\pi/2)*\text{sign}(V'-v_t)$	$V = \lvert V' - v_t\rvert$
orange (left)	$-V'\hat{\mathbf{x}} + v_t\hat{\mathbf{y}}$	$\theta = \pi - \Delta\theta$	$V = \sqrt{V'^2+v_t^2}$
blue (right)	$V'\hat{\mathbf{x}} + v_t\hat{\mathbf{y}}$	$\theta = \Delta\theta$	$V = \sqrt{V'^2+v_t^2}$

Here $v_t = a\omega$ is the turntable upward speed where the launch occurs. The angle θ is the usual polar azimuthal angle measured CCW from the $\hat{\mathbf{x}}$ axis which points to the right, and $\Delta\theta = \tan^{-1}(v_t/V')$.

We treat each of these Frame S velocities as the velocity of a Problem 3 ant flyover. The Frame S' trajectories are then given by (15.3.8) where the Frame S and Frame S' origins coincide at the spindle (b=0) and have aligned axes at time t = 0,

$$x' = Vt\cos(\theta-\omega t) + x_0\cos(\omega t) + y_0\sin(\omega t)$$
$$y' = Vt\sin(\theta-\omega t) - x_0\sin(\omega t) + y_0\cos(\omega t) \quad . \qquad (15.3.8)$$

At t=0 the four projectiles start at $(x_0,y_0) = (x'_0,y'_0) = (a,0)$ so things simplify a bit more,

$$x'(t) = Vt\cos(\theta-\omega t) + a\cos(\omega t)$$
$$y'(t) = Vt\sin(\theta-\omega t) - a\sin(\omega t) \quad . \qquad (15.4.9)$$

These are very simple expressions indeed, considering the coupled differential equations above.

It remains only to have Maple plot the projectile trajectories with the initial velocities installed :

```
xp := V*t*cos(theta-omega*t) + a*cos(omega*t);
yp := V*t*sin(theta-omega*t) - a*sin(omega*t);
```

$$xp := V\,t\cos(\theta - \omega\,t) + a\cos(\omega\,t)$$

$$yp := V\,t\sin(\theta - \omega\,t) - a\sin(\omega\,t)$$

```
omega := 1: Vp := 5: a := 1: tmax := 0.2:
vt := omega*a:
if (Vp=0) then dtheta := Pi/2 else dtheta :=arctan(vt/Vp) fi:

theta := Pi/2:  V := Vp+vt:                          #BLACK
BLACK := plot([xp,yp,t=0..tmax],color=black):

theta := -Pi/2*signum(Vp-vt): V := abs(Vp-vt): #RED
RED := plot([xp,yp,t=0..tmax],color=red):

theta := Pi-dtheta: V := sqrt(Vp^2+vt^2):       #ORANGE
ORANGE := plot([xp,yp,t=0..tmax],color=Orange):

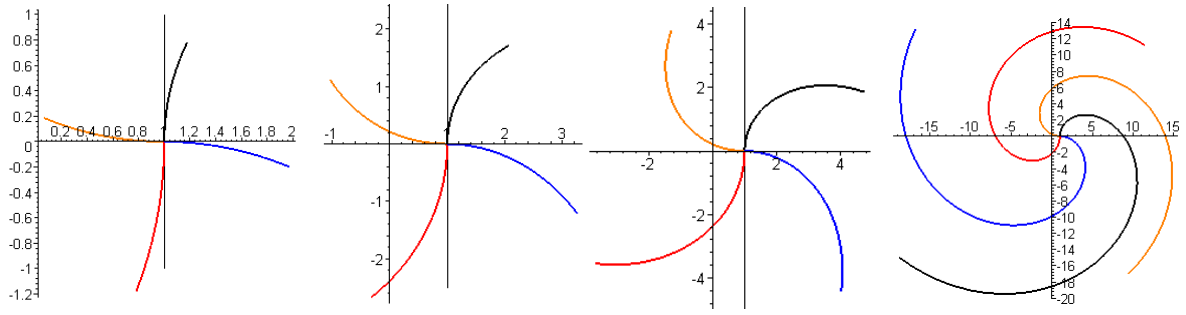
theta := dtheta: V := sqrt(Vp^2+vt^2):
BLUE := plot([xp,yp,t=0..tmax],color=blue):       #BLUE

display(BLACK,RED,ORANGE,BLUE,thickness=2,scaling=constrained);
```

(15.4.10)

Here then are some plots. The first four all have V' = 5 and but have increasing duration. The notion of the deflections being roughly circular (end of Section 8.3) is not too bad for tmax ≤ 1 :

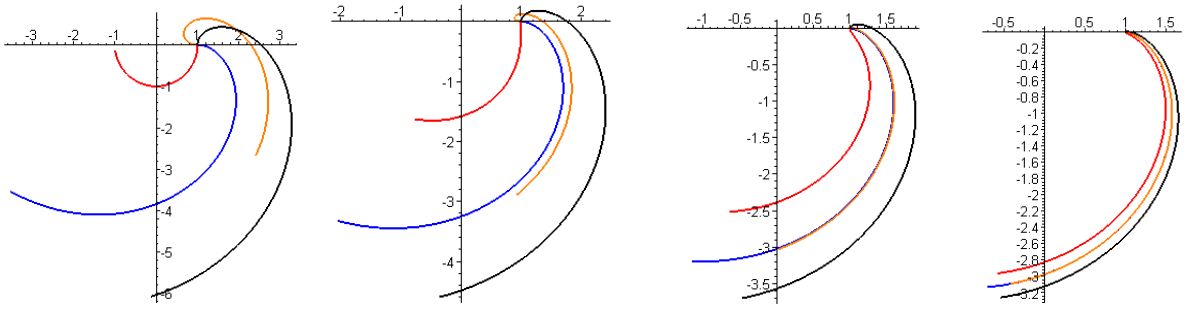
V' = 5: tmax = 0.2 tmax = 0.5 tmax = 1.0 tmax = 4



(15.4.11)

In the next plots we fix tmax = 3 but take V' to ever-decreasing values :

tmax=3: V' = 1 V' = 0.5 V' = 0.2 V' = 0.05



(15.4.12)

As V' → 0, all four curves eventually become the same curve. The blue and orange curves line up early because, with small V', both projectiles accelerate radially outward in Frame S' with no tangential velocity. The orange one first goes to the left, then reverses and goes to the right and follows the blue particle in nearly the same trajectory.

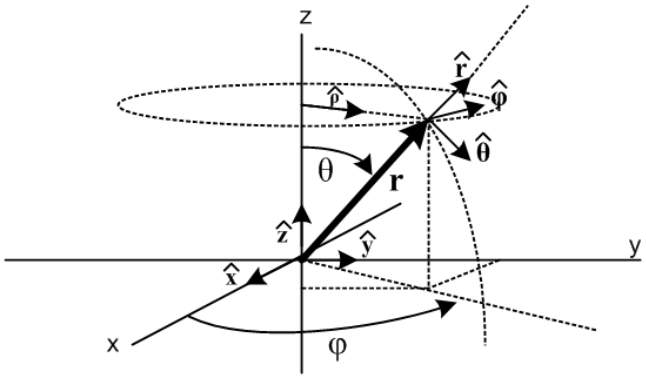
Appendix A: Derivation of R(ξ) and Properties of Rotation Matrices

Here we derive equations (14.4) through (14.6) for spherical coordinates. First, just for reference, here are the three active rotation matrices used below :

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \qquad (A.1)$$

These are called "active" since they rotate a vector forward (counterclockwise) relative to fixed axes by amount θ according to the right hand rule when the thumb is aligned with the axis in question.

From the usual picture of spherical coordinates,



(A.2)

one can see (by staring hard enough) that

$$\begin{aligned} \hat{\mathbf{r}} &= R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{z}} && = \mathcal{R}\hat{\mathbf{z}} && \mathcal{R} \equiv R_z(\varphi)\, R_y(\theta) \\ \hat{\boldsymbol{\theta}} &= R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{x}} && = \mathcal{R}\hat{\mathbf{x}} && \\ \hat{\boldsymbol{\varphi}} &= R_z(\varphi)\, R_y(\theta)\, \hat{\mathbf{y}} && = \mathcal{R}\hat{\mathbf{y}} && \text{// } R_y(\theta) \text{ does nothing here} \end{aligned} \qquad (A.3)$$

which we rewrite as

$$\begin{aligned} \hat{\boldsymbol{e}}_1 &= \mathcal{R}\, \mathbf{e}_3 && \text{where} && \hat{\boldsymbol{e}}_1 = \hat{\mathbf{r}} && \mathbf{e}_3 = \hat{\mathbf{z}} \\ \hat{\boldsymbol{e}}_2 &= \mathcal{R}\, \mathbf{e}_1 && && \hat{\boldsymbol{e}}_2 = \hat{\boldsymbol{\theta}} && \mathbf{e}_1 = \hat{\mathbf{x}} \\ \hat{\boldsymbol{e}}_3 &= \mathcal{R}\, \mathbf{e}_2 && && \hat{\boldsymbol{e}}_3 = \hat{\boldsymbol{\varphi}} && \mathbf{e}_2 = \hat{\mathbf{y}} \quad . \end{aligned} \qquad (A.4)$$

We can repair the ordering of the basis vectors $\mathbf{e}_n$ on the right using $R_2 = (\hat{\mathbf{z}},\hat{\mathbf{x}},\hat{\mathbf{y}})$ as follows

$$\mathbf{e}_3 = R_2\, \mathbf{e}_1 \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = (\hat{\mathbf{z}},\hat{\mathbf{x}},\hat{\mathbf{y}})$$

$$\mathbf{e_1} = R_2\,\mathbf{e_2} \qquad \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\mathbf{e_2} = R_2\,\mathbf{e_3} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} \quad . \tag{A.5}$$

Maple tells us that $R_2^{-1} = R_2^{T}$ and $\det(R_2) = 1$, confirming that R_2 is a rotation. Putting (A.5) into (A.4),

$$\begin{aligned}\hat{\boldsymbol{e}}_1 &= \boldsymbol{\mathcal{R}}\,R_2\,\mathbf{e_1}\\ \hat{\boldsymbol{e}}_2 &= \boldsymbol{\mathcal{R}}\,R_2\,\mathbf{e_2}\\ \hat{\boldsymbol{e}}_3 &= \boldsymbol{\mathcal{R}}\,R_2\,\mathbf{e_3}\end{aligned} \tag{A.6}$$

or

$$\hat{\boldsymbol{e}}_i = R(\xi)\,\mathbf{e}_i \quad . \tag{A.7}$$

Recalling (14.4),

$$\begin{aligned}&\hat{\boldsymbol{e}}_n = R(\xi)\,\mathbf{e}_n \quad \Rightarrow \quad \mathbf{e}_n = R(\xi)_{nm}\hat{\boldsymbol{e}}_m \quad \text{or} \quad \hat{\boldsymbol{e}}_n = [R(\xi)]^{-1}_{nm}\,\mathbf{e}_m\\ &\hat{\boldsymbol{e}}'_n = R'(\xi')\,\mathbf{e}'_n \quad \Rightarrow \quad \mathbf{e}'_n = R'(\xi')_{nm}\hat{\boldsymbol{e}}'_m \quad \text{or} \quad \hat{\boldsymbol{e}}'_n = [R'(\xi')]^{-1}_{nm}\,\mathbf{e}'_m \quad .\end{aligned} \tag{14.4}$$

we have therefore found the sought-after matrix $R(\xi)$ appearing in the top left of (14.4).

$$R(\xi) = \boldsymbol{\mathcal{R}}\,R_2 \quad . \tag{A.8}$$

Specific evaluation gives

$$\boldsymbol{\mathcal{R}} = R_z(\varphi)\,R_y(\theta) = \begin{pmatrix}\cos\varphi&-\sin\varphi&0\\ \sin\varphi&\cos\varphi&0\\ 0&0&1\end{pmatrix}\begin{pmatrix}\cos\theta&0&\sin\theta\\ 0&1&0\\ -\sin\theta&0&\cos\theta\end{pmatrix} = \begin{pmatrix}\cos\varphi\cos\theta&-\sin\varphi&\cos\varphi\sin\theta\\ \sin\varphi\sin\theta&\cos\varphi&\sin\varphi\sin\theta\\ -\sin\theta&0&\cos\theta\end{pmatrix} \tag{A.9}$$

and then

$$R(\xi) = \boldsymbol{\mathcal{R}}\,R_2 = \begin{pmatrix}\cos\varphi\cos\theta&-\sin\varphi&\cos\varphi\sin\theta\\ \sin\varphi\sin\theta&\cos\varphi&\sin\varphi\sin\theta\\ -\sin\theta&0&\cos\theta\end{pmatrix}\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix} = \begin{pmatrix}\cos\varphi\sin\theta&\cos\varphi\cos\theta&-\sin\varphi\\ \sin\varphi\sin\theta&\sin\varphi\cos\theta&\cos\varphi\\ \cos\theta&-\sin\theta&0\end{pmatrix} \tag{A.10}$$

from which we find

$$[R(\xi)]^{-1} = [R(\xi)]^{T} = \begin{pmatrix}\cos\varphi\sin\theta&\sin\varphi\sin\theta&\cos\theta\\ \cos\varphi\cos\theta&\sin\varphi\cos\theta&-\sin\theta\\ -\sin\varphi&\cos\varphi&0\end{pmatrix} \quad . \tag{A.11}$$

From the top right equation of (14.4) (quoted above) we can write

$$\hat{\boldsymbol{e}}_i = [R(\xi)]^{-1}_{ij}\,\mathbf{e}_j = [R(\xi)]^{-1}_{i1}\,\mathbf{e_1} + [R(\xi)]^{-1}_{i2}\,\mathbf{e_2} + [R(\xi)]^{-1}_{i3}\,\mathbf{e_3} \quad . \tag{A.12}$$

This can be written in this matrix notation:

$$\begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{A.13a}$$

or

$$\begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)]^{-1} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \tag{A.13b}$$

or

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\varphi\sin\theta\, \hat{\mathbf{x}} + \sin\varphi\sin\theta\, \hat{\mathbf{y}} + \cos\theta\, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\varphi\cos\theta\, \hat{\mathbf{x}} + \sin\varphi\cos\theta\, \hat{\mathbf{y}} - \sin\theta\, \hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\, \hat{\mathbf{x}} + \cos\varphi\, \hat{\mathbf{y}} \end{aligned} \tag{A.13c}$$

which are the well known expressions for the spherical unit vectors in terms of the Cartesian ones. The inverse of the last three equations can be obtained in the following manner (matrix from (A.10))

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = [R(\xi)] \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} \tag{A.14a}$$

or

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)] \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} \tag{A.14b}$$

or

$$\begin{aligned} \hat{\mathbf{x}} &= \cos\varphi\sin\theta\, \hat{\mathbf{r}} + \cos\varphi\cos\theta\, \hat{\boldsymbol{\theta}} - \sin\varphi\, \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{y}} &= \sin\varphi\sin\theta\, \hat{\mathbf{r}} + \sin\varphi\cos\theta\, \hat{\boldsymbol{\theta}} + \cos\varphi\, \hat{\boldsymbol{\varphi}} \\ \hat{\mathbf{z}} &= \cos\theta\, \hat{\mathbf{r}} - \sin\theta\, \hat{\boldsymbol{\theta}} \ . \end{aligned} \tag{A.14c}$$

Appendix E contains further information on spherical coordinates.

Some Properties of Rotation Matrices

A rotation matrix R is always real orthogonal which means $R^{-1} = R^T$. It follows that

$$\begin{aligned} RR^T = 1 \quad &\Rightarrow \quad \Sigma_k R_{ik}(R^T)_{kj} = \delta_{i'j} \quad \Rightarrow \quad \Sigma_k R_{ik}R_{jk} = \delta_{i,j} \\ R^TR = 1 \quad &\Rightarrow \quad \Sigma_k (R^T)_{ik}R_{kj} = \delta_{i'j} \quad \Rightarrow \quad \Sigma_k R_{ki}R_{kj} = \delta_{i,j} \ . \end{aligned} \tag{A.15}$$

Also

$$RR^T = 1 \quad \Rightarrow \quad \det(RR^T) = \det(1) \quad \Rightarrow \quad [\det(R)]^2 = 1$$

For a rotation matrix, $\det(R) = +1$. (A.16)

The determinant of R may be written using a standard expansion for the determinant of a matrix,

$$1 = \det(R) = \Sigma_{ijk}\varepsilon_{ijk}R_{i1}R_{j2}R_{k3} \,, \qquad (A.17)$$

where the permutation tensor was described in (1.5.3). This can be generalized to read

$$\varepsilon_{abc} = \Sigma_{ijk}\varepsilon_{ijk}R_{ia}R_{jb}R_{kc} \qquad \text{for any a,b,c} \qquad (A.18)$$

where (A.17) is the particular case with abc = 123. We leave the proof of (A.18) as a Reader Exercise.

Now apply $\Sigma_a R_{na}$ to both sides and sum on a to get

$$\Sigma_a R_{na}\,\varepsilon_{abc} = \Sigma_a R_{na}\,\Sigma_{ijk}\varepsilon_{ijk}R_{ia}R_{jb}R_{kc}$$

$$= \Sigma_{ijk}\varepsilon_{ijk}\,[\,\Sigma_a R_{na}R_{ia}]\,R_{jb}R_{kc}$$

$$= \Sigma_{ijk}\varepsilon_{ijk}\,[\delta_{n,i}]\,R_{jb}R_{kc} \qquad \text{// (A.15)}$$

$$= \Sigma_{jk}\,\varepsilon_{njk}R_{jb}R_{kc} \,.$$

We have just derived the following rarely-stated property of any 3x3 rotation matrix R :

$$\Sigma_a R_{na}\varepsilon_{abc} = \Sigma_{jk}\varepsilon_{njk}R_{jb}R_{kc} \qquad \text{for any n,b,c .} \qquad (A.19)$$

Theorem: If **A'** = R**A**, **B'** =R**B** and **C'** = R**C**, then **C** = **A** x **B** $\Leftrightarrow$ **C'** = **A'** x **B'** . (A.20)

Proof $\Leftarrow$: **C'** = **A'** x **B'**

$$\Rightarrow\ C'_n = \Sigma_{jk}\varepsilon_{njk}\,A'_jB'_k \ \Rightarrow\ (R\mathbf{C})_n = \Sigma_{jk}\varepsilon_{njk}\,(R\mathbf{A})_j(R\mathbf{B})_k$$

$$\Rightarrow\ (\Sigma_a R_{na}C_a) = \Sigma_{jk}\varepsilon_{njk}(\Sigma_b R_{jb}A_b)(\Sigma_c R_{kc}B_c) =$$

$$= \Sigma_{bc}\,[\,\Sigma_{jk}\varepsilon_{njk}R_{jb}R_{kc}]\,A_bB_c$$

$$= \Sigma_{bc}\,[\,\Sigma_a R_{na}\varepsilon_{abc}]\,A_bB_c \qquad \text{// (A.19)}$$

$$= \Sigma_a R_{na}\,[\Sigma_{bc}\varepsilon_{abc}A_bB_c\,]$$

$$= \Sigma_a\ R_{na}\,[\mathbf{A}\text{ x }\mathbf{B}\,]_a \,.$$

In vector notation we have just shown that R**C** = R(**A**x**B**). Apply R^{-1} from the left to conclude that **C** = **A** x **B**, QED. Run the steps in reverse to prove $\Rightarrow$.

This theorem confirms the intuitive fact that if **A**,**B**,**C** all transform as normal vectors under R, then if **C** = **A**x**B** in Frame S, then **C'** = **A'**x**B'** in rotated Frame S' . The fact that **A**•**B** = **A'**•**B'** is more obvious and requires only (A.15) : $\Sigma_j(A')_j(B')_j = \Sigma_j(\Sigma_b R_{jb}A_b)(\Sigma_c R_{jc}B_c) = \Sigma_{bc}\,\delta_{bc}A_bB_c = \Sigma_b A_bB_b$.

Appendix B: The G Rule for a Tensor of Rank n

In this section we continue to use these shorthand operator notations,

$$\partial_{\mathbf{S}} \equiv (d/dt)_{\mathbf{S}} \qquad \partial_{\mathbf{S}'} \equiv (d/dt)_{\mathbf{S}'} \qquad \partial_{\mathbf{t}} \equiv (d/dt) \; . \qquad \textit{(1.8.3)} \qquad \text{(B.1)}$$

Recall from (1.10.1) that for a scalar function or for a component of any tensor one has,

$$(\partial_{\mathbf{S}} T_{\mathbf{ijk}\ldots}) = (\partial_{\mathbf{S}'} T_{\mathbf{ijk}\ldots}) = (\partial_{\mathbf{t}} T_{\mathbf{ijk}\ldots}) \; , \qquad \textit{(1.10.1)} \qquad \text{(B.2)}$$

and from (2.5) that

$$\partial_{\mathbf{S}'} \mathbf{e}_{\mathbf{i}} = -\, \boldsymbol{\omega} \times \mathbf{e}_{\mathbf{i}} \; . \qquad \textit{(2.9)} \qquad \text{(B.3)}$$

So far we know all about the G Rule for tensors of rank 0 and 1 (scalar and vector),

$$\partial_{\mathbf{S}} A = \partial_{\mathbf{S}'} A \qquad // = \partial_{\mathbf{t}} A; \quad A \text{ is a scalar function of t} \qquad \textit{(B.2)}$$

$$\partial_{\mathbf{S}} \mathbf{A} = \partial_{\mathbf{S}'} \mathbf{A} \; + \; \boldsymbol{\omega} \times \mathbf{A} \; . \qquad // \; \mathbf{A} \text{ is a vector} \qquad \textit{(2.1)} \qquad \text{(B.4)}$$

What happens for tensors of rank 2 or more? To explore this question, we first collect a few facts.

A tensor of rank-n has the following expansion (where q represents the $n^{\mathbf{th}}$ letter of the alphabet)

$$T = \Sigma_{\mathbf{abc..q}} . T_{\mathbf{abc..q}} \; (\mathbf{e}_{\mathbf{a}} \otimes \mathbf{e}_{\mathbf{b}} \otimes \mathbf{e}_{\mathbf{c}} \ldots\ldots \otimes \mathbf{e}_{\mathbf{q}}) \quad . \qquad \text{(B.5)}$$

Here T has n subscripts and there is a tensor product of n basis vectors $\mathbf{e}_{\mathbf{i}}$. This expansion is a generalization of the expansion of a vector,

$$\mathbf{V} = \Sigma_{\mathbf{i}} V_{\mathbf{i}} \mathbf{e}_{\mathbf{i}} \quad . \qquad \text{(B.6)}$$

The basis vectors are axis-aligned unit vectors, so we have

$$(\mathbf{e}_{\mathbf{n}})_{\mathbf{i}} = \delta_{\mathbf{n,i}} \quad . \qquad \textit{(1.1.3)} \qquad \text{(B.7)}$$

The meaning of the $\otimes$ symbol is nothing more than the following,

$$[A \otimes B \otimes \ldots\ldots \otimes Q \,]_{\mathbf{abc}\ldots\mathbf{q}} = A_{\mathbf{a}} B_{\mathbf{b}} C_{\mathbf{c}} \ldots\ldots Q_{\mathbf{q}} \quad . \qquad \text{(B.8)}$$

This particular tensor $A \otimes B \otimes \ldots\ldots \otimes Q$ happens to be the direct product of the n vectors A,B,C...Q, but there are of course general tensors T which cannot be written as such a direct product.

As a simple tensor product example,

$$(\mathbf{e}_{\mathbf{a}} \otimes \mathbf{e}_{\mathbf{b}})_{\mathbf{ij}} = (\mathbf{e}_{\mathbf{a}})_{\mathbf{i}} \, (\mathbf{e}_{\mathbf{b}})_{\mathbf{j}} = \delta_{\mathbf{a,i}} \, \delta_{\mathbf{b,j}} \quad . \qquad \text{(B.9)}$$

Here is the expansion of the direct product tensor T = **A**⊗**B**,

$$\mathbf{A}\otimes\mathbf{B} = \Sigma_{ab}\,[A\otimes B]_{ab}\;\mathbf{e}_a\otimes\mathbf{e}_b = \Sigma_{ab}\,A_aB_b\;\mathbf{e}_a\otimes\mathbf{e}_b \quad . \tag{B.10}$$

[For more details on such tensor products and expansions, see Lucht *Tensor Products* with $\mathbf{e}_i \rightarrow \mathbf{u}_i$] .

Before continuing, we develop two very similar Lemmas:

Lemma 1: $\partial_s(\mathbf{A}\otimes\mathbf{B}) = (\partial_s\mathbf{A})\otimes\mathbf{B} + \mathbf{A}\otimes(\partial_s\mathbf{B})$ (B.11)

where **A** and **B** are vectors whose components are functions of time.

Proof of Lemma 1:

$\partial_s(\mathbf{A}\otimes\mathbf{B}) = \partial_s(\Sigma_{ab}\,A_aB_b\;\mathbf{e}_a\otimes\mathbf{e}_b)$ // expansion (B.10)

$= \Sigma_{ab}\,\partial_s(A_aB_b)\;\mathbf{e}_a\otimes\mathbf{e}_b$ // since $\partial_s\mathbf{e}_n = 0$ by (1.7.3)

$= \Sigma_{ab}\,\partial_t(A_aB_b)\;\mathbf{e}_a\otimes\mathbf{e}_b$ // (B.2)

$= \Sigma_{ab}\,[(\partial_tA_a)B_b + A_a(\partial_tB_b)]\;\mathbf{e}_a\otimes\mathbf{e}_b$ // regular calculus product rule

$= \Sigma_{ab}\,[(\partial_sA_a)B_b + A_a(\partial_sB_b)]\;\mathbf{e}_a\otimes\mathbf{e}_b$ // restore ∂_s using (B.2)

$= \Sigma_{ab}\,[(\partial_s\mathbf{A})_aB_b + A_a(\partial_s\mathbf{B})_b]\;\mathbf{e}_a\otimes\mathbf{e}_b$ /// commutation rule (1.11.1)

$= \Sigma_{ab}\,[(\partial_s\mathbf{A})_aB_b]\;\mathbf{e}_a\otimes\mathbf{e}_b + \Sigma_{ab}\;A_a(\partial_s\mathbf{B})_b\;\mathbf{e}_a\otimes\mathbf{e}_b$ // write as two terms

$= (\partial_s\mathbf{A})\otimes\mathbf{B} + \mathbf{A}\otimes(\partial_s\mathbf{B})$. // expansion (B.10) QED

Recall from (1.2.1) that one can expand a vector **A** on either the $\mathbf{e}_i$ or the $\mathbf{e}'_i$ basis,

$$\mathbf{A} = \Sigma_iA_i\mathbf{e}_i = \Sigma_iA'_i\mathbf{e}'_i \quad . \qquad (1.2.1) \tag{B.12}$$

Similarly, one can expand the tensor T of (B.5) on the $\mathbf{e}'_i$ basis to get

$$T = \Sigma_{abc..q}\,T'_{abc..q}\;(\mathbf{e}'_a\otimes\mathbf{e}'_b\otimes\mathbf{e}'_c.....\otimes\mathbf{e}'_q) \tag{B.13}$$

with this special case

$$\mathbf{A}\otimes\mathbf{B} = \Sigma_{ab}\,[A\otimes B]'_{ab}\;\mathbf{e}'_a\otimes\mathbf{e}'_b = \Sigma_{ab}\,A'_aB'_b\;\mathbf{e}'_a\otimes\mathbf{e}'_b \quad . \tag{B.14}$$

We then have Lemma 2 which is the same as Lemma 1 but with S→S' ,

Lemma 2: $\partial_{\mathbf{s}'}(\mathbf{A}\otimes\mathbf{B}) = (\partial_{\mathbf{s}'}\mathbf{A})\otimes\mathbf{B} + \mathbf{A}\otimes(\partial_{\mathbf{s}'}\mathbf{B})$ (B.15)

The proof exactly follows that of Lemma 1 but we show it anyway:

$\partial_{\mathbf{s}'}(\mathbf{A}\otimes\mathbf{B}) = \partial_{\mathbf{s}'}(\Sigma_{\mathbf{ab}} A'_{\mathbf{a}}B'_{\mathbf{b}}\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}})$ // expansion (B.14)

$= \Sigma_{\mathbf{ab}} \partial_{\mathbf{s}'}(A'_{\mathbf{a}}B'_{\mathbf{b}})\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // since $\partial_{\mathbf{s}'}\mathbf{e}'_{\mathbf{n}} = 0$ by (1.7.2)

$= \Sigma_{\mathbf{ab}} \partial_{\mathbf{t}}(A'_{\mathbf{a}}B'_{\mathbf{b}})\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // (B.2)

$= \Sigma_{\mathbf{ab}} [(\partial_{\mathbf{t}}A'_{\mathbf{a}})B'_{\mathbf{b}} + A'_{\mathbf{a}}(\partial_{\mathbf{t}}B'_{\mathbf{b}})]\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // regular calculus product rule

$= \Sigma_{\mathbf{ab}} [(\partial_{\mathbf{s}'}A'_{\mathbf{a}})B'_{\mathbf{b}} + A'_{\mathbf{a}}(\partial_{\mathbf{s}'}B'_{\mathbf{b}})]\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // restore $\partial_{\mathbf{s}'}$ using (B.2)

$= \Sigma_{\mathbf{ab}} [(\partial_{\mathbf{s}'}\mathbf{A})'_{\mathbf{a}}B'_{\mathbf{b}} + A'_{\mathbf{a}}(\partial_{\mathbf{s}'}\mathbf{B})'_{\mathbf{b}}]\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // commutation rule (1.11.1)

$= \Sigma_{\mathbf{ab}} [(\partial_{\mathbf{s}'}\mathbf{A})'_{\mathbf{a}}B'_{\mathbf{b}}]\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}} + \Sigma_{\mathbf{ab}}\ A'_{\mathbf{a}}(\partial_{\mathbf{s}'}\mathbf{B})'_{\mathbf{b}}\ \mathbf{e}'_{\mathbf{a}}\otimes\mathbf{e}'_{\mathbf{b}}$ // write as two terms

$= (\partial_{\mathbf{s}'}\mathbf{A})\otimes\mathbf{B} + \mathbf{A}\otimes(\partial_{\mathbf{s}'}\mathbf{B})$. // expansion (B.14) QED

G Rule for a Rank-2 Tensor

Recall from (B.5) that,

$\mathrm{T} = \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$. *(B.5)* (B.16)

Then apply $\partial_{\mathbf{s}}$, using the facts $\partial_{\mathbf{s}}\mathbf{e}_{\mathbf{n}} = 0$ and (B.2),

$(\partial_{\mathbf{s}}\mathrm{T}) = \Sigma_{\mathbf{ab}}(\partial_{\mathbf{s}}\mathrm{T}_{\mathbf{ab}})\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}) = \Sigma_{\mathbf{ab}}(\partial_{\mathbf{t}}\mathrm{T}_{\mathbf{ab}})\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$. (B.17)

On the other hand,

$(\partial_{\mathbf{s}'}\mathrm{T}) = \Sigma_{\mathbf{ab}}\ (\partial_{\mathbf{s}'}\mathrm{T}_{\mathbf{ab}})\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}) + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\ \partial_{\mathbf{s}'}(\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$

$= \Sigma_{\mathbf{ab}}\ (\partial_{\mathbf{t}}\mathrm{T}_{\mathbf{ab}})\ (\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}) + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\ \partial_{\mathbf{s}'}(\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$ // (B.2)

$= (\partial_{\mathbf{s}}\mathrm{T}) + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\ \partial_{\mathbf{s}'}(\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}})$. // (B.17) (B.18)

But,

$\partial_{\mathbf{s}'}(\mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}) = (\partial_{\mathbf{s}'}\mathbf{e}_{\mathbf{a}})\otimes\mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{a}}\otimes(\partial_{\mathbf{s}'}\mathbf{e}_{\mathbf{b}})$ // by Lemma 2 (B.15)

$= -[\ (\boldsymbol{\omega}\ \mathrm{x}\ \mathbf{e}_{\mathbf{a}})\otimes\mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{a}}\otimes(\boldsymbol{\omega}\ \mathrm{x}\ \mathbf{e}_{\mathbf{b}})$. // (B.3) (B.19)

Inserting this last result into (B.18) and swapping sides then gives the G rule for a rank-2 tensor,

$$(\partial_{\mathbf{S}}\mathrm{T}) = (\partial_{\mathbf{S}'}\mathrm{T}) + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\left[(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}}) \otimes \mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{a}} \otimes (\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{b}}) \right] \quad . \tag{B.20}$$

We can write the G rule for a vector $\mathbf{T}$ (rank-1 tensor) in a similar form,

$$\begin{aligned}(\partial_{\mathbf{S}}\mathbf{T}) &= (\partial_{\mathbf{S}'}\mathbf{T}) + \boldsymbol{\omega} \times \mathbf{T} \\ &= (\partial_{\mathbf{S}'}\mathbf{T}) + \boldsymbol{\omega} \times (\Sigma_{\mathbf{a}}\mathrm{T}_{\mathbf{a}}\mathbf{e}_{\mathbf{a}}) \\ &= (\partial_{\mathbf{S}'}\mathbf{T}) + \Sigma_{\mathbf{a}}\mathrm{T}_{\mathbf{a}}\ \boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}} \quad .\end{aligned} \tag{B.21}$$

G Rule for a Rank-n Tensor

It is easy to show that the above general pattern applies to tensors of rank 3 and higher, and one ends up with the following lowest tensor G rules, where the tensor rank is shown on the left .

$$\begin{aligned}
&0 \quad (\partial_{\mathbf{S}}\mathrm{T}) = (\partial_{\mathbf{S}'}\mathrm{T}) \\
&1 \quad (\partial_{\mathbf{S}}\mathbf{T}) = (\partial_{\mathbf{S}'}\mathbf{T}) + \Sigma_{\mathbf{a}}\mathrm{T}_{\mathbf{a}}\ \boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}} \\
&2 \quad (\partial_{\mathbf{S}}\mathrm{T}) = (\partial_{\mathbf{S}'}\mathrm{T}) + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\left[(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}}) \otimes \mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{a}} \otimes (\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{b}}) \right] \\
&3 \quad (\partial_{\mathbf{S}}\mathrm{T}) = (\partial_{\mathbf{S}'}\mathrm{T}) + \Sigma_{\mathbf{abc}}\mathrm{T}_{\mathbf{abc}}\left[(\boldsymbol{\omega}\times\mathbf{e}_{\mathbf{a}})\otimes\mathbf{e}_{\mathbf{b}}\otimes\mathbf{e}_{\mathbf{c}} + \mathbf{e}_{\mathbf{a}}\otimes(\boldsymbol{\omega}\times\mathbf{e}_{\mathbf{b}})\otimes\mathbf{e}_{\mathbf{c}} + \mathbf{e}_{\mathbf{a}}\otimes\mathbf{e}_{\mathbf{b}}\otimes(\boldsymbol{\omega}\times\mathbf{e}_{\mathbf{c}}) \right]
\end{aligned} \tag{B.22}$$

These G rule results may also be expressed in components. For example:

$$\begin{aligned}
1 \quad (\partial_{\mathbf{S}}\mathbf{T})_{\mathbf{i}} &= (\partial_{\mathbf{S}'}\mathbf{T})_{\mathbf{i}} + \Sigma_{\mathbf{a}}\mathrm{T}_{\mathbf{a}}\ (\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}})_{\mathbf{i}} \\
&= (\partial_{\mathbf{S}'}\mathbf{T})_{\mathbf{i}} + \Sigma_{\mathbf{a}}\mathrm{T}_{\mathbf{a}}\ \varepsilon_{\mathbf{irs}}\,\omega_{\mathbf{r}}\,(\mathbf{e}_{\mathbf{a}})_{\mathbf{s}} \\
&= (\partial_{\mathbf{S}'}\mathbf{T})_{\mathbf{i}} + \Sigma_{\mathbf{a}}\mathrm{T}_{\mathbf{a}}\ \varepsilon_{\mathbf{irs}}\,\omega_{\mathbf{r}}\,\delta_{\mathbf{a},\mathbf{s}} \\
&= (\partial_{\mathbf{S}'}\mathbf{T})_{\mathbf{i}} + \varepsilon_{\mathbf{irs}}\,\omega_{\mathbf{r}}\mathrm{T}_{\mathbf{s}} \quad .
\end{aligned} \tag{B.23}$$

$$\begin{aligned}
2 \quad (\partial_{\mathbf{S}}\mathrm{T})_{\mathbf{ij}} &= (\partial_{\mathbf{S}'}\mathrm{T})_{\mathbf{ij}} + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\left[(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}}) \otimes \mathbf{e}_{\mathbf{b}} + \mathbf{e}_{\mathbf{a}} \otimes (\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{b}}) \right]_{\mathbf{ij}} \\
&= (\partial_{\mathbf{S}'}\mathrm{T})_{\mathbf{ij}} + \Sigma_{\mathbf{ab}}\mathrm{T}_{\mathbf{ab}}\left[(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}})_{\mathbf{i}}\,(\mathbf{e}_{\mathbf{b}})_{\mathbf{j}} + (\mathbf{e}_{\mathbf{a}})_{\mathbf{i}}(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{b}})_{\mathbf{j}} \right] \\
&= (\partial_{\mathbf{S}'}\mathrm{T})_{\mathbf{ij}} + \left[\Sigma_{\mathbf{ab}}\ \mathrm{T}_{\mathbf{ab}}\,(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}})_{\mathbf{i}}\,\delta_{\mathbf{b},\mathbf{j}} + \Sigma_{\mathbf{ab}}\ \mathrm{T}_{\mathbf{ab}}\,\delta_{\mathbf{a},\mathbf{i}}\,(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{b}})_{\mathbf{j}} \right] \\
&= (\partial_{\mathbf{S}'}\mathrm{T})_{\mathbf{ij}} + \left[\Sigma_{\mathbf{a}}\ \mathrm{T}_{\mathbf{aj}}\,(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{a}})_{\mathbf{i}} + \Sigma_{\mathbf{b}}\ \mathrm{T}_{\mathbf{ib}}\,(\boldsymbol{\omega} \times \mathbf{e}_{\mathbf{b}})_{\mathbf{j}} \right] \\
&= (\partial_{\mathbf{S}'}\mathrm{T})_{\mathbf{ij}} + \left[\Sigma_{\mathbf{a}}\ \mathrm{T}_{\mathbf{aj}}\,\varepsilon_{\mathbf{irs}}\omega_{\mathbf{r}}(\mathbf{e}_{\mathbf{a}})_{\mathbf{s}} + \Sigma_{\mathbf{b}}\ \mathrm{T}_{\mathbf{ib}}\,\varepsilon_{\mathbf{jrs}}\omega_{\mathbf{r}}(\mathbf{e}_{\mathbf{b}})_{\mathbf{s}} \right]
\end{aligned}$$

$$= (\partial_S T)_{ij} + [\, \Sigma_a\, T_{aj}\, \varepsilon_{irs}\omega_r\delta_{a,s} + \Sigma_b\, T_{ib}\, \varepsilon_{jrs}\omega_r\delta_{b,s}\,]$$

$$= (\partial_S T)_{ij} + [\, T_{sj}\, \varepsilon_{irs}\omega_r + T_{is}\, \varepsilon_{jrs}\omega_r\,]$$

$$= (\partial_S T)_{ij} + \varepsilon_{irs}\omega_r\, T_{sj} + \varepsilon_{jrs}\omega_r T_{is} \quad . \tag{B.24}$$

We shall do one more case to establish the general pattern,

$$3 \quad (\partial_S T)_{ijk} - (\partial_{S'} T)_{ijk} = \Sigma_{abc} T_{abc}\, [\, (\boldsymbol{\omega} \mathrm{x} \mathbf{e}_a)\otimes\mathbf{e}_b\otimes\mathbf{e}_c + \mathbf{e}_a\otimes(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_b)\otimes\mathbf{e}_c + \mathbf{e}_a\otimes\mathbf{e}_b\otimes(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_c)\,]_{ijk}$$

$$= \Sigma_{abc} T_{abc}\, [\, (\boldsymbol{\omega}\mathrm{x}\mathbf{e}_a)_i(\mathbf{e}_b)_j(\mathbf{e}_c)_k + (\mathbf{e}_a)_i(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_b)_j(\mathbf{e}_c)_k + (\mathbf{e}_a)_i(\mathbf{e}_b)_j(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_c)_k\,]$$

$$= \Sigma_{abc} T_{abc}(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_a)_i\delta_{b,j}\delta_{c,k} + \Sigma_{abc} T_{abc}\delta_{a,i}(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_b)_j\delta_{c,k} + \Sigma_{abc} T_{abc}\delta_{a,i}\,\delta_{b,j}(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_c)_k$$

$$= \Sigma_a T_{ajk}(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_a)_i + \Sigma_b T_{ibk}(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_b)_j + \Sigma_c T_{ijc}(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_c)_k$$

$$= \Sigma_a T_{ajk}\varepsilon_{irs}\omega_r\,(\mathbf{e}_a)_s + \Sigma_b T_{ibk}\varepsilon_{jrs}\omega_r\,(\mathbf{e}_b)_s + \Sigma_c T_{ijc}\varepsilon_{krs}\omega_r\,(\mathbf{e}_c)_s$$

$$= \Sigma_a T_{ajk}\varepsilon_{irs}\omega_r\,\delta_{a,s} + \Sigma_b T_{ibk}\varepsilon_{jrs}\omega_r\,\delta_{b,s} + \Sigma_c T_{ijc}\varepsilon_{krs}\omega_r\delta_{c,s}$$

$$= T_{sjk}\varepsilon_{irs}\omega_r + T_{isk}\varepsilon_{jrs}\omega_r + T_{ijs}\varepsilon_{krs}\omega_r$$

so we conclude that

$$(\partial_S T)_{ijk} = (\partial_{S'} T)_{ijk} + \varepsilon_{irs}\omega_r T_{sjk} + \varepsilon_{jrs}\omega_r T_{isk} + \varepsilon_{krs}\omega_r T_{ijs} \quad . \tag{B.25}$$

Looking at the last two results we can infer the rank-4 result

$$(\partial_S T)_{ijk\ell} = (\partial_{S'} T)_{ijk\ell} + \varepsilon_{irs}\omega_r T_{sjk\ell} + \varepsilon_{jrs}\omega_r T_{isk\ell} + \varepsilon_{krs}\omega_r T_{ijs\ell} + \varepsilon_{\ell rs}\omega_r T_{ijks} \quad . \tag{B.26}$$

Summary of G rule for all ranks (tensor form) (B.27)

$$0 \quad (\partial_S T) = (\partial_{S'} T)$$

$$1 \quad (\partial_S \mathbf{T}) = (\partial_{S'} \mathbf{T}) + \boldsymbol{\Sigma}_a T_a\ \boldsymbol{\omega}\ \mathrm{x}\ \mathbf{e}_a$$

$$2 \quad (\partial_S T) = (\partial_{S'} T) + \Sigma_{ab} T_{ab}\, [\, (\boldsymbol{\omega}\ \mathrm{x}\ \mathbf{e}_a)\otimes\mathbf{e}_b + \mathbf{e}_a\otimes(\boldsymbol{\omega}\ \mathrm{x}\ \mathbf{e}_b)\,]$$

$$3 \quad (\partial_S T) = (\partial_{S'} T) + \Sigma_{abc} T_{abc}\, [\, (\boldsymbol{\omega}\mathrm{x}\mathbf{e}_a)\otimes\mathbf{e}_b\otimes\mathbf{e}_c + \mathbf{e}_a\otimes(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_b)\otimes\mathbf{e}_c + \mathbf{e}_a\otimes\mathbf{e}_b\otimes(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_c)\,]$$

$$4 \quad (\partial_S T) = (\partial_{S'} T) + \Sigma_{abcd} T_{abcd}\, [\, (\boldsymbol{\omega}\mathrm{x}\mathbf{e}_a)\otimes\mathbf{e}_b\otimes\mathbf{e}_c\otimes\mathbf{e}_d + \mathbf{e}_a\otimes(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_b)\otimes\mathbf{e}_c\otimes\mathbf{e}_d + \mathbf{e}_a\otimes\mathbf{e}_b\otimes(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_c)\otimes\mathbf{e}_d + \mathbf{e}_a\otimes\mathbf{e}_b\otimes\mathbf{e}_c\otimes(\boldsymbol{\omega}\mathrm{x}\mathbf{e}_d)\,]$$

etc.

Summary of G rule for all ranks (component form) (B.28)

0 $(\partial_S T) = (\partial_{S'} T)$

1 $(\partial_S \mathbf{T})_i = (\partial_{S'} \mathbf{T})_i + \varepsilon_{irs}\,\omega_r T_s = (\partial_{S'} \mathbf{T})_i + (\boldsymbol{\omega} \times \mathbf{T})_i$

2 $(\partial_S T)_{ij} = (\partial_{S'} T)_{ij} + \varepsilon_{irs}\omega_r T_{sj} + \varepsilon_{jrs}\omega_r T_{is}$

3 $(\partial_S T)_{ijk} = (\partial_{S'} T)_{ijk} + \varepsilon_{irs}\omega_r T_{sjk} + \varepsilon_{jrs}\omega_r T_{isk} + \varepsilon_{krs}\omega_r T_{ijs}$

4 $(\partial_S T)_{ijk\ell} = (\partial_{S'} T)_{ijk\ell} + \varepsilon_{irs}\omega_r T_{sjk\ell} + \varepsilon_{jrs}\omega_r T_{isk\ell} + \varepsilon_{krs}\omega_r T_{ijs\ell} + \varepsilon_{\ell rs}\omega_r T_{ijks}$

etc.

In the main body of our document we deal only with vector quantities **r**, **v** and **a** , but one can easily imagine "frames of reference problems" which deal also with tensors, such as the inertia tensor. Applications involving continuum mechanics abound in tensors.

Appendix C: The Foucault Pendulum

This is a long appendix so we provide an overview:

The opening section reviews the traditional brief treatment of the Foucault pendulum precession.

Section C.1 sets up the kinematics of the spherical pendulum.

Section C.2 gives a quick qualitative explanation of the Foucault precession.

Section C.3 derives the Foucault pendulum angular equations of motion, stated in (C.3.9).

Section C.4 derives the exact solution to the simple (plane) pendulum in (C.4.20) or (C.4.26).

Section C.5 reviews the exact solution of spherical pendulum, and also gives qualitative hints as to the nature of the pendulum motion. After stating the simple conical motion solution, the motion of thin ellipses is reviewed and the Airy precession appears. After reviewing the Foucault mode of the spherical pendulum, the interference between the Airy and Foucault precessions is analyzed. A detailed analysis of the Foucault pendulum at the Pantheon in Paris is presented. Finally, it is shown how to numerically obtain 2D and 3D plots of arbitrary motions of a spherical pendulum using Maple.

Section C.6 derives the Foucault equations of motion in Cartesian coordinates in (C.6.9).

Section C.7 shows that the x,y,z equations of motion of Section C.6 are entirely equivalent to the θ,φ equations of motion of Section C.3.

Section C.8 studies numerical solutions of both the Foucault and Spherical pendulums. For the latter the Airy precession is demonstrated.

Nutshell Analysis of the Foucault Pendulum

This subject is usually treated in the small oscillation limit in Cartesian coordinates (e.g. Taylor, Marion, Thornton and Marion). Applying Newton's Law **F** = m**a** including the Coriolis force one quickly obtains a pair of coupled linear ODE's which, in "top view" (x,y) notation, are

$$\ddot{x} - 2\omega\cos\beta\,\dot{y} + \Omega^2 x = 0 \qquad \omega = \text{Earth rotation rate},\ \beta = \text{polar angle (colatitude) from North Pole}$$
$$\ddot{y} + 2\omega\cos\beta\,\dot{x} + \Omega^2 y = 0 \qquad \Omega = \sqrt{g/\ell} = \text{swing rate},\ \omega << \Omega,\ \ell = \text{string length, g=gravity .} \qquad \text{(C.1)}$$

If the second equation is multiplied by i and added to the first, one gets

$$\ddot{\eta} + 2i\omega\cos\beta\dot{\eta} = -\Omega^2\eta \ . \qquad \eta \equiv x + iy \qquad \text{(C.2)}$$

We ask Maple to try out the following candidate solution to (C.2), keeping in mind that ω << Ω,

$$\eta(t) = A\, e^{-i(\omega\cos\beta)t}\cos(\Omega t) \ . \qquad \text{(C.3)}$$

```
eq := diff(eta(t),t,t) + 2*I*omega*cos(beta)*diff(eta(t),t) = - Omega^2*eta(t);
```

$$eq := \left(\frac{\partial^2}{\partial t^2}\eta(t)\right) + 2\,I\,\omega\cos(\beta)\left(\frac{\partial}{\partial t}\eta(t)\right) = -\Omega^2\,\eta(t)$$

```
eta := (t) ->A*exp(-I*omega*cos(beta)*t)*cos(Omega*t);
```

$$\eta := t \to A\,e^{(-I\,\omega\cos(\beta)\,t)}\cos(\Omega\,t)$$

```
eq/Omega^2 : expand(%);
```

$$\frac{A\,\omega^2\cos(\beta)^2\cos(\Omega\,t)}{\Omega^2\,e^{(I\,\omega\cos(\beta)\,t)}} - \frac{A\cos(\Omega\,t)}{e^{(I\,\omega\cos(\beta)\,t)}} = -\frac{A\cos(\Omega\,t)}{e^{(I\,\omega\cos(\beta)\,t)}}$$

(C.4)

Thus the equation (C.2) is solved by the candidate solution (C.3) to order $(\omega/\Omega)^2$ and so is a good approximate solution. Writing out x = Reη and y = Imη one finds that y/x = tanφ where

$$\varphi = -\omega\cos\beta\, t \qquad \dot{\varphi} = -\omega\cos\beta \tag{C.5}$$

which indicates a slow clockwise rotation of the swing axis for cosβ > 0 (Northern Hemisphere). This is in agreement with one's knowledge that, in said Hemisphere, on each swing the pendulum is deflected a little to the right by the Coriolis force (see Fig (C.2.1) or Fig (C.8.7) below).

Our treatment below does not assume small oscillation. The pendulum is treated as a full-blown "spherical pendulum" which is influenced by the rotation of the Earth. We derive the pendulum equations of motion first in spherical coordinates in Section C.3, then later in Cartesian coordinates in Section C.6, since these are better for the numerical work of Section C.8. We show in (C.6.14) that equations (C.1) above are in fact the small-oscillation limit of the general equations of motion.

C.1 Drawings, Notation, Coordinates and Basis Vectors

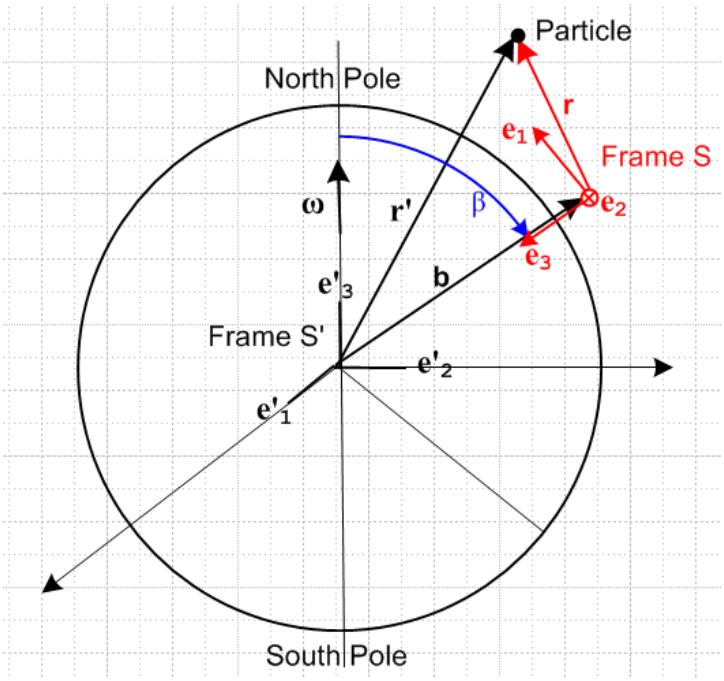
Fig (4.7.1) showed a typical Earth problem kinematic scenario in the "non-swap" notation where Frame S' is the rotating frame. Here, we choose instead to use the "swap" notation where Frame S is the rotating frame, since this eliminates the need for scores of prime symbols which would otherwise clutter the equations. So now Frame S' is at the center of the Earth and is an inertial frame, and Frame S is on the surface of the Earth and is a non-inertial frame. This use of primes is then in accordance with the Goldstein and Marion references discussed in Sections 9 and 10.

In addition to this S↔S' swap, we make a few other changes compared to Fig (4.7.1).

First, we select the basis vectors $\mathbf{e_n}$ differently. Since we are going to be dealing with a spherical pendulum, we would like the angle between the pendulum string and local vertical to be the polar angle of a spherical coordinate system for Frame S. This means that we want the $\hat{\mathbf{z}} = \mathbf{e_3}$ axis associated with this spherical system to be pointed toward the center of the Earth, rather than pointing "up". This then suggests that we take $\hat{\mathbf{x}} = \mathbf{e_1}$ as pointing "north", and $\hat{\mathbf{y}} = \mathbf{e_2}$ as pointing "east" and then the resulting vectors $\mathbf{e_n}$ form a right-handed coordinate system.

Second, we place the origin of Frame S at a height $\boldsymbol{\ell}$ above the surface of the Earth, where $\boldsymbol{\ell}$ is the length of the spherical pendulum's string. Then the length of vector **b** is $b = R_e + \ell$, where R_e is the radius of the Earth. The Frame S origin is then at the pendulum pivot point.

The new kinematic picture is then,



(C.1.1)

The next step is to use spherical coordinates in both Frames S and S'.

For Frame S', with origin at the center of the Earth and assumed fixed relative to distant stars, we select spherical coordinates, so $\mathbf{r'} = (r',\theta',\varphi')$. We shall assume that the origin of Frame S is located at $\mathbf{b} = (b, \beta, \alpha)$ where β is the polar angle measured down from the North Pole (colatitude, range 0 to π), α is the origin's azimuth (longitude), and $b = R_e + \ell$ where R_e is the radius of the Earth and ℓ is the length of the pendulum string (picture coming soon). Parameters α and b will play no role in what follows.

Note that $\boldsymbol{\omega} = \omega \mathbf{e'}_3$ with $\omega > 0$ since the Earth rotates counterclockwise as viewed from above the North Pole, causing the Sun to rise in the east. The same Earth, viewed looking upward from beneath the South Pole, appears to rotate clockwise (and the Sun still rises in the east).

For Frame S (red, shown in the Northern Hemisphere) we select another set of spherical coordinates $\mathbf{r} = (r,\theta,\varphi)$ as will be described momentarily.

The Cartesian unit basis vectors $\mathbf{e}_n$ of Frame S can be expressed in terms of the Frame S' spherical unit basis vectors $\hat{\mathbf{r}}', \hat{\boldsymbol{\theta}}', \hat{\boldsymbol{\varphi}}'$ evaluated at the position $\mathbf{r'} = \mathbf{b} = (b, \beta, \alpha)$ as follows,

$\mathbf{e}_1 = -\hat{\boldsymbol{\theta}}'$ = "north" = $\hat{\mathbf{x}}$

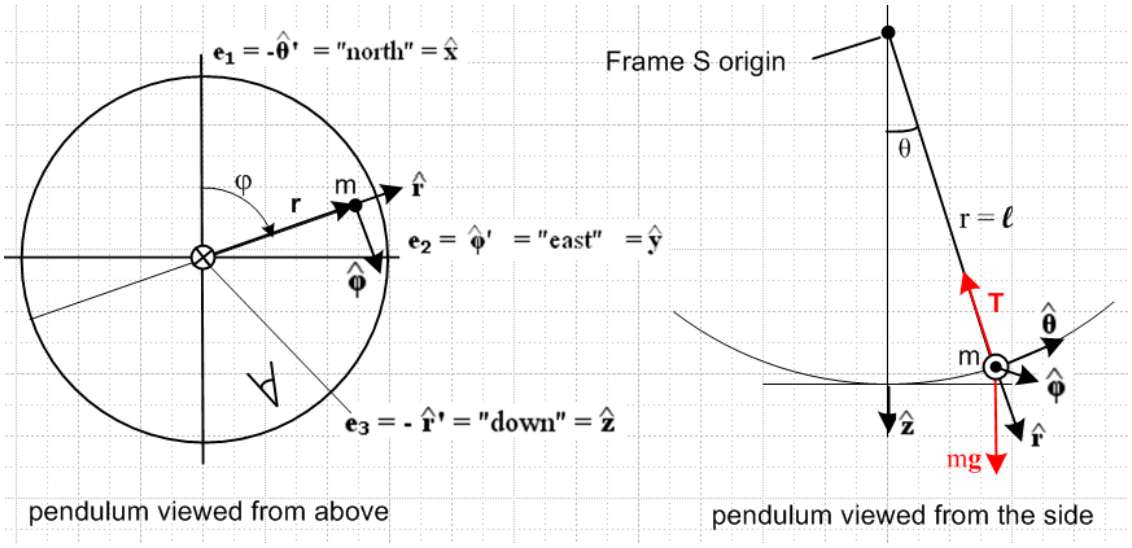
$\mathbf{e}_2 = \hat{\boldsymbol{\varphi}}'$ = "east" = $\hat{\mathbf{y}}$

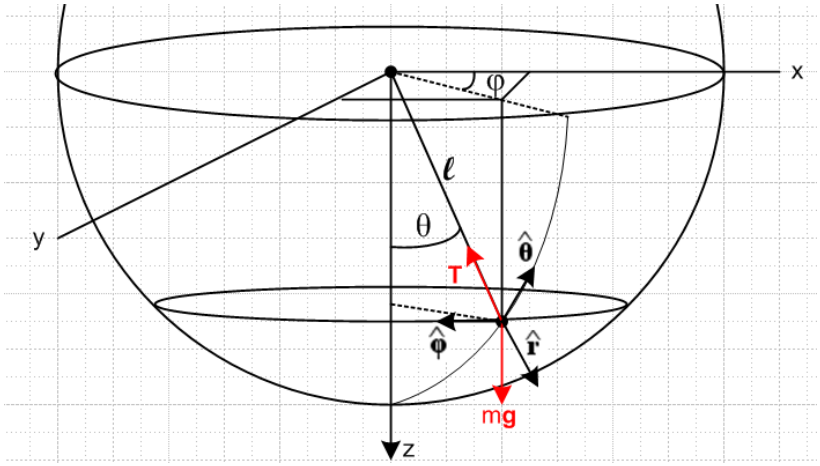
$\mathbf{e}_3 = -\hat{\mathbf{r}}'$ = "down" = $\hat{\mathbf{z}}$.

According to the picture above, $\boldsymbol{\omega} = \omega \mathbf{e'}_3$ can be expanded on these Frame S basis vectors,

$$\boldsymbol{\omega} = -\omega\cos\beta\, \mathbf{e}_3 + \omega\sin\beta\, \mathbf{e}_1 \quad . \tag{C.1.2}$$

As noted, the origin of Frame S is located distance ℓ above the surface of the Earth. If we position ourselves at this origin and gaze down onto the Earth's surface we see what is shown on the left below. A side view is presented on the right, and a 3D view below:





(C.1.3)

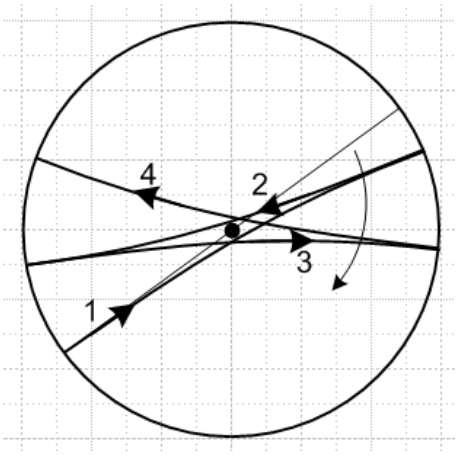
The meaning of the Frame S spherical coordinates r,θ,φ should be clear. These are the standard coordinates one would use to study a spherical pendulum in the absence of the Coriolis force.

The Cartesian basis vectors $\mathbf{e_n}$ for Frame S are related to the spherical ones for Frame S as follows (see (A.14c) or (E.2.7)),

$$\mathbf{e}_1 = \hat{\mathbf{x}} = \sin\theta\cos\varphi\ \hat{\mathbf{r}} + \cos\theta\cos\varphi\ \hat{\boldsymbol{\theta}} - \sin\varphi\ \hat{\boldsymbol{\varphi}}$$
$$\mathbf{e}_2 = \hat{\mathbf{y}} = \sin\theta\sin\varphi\ \hat{\mathbf{r}} + \cos\theta\sin\varphi\ \hat{\boldsymbol{\theta}} + \cos\varphi\ \hat{\boldsymbol{\varphi}}$$
$$\mathbf{e}_3 = \hat{\mathbf{z}} = \cos\theta\ \hat{\mathbf{r}} - \sin\theta\ \hat{\boldsymbol{\theta}}\ . \qquad \text{(C.1.4)}$$

C.2 Qualitative Solution

On each swing the pendulum veers "a little to the right" in the Northern Hemisphere due to the Coriolis fictitious force -2m **ω** x **v**, so the problem is to compute the time for a full 360 degree rotation. Here we show the first four swings of the pendulum (viewed from above)



(C.2.1)

The net effect is that the plane of the swinging pendulum precesses clockwise. The amount of veering shown in the drawing is highly exaggerated since we know that when the pendulum is located at the North Pole the period will be one sidereal day (≈ 23 hours 56 minutes). If each swing takes 10 seconds, that would be about 24*3600/10 = 8,640 swings for a full revolution, so each swing would show only 360/8640 ≈ .04 degrees of precession. Away from the North Pole we know the precession rate will be even slower, at the equator it will be zero, and in the Southern Hemisphere the pendulum will precess in the opposite direction. These qualitative facts may be deduced from the direction and magnitude of the Coriolis force as discussed in Section 8.3. The general pattern was noted for the small-angle limit of the theory in the solution (C.3).

C.3 Equations of Motion for the Foucault Pendulum (Spherical Coordinates)

In our "swap" notation context, the bogus Newton's law for non-inertial Frame S is,

$$\mathbf{F}_{\mathtt{eff}} = m\mathbf{a} \quad . \qquad \textit{(12.2.3)} \qquad \text{(C.3.1)}$$

The effective force $\mathbf{F}_{\mathtt{eff}}$ consists of real forces and fictitious forces. At the end of Section 8.5 in $(8.5.7)_{\mathtt{s}}$ we showed that, for surface-of-the-Earth problems (which are Special Case #1), and in the "swap" notation,

$$\mathbf{F}_{\mathtt{eff}} = (m\mathbf{g} + \text{possible other real forces}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} \quad , \qquad \textit{(8.5.7)}_{\mathtt{s}}$$

where force m**g** already incorporates the effect of the fictitious centrifugal force. In the spherical Foucault pendulum problem, we have "possible other real forces" = **T**, the tension of the massless pendulum string pulling on the pendulum of mass m. Thus,

$$\mathbf{F}_{\mathtt{eff}} = m\mathbf{g} + \mathbf{T} - 2m\,\boldsymbol{\omega} \times \mathbf{v} \quad . \qquad \text{(C.3.2)}$$

Although **g** does not point to the exact center of the Earth, we shall assume that it does, so $\mathbf{g} = g\hat{\mathbf{z}}$. The directional error in making this assumption is on the order of $\Delta\mathbf{g}/g_0 \sim .003\ \hat{\boldsymbol{\rho}}$ radians (see (8.5.8,9)) which is insignificant in our analysis of the Foucault pendulum. The error is probably even less than this when one considers that the Earth's surface is perpendicular to **g** and not $\mathbf{g}_0$, but we don't want to get involved with such fine details. Using $\mathbf{g} = g\hat{\mathbf{z}}$ in (C.3.2) and using the latter in (C.1.1) we find

$$m\mathbf{a} = mg\hat{\mathbf{z}} + \mathbf{T} - 2m\,\boldsymbol{\omega} \times \mathbf{v} \tag{C.3.3}$$

which is our vector "equation of motion" for the pendulum. Our task now is to evaluate the terms in (C.3.3) and then to balance the components on both sides to come up with three scalar equations of motion.

The $\boldsymbol{\omega}$ vector (C.1.2) can be expanded, using (C.1.4) for $\mathbf{e}_3$ and $\mathbf{e}_1$,

$$\begin{aligned}\boldsymbol{\omega} &= -\omega\cos\beta\,\mathbf{e}_3 + \omega\sin\beta\,\mathbf{e}_1 \\ &= -\omega\cos\beta[\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] + \omega\sin\beta[\cos\varphi\sin\theta\,\hat{\mathbf{r}} + \cos\varphi\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] \\ &= \omega(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\,\hat{\mathbf{r}} + \omega(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\,\hat{\boldsymbol{\theta}} + \omega(-\sin\beta\sin\varphi)\,\hat{\boldsymbol{\varphi}}\ .\end{aligned} \tag{C.3.4}$$

The string tension vector may be written as follows, where $T > 0$,

$$\mathbf{T} = -T\,\hat{\mathbf{r}}\ . \tag{C.3.5}$$

If one wants to consider unusual initial conditions for the pendulum which would cause negative string tension, such as starting it at position $\mathbf{r} = -\boldsymbol{\ell}\hat{\mathbf{z}}$ with some tiny velocity, one can imagine the string to be replaced with a massless rigid stick of length $\boldsymbol{\ell}$. Such a stick pendulum then works for either sign of tension T and then matches the equations of motion.

It remains to find expressions for **v** and **a** expanded on the spherical unit vectors. These are developed in (E.2.11) from which we quote:

$$\begin{aligned}(d\hat{\mathbf{r}}/dt)_S &= \dot{\hat{\mathbf{r}}} = \dot\theta\,\hat{\boldsymbol{\theta}} + \sin\theta\,\dot\varphi\,\hat{\boldsymbol{\varphi}} \\ (d\hat{\boldsymbol{\theta}}/dt)_S &= \dot{\hat{\boldsymbol{\theta}}} = -\dot\theta\,\hat{\mathbf{r}} + \cos\theta\,\dot\varphi\,\hat{\boldsymbol{\varphi}} \\ (d\hat{\boldsymbol{\varphi}}/dt)_S &= \dot{\hat{\boldsymbol{\varphi}}} = -\sin\theta\,\dot\varphi\,\hat{\mathbf{r}} - \cos\theta\,\dot\varphi\,\hat{\boldsymbol{\theta}}\ .\end{aligned} \qquad \textit{(E.2.11)} \tag{C.3.6}$$

Using (E.3.6) for acceleration **a** one then has,

$$\mathbf{r} = \ell\,\hat{\mathbf{r}}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \ell\,\dot{\hat{\mathbf{r}}} = \ell\,[\dot\theta\,\hat{\boldsymbol\theta} + \sin\theta\,\dot\varphi\,\hat{\boldsymbol\varphi}]$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ell\,(-\dot\theta^2 - \sin^2\theta\,\dot\varphi^2)\,\hat{\mathbf{r}} + \ell\,(\ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2)\,\hat{\boldsymbol\theta} + \ell\,(2\cos\theta\,\dot\theta\,\dot\varphi + \sin\theta\,\ddot\varphi)\,\hat{\boldsymbol\varphi} \qquad \text{(C.3.7)}$$

so we now have a viable **v** and **a** to use in (C.3.3). The Coriolis cross product can now be computed from (C.3.4) and (C.3.7),

$$\boldsymbol\omega \times \mathbf{v} =$$
$$[\omega(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\,\hat{\mathbf{r}} + \omega(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\,\hat{\boldsymbol\theta} + \omega(-\sin\beta\sin\varphi)\,\hat{\boldsymbol\varphi}]$$
$$\times\ [\ell\,[\dot\theta\,\hat{\boldsymbol\theta} + \sin\theta\,\dot\varphi\,\hat{\boldsymbol\varphi}]$$

or

$$\boldsymbol\omega \times \mathbf{v}\,/(\omega\ell) = (-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\,\hat{\mathbf{r}} \times [\dot\theta\,\hat{\boldsymbol\theta} + \sin\theta\,\dot\varphi\,\hat{\boldsymbol\varphi}]$$
$$+ (\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\,\hat{\boldsymbol\theta} \times [\dot\theta\,\hat{\boldsymbol\theta} + \sin\theta\,\dot\varphi\,\hat{\boldsymbol\varphi}]$$
$$+ (-\sin\beta\sin\varphi)\,\hat{\boldsymbol\varphi} \times [\dot\theta\,\hat{\boldsymbol\theta} + \sin\theta\,\dot\varphi\,\hat{\boldsymbol\varphi}]$$

$$= (-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\,[\dot\theta\,\hat{\boldsymbol\varphi} - \sin\theta\,\dot\varphi\,\hat{\boldsymbol\theta}] \qquad \text{// see (E.2.12)}$$
$$+ (\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\,[\sin\theta\,\dot\varphi\,\hat{\mathbf{r}}]$$
$$+ (-\sin\beta\sin\varphi)\,[-\dot\theta\,\hat{\mathbf{r}}]$$

$$= [(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\sin\theta\,\dot\varphi + \sin\beta\sin\varphi\,\dot\theta\,]\,\hat{\mathbf{r}}$$
$$- [(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\sin\theta\,\dot\varphi]\,\hat{\boldsymbol\theta}$$
$$+ [(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\,\dot\theta\,]\,\hat{\boldsymbol\varphi}\ . \qquad \text{(C.3.8)}$$

We can now assemble the pieces from (C.1.4), (C.3.5), (C.3.7) and (C.3.8) to write the equation of motion (C.3.3) divided by m,

$$\mathbf{a} = g\hat{\mathbf{z}} + \mathbf{T}/m - 2\,\boldsymbol\omega \times \mathbf{v}\,, \qquad \textit{(C.3.3)}$$

as

$$\ell(-\dot\theta^2 - \sin^2\theta\,\dot\varphi^2)\,\hat{\mathbf{r}} + \ell(\ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2)\,\hat{\boldsymbol\theta} + \ell(2\cos\theta\,\dot\theta\,\dot\varphi + \sin\theta\,\ddot\varphi)\,\hat{\boldsymbol\varphi}$$
$$= g(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol\theta}) - (T/m)\,\hat{\mathbf{r}}$$
$$- 2\omega\ell\,[(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\sin\theta\,\dot\varphi + \sin\beta\sin\varphi\,\dot\theta\,]\,\hat{\mathbf{r}}$$
$$+ 2\omega\ell\,[(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\sin\theta\,\dot\varphi]\,\hat{\boldsymbol\theta}$$
$$- 2\omega\ell\,[(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\,\dot\theta\,]\,\hat{\boldsymbol\varphi}\ .$$

Matching components gives these three scalar equations of motion (the unit vectors on the right are just reminders of the origin of the equations),

$$\boldsymbol{\ell}(-\dot\theta^2 - \sin^2\theta\,\dot\varphi^2) = g\cos\theta - (T/m) - 2\omega\boldsymbol{\ell}\,[(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\sin\theta\,\dot\varphi + \sin\beta\sin\varphi\,\dot\theta\,] \qquad \hat{\mathbf{r}}$$

$$\boldsymbol{\ell}(\ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2) = -g\sin\theta + 2\omega\boldsymbol{\ell}\,[(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\sin\theta\,\dot\varphi] \qquad \hat{\boldsymbol{\theta}}$$

$$\boldsymbol{\ell}(2\cos\theta\,\dot\theta\,\dot\varphi + \sin\theta\,\ddot\varphi) = -2\omega\boldsymbol{\ell}\,[(-\cos\beta\cos\theta + \sin\beta\cos\varphi\sin\theta)\,\dot\theta\,] \qquad \hat{\boldsymbol{\varphi}}$$

which we can simplify slightly to get

$$\dot\theta^2 + \sin^2\theta\,\dot\varphi^2 = -(g/\boldsymbol{\ell})\cos\theta + T/(m\boldsymbol{\ell}) + 2\omega\,[(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\sin\theta\,\dot\varphi + \sin\beta\sin\varphi\,\dot\theta\,]$$

$$\ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 = -(g/\boldsymbol{\ell})\sin\theta - 2\omega\sin\theta\,\dot\varphi\,(\cos\beta\cos\theta - \sin\beta\cos\varphi\sin\theta)$$

$$2\cos\theta\,\dot\theta\,\dot\varphi + \sin\theta\,\ddot\varphi = 2\omega\dot\theta\,(\cos\beta\cos\theta - \sin\beta\cos\varphi\sin\theta)\,. \qquad \text{(C.3.9)}$$

These are the equations of motion for a spherical pendulum operating on the rotating Earth with no approximations other than the modest ones that g does not vary over the small distance $\boldsymbol{\ell}$ and that $\mathbf{g} = g\hat{\mathbf{z}}$. Of course there is no air friction and the string is massless and lossless where it attaches.

Reader Exercise: Can equations (C.3.9) be obtained from the conventional Lagrangian or Hamiltonian formalisms? Are these approaches valid in non-inertial frames, or are modifications needed?

C.4 The Simple Pendulum

In this section we temporarily turn off the rotation of the Earth to study the behavior of pendulum without that complication. Setting $\omega = 0$ in (C.3.9) the pendulum equations of motion become

$$\dot\theta^2 + \sin^2\theta\,\dot\varphi^2 = -(g/\boldsymbol{\ell})\cos\theta + T/(m\boldsymbol{\ell}) \qquad \hat{\mathbf{r}}$$

$$\ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 = -(g/\boldsymbol{\ell})\sin\theta \qquad \hat{\boldsymbol{\theta}}$$

$$2\cos\theta\,\dot\theta\,\dot\varphi + \sin\theta\,\ddot\varphi = 0\,. \qquad \hat{\boldsymbol{\varphi}} \qquad \text{(C.4.1)}$$

We now seek a solution of (C.4.1) for which $\dot\varphi = 0$. In this case the last equation requires $\ddot\varphi = 0$ and we are left with just two equations,

$$\dot\theta^2 = -(g/\boldsymbol{\ell})\cos\theta + T/(m\boldsymbol{\ell}) \qquad \hat{\mathbf{r}}$$

$$\ddot\theta = -(g/\boldsymbol{\ell})\sin\theta\,. \qquad \hat{\boldsymbol{\theta}} \qquad \text{(C.4.2)}$$

These equations describe a pendulum that seems to swing in the plane φ = constant and so this is an example of a plane pendulum, also known as a simple pendulum. One can solve the second equation for $\theta(t)$, and then the first equation gives the tension $T(t)$.

If the pendulum is started at some polar angle θ_0 and released perfectly so $\varphi = \varphi_0$ and $\dot\varphi = 0$, the pendulum swings in a plane, but at it swings through $\theta = 0$ the coordinate $\varphi = \varphi_0$ discontinuously jumps to $\varphi = \varphi_0 + \pi$, so there is a continuity issue for φ at $\theta = 0$. We could treat this technically using Heaviside

and delta functions, but shall not go down that road. It is a simple fact that, for spherical coordinates, points on the z-axis where $\theta = 0$ have an undefined value of φ.

For small angles θ, one finds for the above initial condition that

$$\ddot{\theta} + (g/\ell)\theta = 0 \quad \Rightarrow \quad \theta(t) = \theta_0 \cos(\Omega t) \quad \Omega \equiv \sqrt{g/\ell} \qquad \{\theta(0) = \theta_0 \, . \dot{\theta}(0) = 0\} \tag{C.4.3}$$

On the other hand, if the pendulum is initialized with $\theta = 0$ and some velocity sufficient to take it up to a maximum angle θ_0 we get

$$\ddot{\theta} + (g/\ell)\theta = 0 \quad \Rightarrow \quad \theta(t) = \theta_0 \sin(\Omega t) \quad \Omega \equiv \sqrt{g/\ell} \qquad \{\theta(0) = 0, \, \theta_{max} = \theta_0 \} \tag{C.4.4}$$

Exact Solution for the Simple Pendulum

We provide this detailed solution because it generally does not appear in textbooks.

For larger θ_0 the simple pendulum equation $\ddot{\theta} + (g/\ell)\sin\theta = 0$ falls into a class of second order non-linear ODE's which have the form $\ddot{x} = f(x)$ where $\ddot{x}$ means $\partial_t^2 x$ and we seek x(t). The solution is not hard to obtain, as we now outline. The first step is to define $v \equiv \dot{x}$:

$$v \equiv \dot{x} \qquad \Rightarrow \qquad \ddot{x} = (dv/dt) = (dv/dx)(dx/dt) = (dv/dx)v \qquad \Rightarrow \qquad (dv/dx)v = f(x)$$

$$\Rightarrow \quad vdv = f(x)dx \qquad \Rightarrow \qquad (v^2/2) = \int^x f(x')dx' + C$$

$$\Rightarrow \quad (dx/dt) = \sqrt{2} \qquad \Rightarrow \qquad dt = \frac{1}{\sqrt{2}} \, \frac{dx}{\sqrt{\int^x f(x')dx' + C}}$$

$$\Rightarrow \quad t(x) = \left(\frac{1}{\sqrt{2}} \int^x \frac{dx''}{\sqrt{\int^{x''} f(x')dx' + C}} \right) + C' \tag{C.4.5}$$

where integration constants C and C' are determined by initial conditions. This solution gives t = t(x) which one must then "invert" to obtain x = x(t).

If we take $f(x) = -\Omega^2 \sin x$ and then $x \rightarrow \theta$ the $(v^2/2)$ result of (C.4.5) becomes

$$(\dot{\theta}^2/2) = \int^\theta f(\theta')d\theta' + C = -\Omega^2 \int^\theta \sin(\theta')d\theta' + C = \Omega^2 \cos\theta + C \, . \tag{C.4.6}$$

We set the zero of potential energy for the pendulum at the bottom, $\theta = 0$. At angle θ the pendulum has risen a height $h = \ell - \ell\cos\theta$ so the potential energy at θ is then $V = mgh = mg\ell(1-\cos\theta)$.

We shall now apply the boundary conditions of (C.4.4). At $t = 0$ the pendulum is at $\theta = 0$ and we give it a kick in the $+\theta$ direction with some initial velocity $\ell\dot{\theta}(0)$. We assume that this causes the pendulum to rise up to some max angle $\theta_0 < \pi$. A too-large kick results in over-the-top behavior which we exclude.

At $t = 0$ the total pendulum energy is $(1/2)m[\ \ell\dot{\theta}(0)]^2$.

At the top of the swing the total energy is $mg\ell(1-\cos\theta_0)$. Therefore from energy conservation,

$$(1/2)m[\ \ell\dot{\theta}(0)]^2 = mg\ell(1-\cos\theta_0)$$

or

$$(1/2)[\ \dot{\theta}(0)]^2 = (g/\ell)(1-\cos\theta_0) = \Omega^2(1-\cos\theta_0)$$

so

$$\dot{\theta}(0) = \sqrt{2}\,\Omega\sqrt{1-\cos\theta_0} = 2\Omega\sin(\theta_0/2) \ . \qquad \text{(C.4.7)}$$

Evaluating (C.4.6) at $t = 0$ gives

$$(1/2)[\ \dot{\theta}(0)]^2 = \Omega^2\cos(0) + C = \Omega^2 + C \ .$$

Comparing this last equation with the middle line of (C.4.7) gives,

$$C = -\Omega^2\cos\theta_0 \ . \qquad \text{(C.4.8)}$$

From (C.4.5) the solution for $t(\theta)$ is then

$$\Rightarrow\ t(\theta) = \left(\frac{1}{\sqrt{2}}\int^{\theta}\frac{d\theta''}{\sqrt{\int^{\theta''} f(\theta')d\theta' + C}}\right) + C' = \left(\frac{1}{\sqrt{2}}\int^{\theta}\frac{d\theta''}{\sqrt{\Omega^2\cos\theta'' + C}}\right) + C'$$

$$= \left(\frac{1}{\sqrt{2}\,\Omega}\int^{\theta}\frac{d\theta''}{\sqrt{\cos\theta'' - \cos\theta_0}}\right) + C' \qquad \text{(C.4.9)}$$

The integral of interest appears on page 179 of GR7 as 2.571.4 (where we shall use the second form),

4. $$\int \frac{dx}{\sqrt{a+b\cos x}} = \frac{2}{\sqrt{a+b}}F\left(\frac{x}{2}, r\right) \qquad [a > b > 0, \quad 0 \le x \le \pi]$$

$$= \sqrt{\frac{2}{b}}F\left(\gamma, \frac{1}{r}\right) \qquad \left[b \ge |a| > 0, \quad 0 \le x < \arccos\left(-\frac{a}{b}\right)\right]$$

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$$\gamma = \arcsin\sqrt{\frac{b(1-\cos x)}{a+b}}, \quad r = \sqrt{\frac{2b}{a+b}} \qquad \text{(C.4.10)}$$

and where F(φ,k) is "the elliptic integral of the first kind". In our case b = 1 and a = -cosθ₀ so

$$r = \sqrt{\frac{2}{a+1}} = \sqrt{\frac{2}{1-\cos\theta_0}} = \frac{1}{\sin(\theta_0/2)}$$

$$\gamma = \sin^{-1}\left[\sqrt{\frac{1-\cos\theta}{1-\cos\theta_0}}\right] = \sin^{-1}\left[\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right] . \qquad \text{(C.4.11)}$$

Using integral evaluation (C.4.10) in (C.4.9) one finds that,

$$t(\theta) = \frac{1}{\Omega} F\left(\sin^{-1}\left[\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\right], \sin(\theta_0/2)\right) + C'$$

$$= \frac{1}{\Omega} F(\varphi, k) + C' \qquad \text{where} \quad \varphi = \sin^{-1}\left[\frac{\sin(\theta/2)}{k}\right], \quad k = \sin(\theta_0/2) . \qquad \text{(C.4.12)}$$

The function F is defined in GR7 page 860 as 8.111.2,

2. The elliptic integral of the first kind:

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} = \int_0^{\sin\varphi} \frac{dx}{\sqrt{(1 - x^2)(1 - k^2 x^2)}} \qquad \text{(C.4.13)}$$

from which we see that F(0,k) = 0. Recall our boundary condition that θ(0) = 0 with some $\dot{\theta}(0) > 0$. From (C.4.12) we find at t = 0 and θ = 0 that

$$0 = \frac{1}{\Omega} F(0, k) + C' = 0 + C' \qquad \Rightarrow \qquad C' = 0 . \qquad \text{(C.4.14)}$$

Our <u>final solution before inversion</u> is then

$$t(\theta) = \frac{1}{\Omega} F(\varphi, k) \qquad \text{where} \quad \varphi = \sin^{-1}\left[\frac{\sin(\theta/2)}{k}\right], \quad k = \sin(\theta_0/2), \quad \Omega \equiv \sqrt{g/\ell} . \qquad \text{(C.4.15)}$$

The Jacobi elliptic function sn(u) is usually defined in this rather obscure manner,

8.144

1. $$u = \int_0^{\operatorname{sn} u} \frac{dt}{\sqrt{(1 - t^2)(1 - k^2 t^2)}} \qquad \text{(C.4.16)}$$

If one writes sn(u) = sin(φ), then the right sides of (C.4.13) and (C.4.16) are the same, so

$$F(\varphi,k) = u = \operatorname{sn}^{-1}(\sin\varphi) \qquad \text{or} \qquad \operatorname{sn}(F(\varphi,k)) = \sin\varphi . \qquad \text{(C.4.17)}$$

We now apply this last equation to (C.4.15) as follows:

$$\mathrm{sn}(F(\varphi,k)) = \sin\varphi$$

or

$$\mathrm{sn}(\Omega t) = \sin\left\{ \sin^{-1}\left[\frac{\sin(\theta/2)}{k} \right] \right\} = \frac{\sin(\theta/2)}{k}$$

so

$$\sin(\theta/2) = k\,\mathrm{sn}(\Omega t) \qquad k = \sin(\theta_0/2)$$

and

$$\theta(t) = 2\sin^{-1}(k\,\mathrm{sn}(\Omega t)) \qquad k = \sin(\theta_0/2) \qquad \text{(C.4.18)}$$

and the inversion of (C.4.15) is now complete. Following the convention of GR7, since sn(u) has an implicit parameter k, we make it explicit by writing sin(u) = sin(u,k) and then our solution is

$$\sin(\theta/2) = k\,\mathrm{sn}(\Omega t,k) \qquad k = \sin(\theta_0/2)$$
$$\theta(t) = 2\sin^{-1}(k\,\mathrm{sn}(\Omega t,k)) \qquad k = \sin(\theta_0/2) \qquad \text{(C.4.19)}$$

or

$$\sin(\theta/2) = \sin(\theta_0/2)\,\mathrm{sn}(\Omega t, \sin(\theta_0/2)) \qquad \text{// boundary conditions below} \Rightarrow \theta_{\mathbf{max}} = \theta_0$$
$$\theta(t) = 2\sin^{-1}[\sin(\theta_0/2)\,\mathrm{sn}(\Omega t, \sin(\theta_0/2))] \qquad \text{for } \theta(0) = 0,\ \dot{\theta}(0) = 2\Omega\sin(\theta_0/2)\,. \qquad \text{(C.4.20)}$$

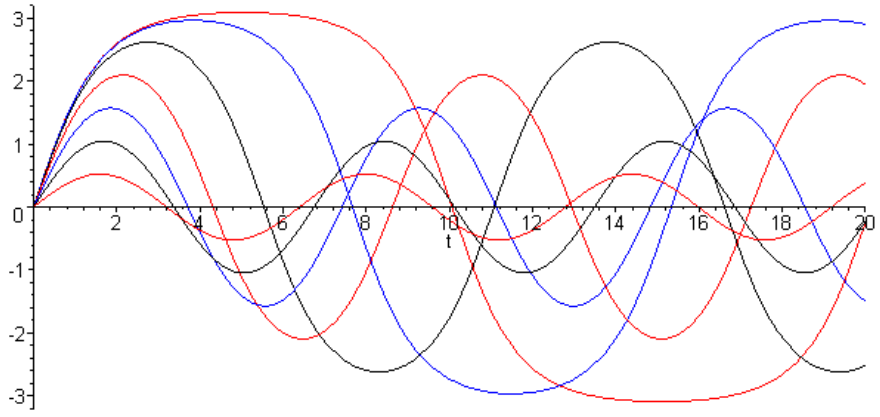
When k is not too large, one has sn(x,k) ≈ sin(x). If θ_0 is small, then so is θ and the first line of (C.4.20) reads,

$$\theta/2 \approx (\theta_0/2)\sin(\Omega t) \qquad \Rightarrow \qquad \theta(t) = \theta_0 \sin(\Omega t) \qquad \text{(C.4.21)}$$

which is the correct small-angle solution for θ(0) = 0 and max angle θ_0 as was stated in (C.4.4).

Here is a Maple plot of θ(t) from (C.4.20) with a selection of peak angles θ_0 with Ω = 1 so that the small-angle period is T = 2π/Ω = 2π :

```
restart; alias(sn=JacobiSN): with(plots):
theta := 2*arcsin(k*sn(Omega*t,k));
                                θ := 2 arcsin(k sn(Ω t, k))
k := sin(theta0/2):  Omega := 1:
theta0 := th0deg * (Pi/180):
deg := [30,60,90,120,150,170,177]:  n := 7:
col := [red,black,blue,red,black,blue,red]:
for i from 1 to n do
   th0deg := deg[i];
   s[i] := plot(theta,t=0..20, color=[col[i]]):
od:
display(seq(s[i],i=1..n));
```



(C.4.22)

For small θ_0 the function $\theta(t)$ is very sine-like with period $T = 2\pi$, but for larger θ_0 the period increases and the top flattens out. For $\theta_0 = 177°$ the top is quite flat, indicating a long "hang time" when the pendulum (on its rigid massless stick) lingers in the near-vertical position.

The *complete* elliptic integral of the first kind is defined by (some sources write K as **K**),

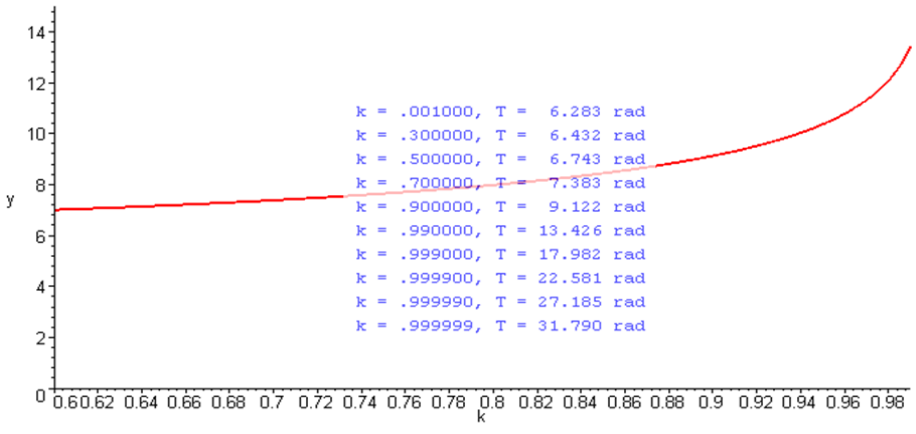
$$K(k) \equiv F(\pi/2,k) = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2\sin^2\alpha}} \qquad \text{// see (C.4.13), and note } K(0) = \pi/2 \tag{C.4.23}$$

Since $sn(u) = \sin(\varphi)$ as shown above (C.4.17), we know that the peak value of sn(u) is 1 and this occurs one quarter of a wave into the sn(u) waveform. From (C.4.17) we know that

$$sn(F(\varphi,k)) = \sin\varphi \quad \Rightarrow \quad sn(F(\pi/2,k)) = 1 \quad \Rightarrow \quad sn(K(k)) = 1 \tag{C.4.24}$$

Thus, a quarter period of the function sn(u,k) along the real u axis is just K(k) and so the full period is then T = 4K(k). Here is a plot of 4K(k) showing how it increases from 2π to larger values *slowly* approaching the limit $4K(\infty) = \infty$:

```
plot(4*EllipticK(k),k = 0..0.99,y=0..10,thickness=2);
```



(C.4.25)

Since a quarter period of sn(x,k) is K(k), a quarter period of sn(Ωt,k) is then K(k)/Ω. For the more standard boundary conditions shown in (C.4.3) where $\theta(0) = \theta_0$ and $\dot{\theta}(0) = 0$, the solution is obtained by shifting (C.4.20) ahead by a quarter period, which means taking $t \to t + K(k)/\Omega$ or $\Omega t \to \Omega t + K(k)$. The result is then, with $k = \sin(\theta_0/2)$,

$$\sin(\theta/2) = \sin(\theta_0/2)\, sn(\Omega t + K(\sin(\theta_0/2)), \sin(\theta_0/2))$$

$$\theta(t) = 2 \sin^{-1}[\, \sin(\theta_0/2)\, sn(\Omega t + K(\sin(\theta_0/2)), \sin(\theta_0/2))\,] \qquad \theta(0) = \theta_0\ , \quad \dot{\theta}(0) = 0\ . \qquad \text{(C.4.26)}$$

Now for small angles the first line gives

$$\theta/2 \approx \theta_0/2 \sin(\Omega t + \pi/2) = \theta_0/2 \cos(\Omega t) \quad \Rightarrow \quad \theta(t) = \theta_0 \cos(\Omega t) \qquad \text{(C.4.27)}$$

in agreement with (C.4.3)

Conventions: We have adopted the notation for K(k) and sn(u,k) which is used by the 2010 *NIST Handbook of Mathematical Functions*, by Gradshteyn and Ryzhik 7th edition (2007), and by the Bateman Manuscript Project (1953), though the last two sources write K as **K**. The reader is warned that there is another common convention which appears in the precursor to the NIST Handbook, namely the heavily-used 1964 *Handbook of Mathematical Functions* of Abramowitz and Segun (AS). The connection is this

$$K(k) = K_{AS}(k^2) \qquad // \ k^2 = m$$
$$sn(u,k) = sn_{AS}(u,k^2) = sn(u \mid k^2) \qquad // \text{ sometimes } = sn(u;\, k^2) \qquad \text{(C.4.28)}$$

In using these functions one must be very careful to learn the convention used by a given source. For example, our ancient Maple V states that (notice the k^2 in the integral but k in the argument),

- JacobiAM is the inverse of the trigonometric form of the elliptic integral of the first kind, EllipticF:

```
EllipticF(phi,k) = int(1/sqrt(1-k^2*sin(theta)^2),theta=0..phi)
JacobiAM(EllipticF(sin(phi),k),k) = phi
```

```
JacobiSN(z,k) = sin(JacobiAM(z,k))
```

// $am(z,k) = \sin^{-1} sn(z,k)$

So this version of Maple is using our "modern" NIST 2010 notation for argument k.

C.5 The Spherical and Foucault Pendulums

Having studied the simple pendulum, we now return the more general spherical pendulum and at the end we look at the Foucault mode of this pendulum. Since there are many subsections below, here is a list of their headings:

(a) The equations of motion for the Spherical Pendulum
(b) L_z as constant of the motion
(c) E as another constant of the motion
(d) Exact solution to the Spherical Pendulum (outline)
(e) The nature of the general solution for the Spherical Pendulum
(f) The Conical Motion solution of the Spherical Pendulum
(g) The thin ellipse scenario for the Spherical Pendulum
(h) The Intrinsic Airy Precession of the Spherical Pendulum
(i) The Foucault Mode of the Spherical Pendulum
(j) Interference between the Airy and Foucault precession
(k) The Foucault Pendulum at the Pantheon in Paris
(ℓ) General Numerical solutions of the Spherical Pendulum

The French name Foucault is pronounced foo-ko' with accent on the ko (sounds like so).

(a) The equations of motion for the Spherical Pendulum

As before, we turn off the rotation of the Earth by setting $\omega = 0$, so equations (C.3.9) become

$$\begin{aligned} &\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 = -(g/\ell)\cos\theta + T/(m\ell) && \hat{\mathbf{r}} \\ &\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = -(g/\ell)\sin\theta && \hat{\boldsymbol{\theta}} \\ &2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} = 0 \quad . && \hat{\boldsymbol{\varphi}} \end{aligned} \qquad \text{(C.5.1)}$$

These are the equations of motion for a spherical pendulum in the presence of a uniform gravitational field of strength g. It is useful to know something about the solution of these equations before we turn the Earth's rotation back on.

Operationally, in the sense of a numerical solution, one can regard the last two equations of (C.5.1) as a pair of coupled second-order non-linear ODE's for functions $\theta(t)$ and $\varphi(t)$ subject to initial conditions θ_0, φ_0, $\dot{\theta}_0$ and $\dot{\varphi}_0$. Once these equations are solved (physically we know a unique solution must exist), the first equation may be used to determine T(t), the string tension.

(b) L_z as constant of the motion

The last equation in (C.5.1) may be written in this form

$$d/dt\,(\sin^2\theta\,\dot{\varphi}) = 0 \qquad \text{(C.5.2)}$$

which says that $\sin^2\theta\,\dot{\varphi}$ must be a "constant of the motion". To understand the meaning of this constant, we first compute the torque about the string pivot point due to the gravitational force on the mass m,

$$\mathbf{N} = \mathbf{r} \times m\mathbf{g} = \mathbf{r} \times mg\hat{\mathbf{z}} = mg\ell\,\hat{\mathbf{r}} \times [\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] = -mg\ell\sin\theta\,\hat{\boldsymbol{\varphi}} \ . \tag{C.5.3}$$

Since this torque lies in a plane normal to the z axis, as in (C.1.3), we may conclude that the Cartesian torque component $N_z = 0$. The angular version of Newton's Law says $\mathbf{N} = d\mathbf{L}/dt$, and therefore we expect that the quantity L_z will be a constant of the motion. Direct calculation shows that

$$L_z = \mathbf{L} \bullet \hat{\mathbf{z}} = m\,(\mathbf{r} \times \mathbf{v}) \bullet \hat{\mathbf{z}} = m\,(\hat{\mathbf{z}} \times \mathbf{r}) \bullet \mathbf{v} = m\,\ell\sin\theta\,\hat{\boldsymbol{\varphi}} \bullet (\ell\,[\dot{\theta}\,\hat{\boldsymbol{\theta}} + \sin\theta\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}}])$$
$$= m\ell^2\sin^2\theta\,\dot{\varphi} \tag{C.5.4}$$

where we have made use of (C.1.4) for $\hat{\mathbf{z}}$ and (C.3.7) for $\mathbf{v}$. Thus, we see that our third equation of motion in (C.5.1), rewritten as in (C.5.2), is just the statement that $dL_z/dt = 0$. In the Lagrangian formulation one finds that L_z is one of the canonical momenta which is constant because φ is a cyclic coordinate, meaning it does not appear in the Lagrangian (see Comment below).

In terms of the spherical unit vectors, one finds (again using (C.3.7) for **v)** that the vector angular momentum of the pendulum mass is given by

$$\mathbf{L} = m\,\mathbf{r} \times \mathbf{v} = m\ell^2\,[\dot{\theta}\,\hat{\boldsymbol{\varphi}} - \sin\theta\,\dot{\varphi}\,\hat{\boldsymbol{\theta}}] \ . \tag{C.5.5}$$

Then application of $\mathbf{N} = d\mathbf{L}/dt$, with **N** as in (C.5.3) and unit-vector change rates as in (C.3.6), simply reproduces the last two equations of motion in (C.5.1).

So we conclude that :

$$h \equiv L_z/(m\ell^2) = \sin^2\theta\,\dot{\varphi} \tag{C.5.6}$$

is a constant of the motion of the spherical pendulum.

Comment: In Lagrangian dynamics one has $\mathcal{L}$ = KE-PE = $(1/2)mv^2 + m\ell\cos\theta$ with $v^2 = \ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\varphi}^2)$. The Euler Lagrange equations $d_t(\partial\mathcal{L}/\partial\dot{\theta}) = (\partial\mathcal{L}/\partial\theta)$ and $d_t(\partial\mathcal{L}/\partial\dot{\varphi}) = (\partial\mathcal{L}/\partial\varphi)$ produce the last two equations of (C.5.1). Since φ is cyclic (does not appear in $\mathcal{L}$) the second equation says $d_t(\partial\mathcal{L}/\partial\dot{\varphi}) = 0$ which says $d_tL_z = 0$. The first equation in (C.5.1) does not appear since it is an equation of constraint.

(c) E as another constant of the motion

We have shown in (C.5.6) that $\dot{\varphi} = h/\sin^2\theta$. If this is substituted into the second equation of (C.5.1) one obtains

$$\ddot{\theta} - h^2\cos\theta/\sin^3\theta + (g/\ell)\sin\theta = 0 \tag{C.5.7}$$

and we just put this equation on hold for a moment, noting that it is a second-order non-linear ODE.

Another constant of the motion is the total energy E which may be regarded as E = T + V = kinetic energy + potential energy (here the zero point of potential is put at the pendulum pivot point),

$$E = (1/2)mv^2 - mg\ell\cos\theta = (1/2)\, m\, \ell^2(\dot\theta^2 + \sin^2\theta\dot\varphi^2) - mg\ell\cos\theta \qquad (C.5.8)$$

where $v^2 = \ell^2(\dot\theta^2 + \sin^2\theta\,\dot\varphi^2)$ according to (C.3.7) for **v**. By rescaling the energy, we can take this constant of the motion to be (note that E and $\mathcal{E}$ could have either sign),

$$\mathcal{E} = \dot\theta^2/2 + \sin^2\theta\dot\varphi^2/2 - (g/\ell)\cos\theta \quad . \qquad E = m\,\ell^2\mathcal{E} \qquad (C.5.9)$$

Using (C.5.6) to eliminate $\dot\varphi$ gives,

$$\mathcal{E} = \dot\theta^2/2 + (h^2/2\sin^2\theta) - (g/\ell)\cos\theta$$

or (C.5.10)

$$\mathcal{E} = \dot\theta^2/2 + V_e(\theta) \qquad \text{where} \qquad V_e(\theta) \equiv (h^2/2\sin^2\theta) - (g/\ell)\cos\theta \quad .$$

Unlike (C.5.7), equation (C.5.10) is a first-order ODE so we prefer it to (C.5.7). In fact, if one multiplies (C.5.7) by $\dot\theta$ and notes that $\dot\theta\,\ddot\theta = (1/2)\,\partial_t(\dot\theta^2)$ and does a few easy integrals, one obtains an equation of the form ∂_t[stuff] = 0 so then stuff = constant. That "stuff" is the right side of (C.5.10) and we have then provided an interpretation for the constant (energy) [$\dot\theta$ is an "integrating factor" for (C.5.7)].

(d) Exact solution to the Spherical Pendulum (outline)

Equation (C.5.10) is a first order non-linear ODE for θ(t) which we know how to solve:

$$\dot\theta = (2\mathcal{E} - 2V_e(\theta))^{1/2} \qquad \Rightarrow \qquad d\theta = (2\mathcal{E} - 2V_e(\theta))^{1/2}dt$$

$$\Rightarrow\ dt = (2\mathcal{E} - 2V_e(\theta))^{-1/2}\, d\theta$$

boundary condition

$$\Rightarrow\quad t(\theta) = \int_{\theta_0}^{\theta} d\theta' \, \frac{1}{\sqrt{2\mathcal{E} - (h^2/\sin^2\theta') + (2g/\ell)\cos\theta'}} \quad . \qquad //\ t(\theta_0) = 0 \text{ so } \theta(0) = \theta_0 \qquad (C.5.11)$$

Letting $z' = \cos\theta'$, so $dz' = -\sin\theta'\, d\theta' = -\sqrt{1-z'^2}\, d\theta'$ one finds

$$\frac{d\theta'}{\sqrt{2\mathcal{E} - (h^2/\sin^2\theta') + (2g/\ell)\cos\theta'}} = -\frac{dz'}{\sqrt{1-z'^2}}\,\frac{1}{\sqrt{2\mathcal{E} - (h^2/[1-z'^2]) + (2g/\ell)z'}}$$

$$= -\frac{dz'}{\sqrt{2\mathcal{E}\,[1-z'^2] - h^2 - (2g/\ell)z'[1-z'^2]}} = -\frac{dz'}{\sqrt{2\mathcal{E} - 2\mathcal{E}z'^2 - h^2 - (2g/\ell)z' + (2g/\ell)\,z'^3}}$$

$$= - \frac{1}{\sqrt{(2g/\ell)\,[\ z'^3 - (\mathcal{E}\ell/g)\,z'^2 - z' + (\ell/g)\,(\mathcal{E} - h^2/2\ )\ ]}}\ . \qquad \text{(C.5.12)}$$

Therefore

$$t(\theta) = \int_z^{z_0} dz' \frac{1}{\sqrt{(2g/\ell)\,[\ z'^3 - (\mathcal{E}\ell/g)\,z'^2 - z' + (\ell/g)\,(\mathcal{E} - h^2/2\ )\ ]}} \qquad // \ z = \cos\theta,\ z_0 = \cos\theta_0$$

$$= \sqrt{\ell/(2g)}\ \int_z^{z_0} dx\ \frac{1}{\sqrt{(x-a)(x-b)(x-c)}} \qquad \text{(C.5.13)}$$

where a,b,c are the roots of the cubic equation $x^3 - (\mathcal{E}\ell/g)\,x^2 - x + (\ell/g)\,(\mathcal{E} - h^2/2\ ) = 0$. The dimensionless integral appearing on the last line can be evaluated in closed form using this indefinite integral,

```
Int(((x-a)*(x-b)*(x-c))^(-1/2),x);
```

$$\int \frac{1}{\sqrt{(x-a)\,(x-b)\,(x-c)}}\,dx$$

```
value(%);
```

$$2\,\frac{(-a+b)\sqrt{\frac{x-a}{-a+b}}\sqrt{\frac{x-c}{a-c}}\sqrt{\frac{x-b}{a-b}}\ \text{EllipticF}\left(\sqrt{\frac{x-a}{-a+b}},\sqrt{\frac{a-b}{a-c}}\right)}{\sqrt{x^3 - x^2 c - b x^2 + b x c - a x^2 + a x c + a b x - a b c}} \qquad \text{(C.5.14)}$$

Here EllipticF is the incomplete elliptic integral of the first kind as was defined in (C.4.13) and (C.4.17),

$$\text{EllipticF}(\sin(\varphi),k) = \int_0^{\sin(\varphi)} dt\ \frac{1}{\sqrt{(1-t^2)(1-k^2t^2)}} = F(\varphi,k) = sn^{-1}(\sin\varphi,k)\ . \qquad \text{(C.5.15)}$$

There are standard (albeit complicated) formulas for the roots of a cubic. Therefore, we have a closed form result for $t(\theta)$ which can then be "inverted" to obtain a solution for $\theta(t)$. Then from (C.5.6) we get

$$\dot{\varphi} = h/\sin^2(\theta(t)) \qquad \Rightarrow \qquad \varphi(t) = \varphi_0 + h\ \int_0^t dt'\ /\sin^2(\theta(t')) \qquad \text{(C.5.16)}$$

and the problem of the spherical pendulum is completely solved more or less in closed form. The string tension is then determined by the first equation in (C.5.1). The solution $\theta(t)$ is a function of θ_0, ℓ, g, m and the two constants of the motion $E = m\ell^2\mathcal{E}$ and $L_z = m\ell^2 h$. Somehow it must describe both low-amplitude pendulum motions ($E < 0$) as well as violent high-speed over-the-top maneuvers (large $E > 0$).

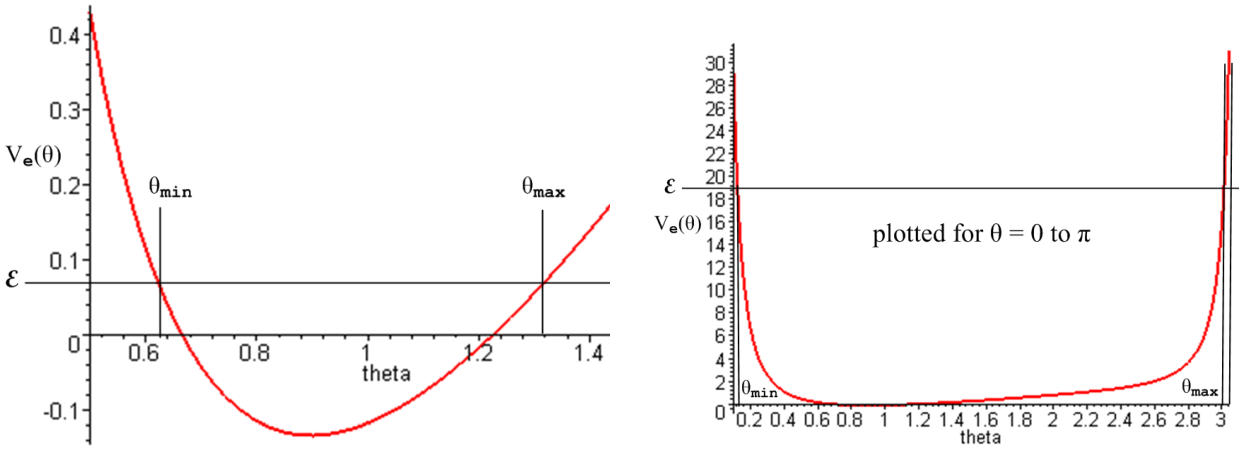
(e) The nature of the general solution for the Spherical Pendulum

Recall (C.5.10) where $\mathcal{E}$ is the total scaled energy for the pendulum mass,

$$\mathcal{E} = \dot{\theta}^2/2 + V_e(\theta) \qquad \text{where} \qquad V_e(\theta) \equiv (h^2/2\sin^2\theta) - (g/\ell)\cos\theta \ . \qquad \textit{(C.5.10)}$$

The "effective potential" $V_e(\theta)$ has the following shape (plotted here for $h^2/2 = .3$ and $(g/\ell) = 1$)

```
plot(.3/sin(theta)^2 - cos(theta),theta=0.5..Pi/2,thickness=2);
```



(C.5.17)

Since $\dot{\theta}^2/2 = (\mathcal{E} - V_e(\theta))$ must be positive, one must have $V_e(\theta) \le \mathcal{E}$ which is valid only between θ_{min} and θ_{max} as shown. These are the "turning points" where $\dot{\theta} = 0$. Differentiation of (C.5.10) tells us that,

$$0 = \dot{\theta}\,\ddot{\theta} + V_e'(\theta)\,\dot{\theta} \qquad // \ V_e'(\theta) \equiv dV_e(\theta)/d\theta = \text{slope of } V_e(\theta)$$

or

$$\ddot{\theta} = -V_e'(\theta) \ . \qquad \text{(C.5.18)}$$

At the right turning point the slope $V_e'(\theta)$ is positive so $\ddot{\theta} < 0$ which accelerates the particle to the left. At the left turning point the slope $V_e'(\theta)$ is negative so $\ddot{\theta} > 0$ which accelerates the particle to the right. The particle therefore bounces back and forth between these two turning points in some manner. Thus the motion of the pendulum is constrained on the spherical surface $r = \ell$ between two horizontal circles of angles θ_{min} and θ_{max} and hits both these angles once per "oscillation".

Meanwhile, from (C.5.16) the action in the φ dimension of the problem is controlled by

$$\dot{\varphi} = h/\sin^2(\theta(t)) \qquad \Rightarrow \qquad \varphi(t) = \varphi_0 + h \int_0^t dt' /\sin^2(\theta(t')) \qquad \textit{(C.5.16)} \qquad \text{(C.5.19)}$$

so as θ(t) does the oscillation just discussed between θ_{min} and θ_{max}, φ(t) increases as shown in its own complicated functional manner.

A quick tour of web animations of the spherical pendulum shows the amazing complexity of the possible motions. If the string is replaced with a massless stiff rod, over-the-top motions are included. Some examples (search youtube if these are dead links) :

http://www.youtube.com/watch?v=6hCLkTENfSA .
http://www.youtube.com/watch?v=VS1dU5HpfOM&feature=relmfu

Both animations demonstrate the $\theta_{min} \le \theta \le \theta_{max}$ idea, though it takes a while to see in the first animation.

(f) The Conical Motion solution of the Spherical Pendulum

The spherical pendulum has an obvious simple solution where $\theta = \theta_0$ = constant, so the string motion traces out a cone. In this case the equations of motion (C.5.1) become

$$\begin{aligned} \sin^2\theta_0\,\dot{\varphi}^2 &= -(g/\ell)\cos\theta_0 + T/(m\ell) && \hat{\mathbf{r}} \\ \cos\theta_0\,\dot{\varphi}^2 &= (g/\ell) && \hat{\boldsymbol{\theta}} \\ \ddot{\varphi} &= 0 \quad . && \hat{\boldsymbol{\varphi}} \end{aligned} \qquad \text{(C.5.20)}$$

The second equation says $\omega_\varphi \equiv \dot{\varphi} = \sqrt{(g/\ell)\sec\theta_0}$. Since this is a constant, the third equation is satisfied as well, and then the first says $T = mg\sec\theta_0$. In the small angle limit for θ_0, ω_φ slows down to its smallest possible value $\omega_\varphi \equiv \sqrt{g/\ell} = \Omega$ which is the frequency of a small-angle plane pendulum. Conversely, as we try to achieve $\theta_0 \to \pi/2$, $\sec\theta_0 \to \infty$ and both ω_φ and tension T become infinite, which seems pretty reasonable. This solution can of course be obtained by elementary methods as well.

Reader Exercise: For the conical motion solution, $\dot{\theta} = 0$ so $\mathcal{E} = V_e(\theta)$ in (C.5.10). How does this fit in with Fig (C.5.17)? Is the pendulum stuck at a turning point? Are there two conical solutions for any given energy $\mathcal{E}$? Can one have a small conical motion with θ near π (string = stick)?

(g) The thin ellipse scenario for the Spherical Pendulum

We now return to the general spherical pendulum equations

$$\begin{aligned} \dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 &= -(g/\ell)\cos\theta + T/(m\ell) && \hat{\mathbf{r}} \\ \ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 &= -(g/\ell)\sin\theta && \hat{\boldsymbol{\theta}} \\ 2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} &= 0 \quad . && \hat{\boldsymbol{\varphi}} \end{aligned} \qquad \textit{(C.5.1)} \qquad \text{(C.5.21)}$$

In the plane pendulum analysis of Section C.4 we set $\dot{\varphi} = 0$ to simplify these equations. However, once the plane swing path is opened even slightly into a thin elliptical path, the $\varphi(t)$ function becomes just as active as the $\theta(t)$ function. For such a thin ellipse, θ bounces between the θ_{max} and θ_{min} turning points and θ_{min} will be very small. Roughly speaking, for each full elliptical swing cycle of θ, $\varphi(t)$ wraps 2π

about the origin, so the functions have similar "frequencies". From (C.5.16) the velocity $\dot{\varphi} = h/\sin^2(\theta(t))$ is very uneven and will have large peaks when θ is near $\theta_{\mathbf{min}}$.

It is useful to look at an actual simulation to see the action of the two angle variables. We start by entering the last two equations in (C.5.21) using $\Omega^2 = g/\ell$:

```
restart; with(plots):
st := sin(theta(t)):      ct := cos(theta(t)):
td := diff(theta(t),t):   tdd := diff(theta(t),t,t):
pd := diff(phi(t),t):     pdd := diff(phi(t),t,t):
eq1 := tdd - st*ct*pd^2 + Omega^2*st = 0;
```

$$eq1 := \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) - \sin(\theta(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 + \Omega^2\sin(\theta(t)) = 0$$

```
eq2 := 2*ct*td*pd + st*pdd = 0;
```

$$eq2 := 2\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right) + \sin(\theta(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right) = 0$$

```
Omega := 1:
```

(C.5.22)

Then we set in some initial conditions and create the simulation (numerical solution). Here we start at $\theta(0) = 90°$ and some small amount of $\dot{\varphi}(0) = 0.4$ which results in an ellipse that is thin but not very thin :

```
eqs := {eq1,eq2}:
funcs := {theta(t),phi(t)};    Digits := 12:
```

$$funcs := \{\theta(t), \phi(t)\}$$

```
inits := { theta(0)=Pi/2,phi(0)=0,D(theta)(0)=0,D(phi)(0)=0.4};
```

$$inits := \left\{\theta(0) = \frac{1}{2}\pi, D(\phi)(0) = .4, \phi(0) = 0, D(\theta)(0) = 0\right\}$$

```
f := dsolve(eqs union inits,funcs,type=numeric,method=gear,output=listprocedure):
odeplot(f,[t,theta(t)],0..9,numpoints = 1000);
```

(C.5.23)

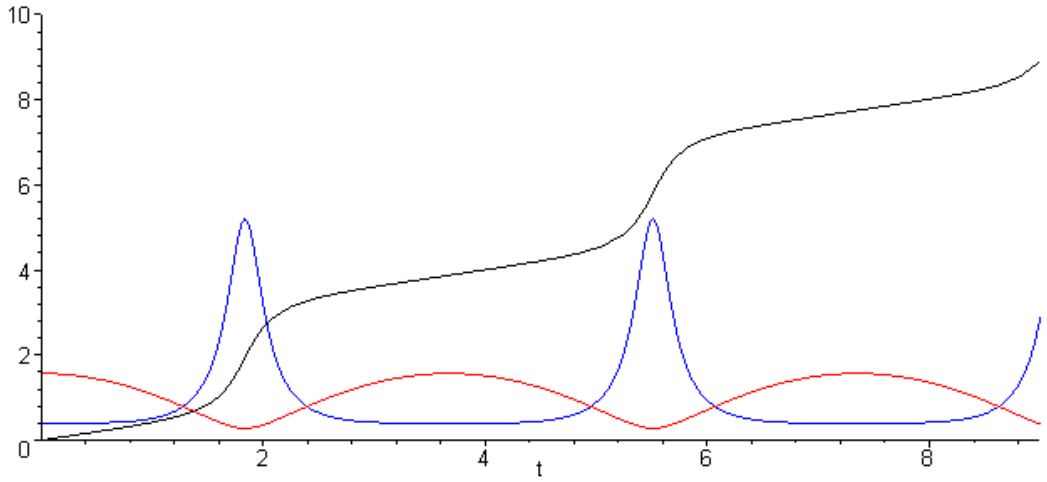
Next we extract the four solution functions of interest which are $\theta(t)$,$\varphi(t)$,$\dot{\theta}(t)$ and $\dot{\varphi}(t)$ [see our Maple User's Guide] ,

```
Theta := subs(f,theta(t)):
Phi := subs(f,phi(t)):
Thetad := subs(f,diff(theta(t),t)):
Phid := subs(f,diff(phi(t),t)):
```

(C.5.24)

We are now ready to make plots:

```
s1 := plot('Theta(t)',t=0..9,y=0..3,color=red):
s2 := plot('Phi(t)',t=0..9,y=0..10,color=black):
s3 := plot('Phid(t)',t=0..9,y=0..10,color=blue):
display(s1,s2,s3);
```

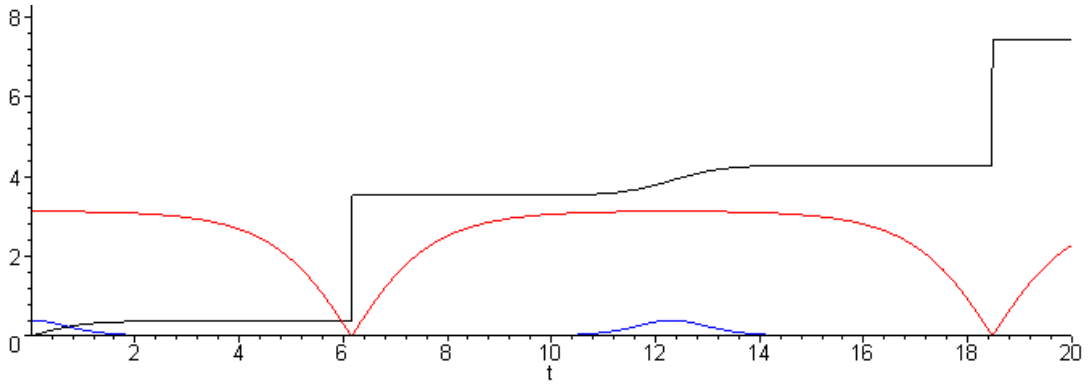


θ = red φ = black $\dot{\varphi}$ = blue (C.5.25)

We see in red the expected θ(t) bouncing between $\theta_{max} = \pi/2$ and $\theta_{min} \approx 0.3$. The blue $\dot{\varphi}(t)$ has peaks when θ is small, as noted above. The black φ just winds around to ever-increasing φ as the pendulum bob goes in its elliptical path (not an exact ellipse).

We conjecture that for a thin elliptical orbit, the behavior of the pendulum for small or large θ_{max} is very similar to what we saw for the plane pendulum in Section C.4. In particular if we set θ_{max} very close to π, we expect to see the top of the red θ curve become flat, corresponding to the long hang time mentioned in that Section. Here we set $\theta_{max} = 0.995*\pi$:

```
inits := { theta(0)=.995*Pi,phi(0)=0,D(theta)(0)=0,D(phi)(0)=0.4};
```



θ = red φ = black $\dot{\varphi}$ = blue (C.5.26)

The period of the θ motion is correspondingly increased.

We have no similar conjecture to make about the φ(t) behavior of the pendulum!

(h) The Intrinsic Airy Precession of the Spherical Pendulum

Consider again the two equations of motion.

$$\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = -\Omega^2\sin\theta \qquad \hat{\theta}$$
$$2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} = 0 \quad . \qquad \hat{\varphi} \qquad (C.5.21) \qquad (C.5.27)$$

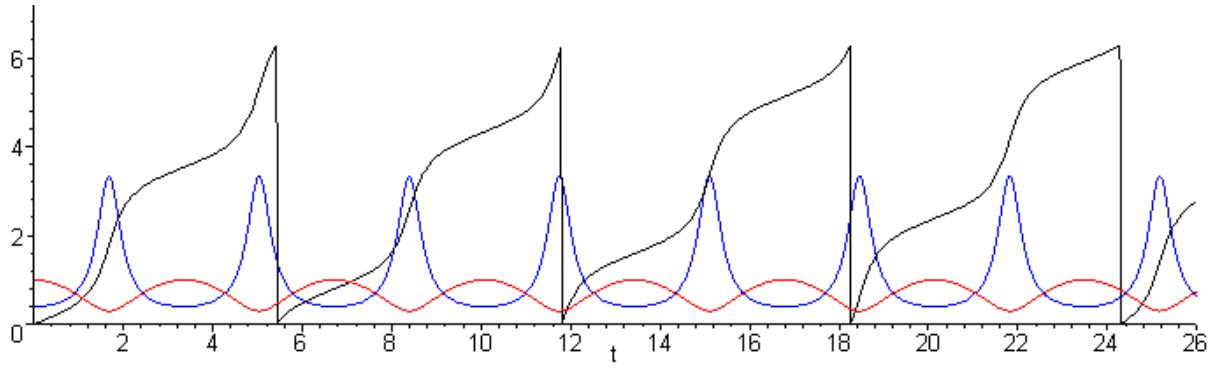
A pair of equivalent equations was noted earlier,

$$\ddot{\theta} - h^2\cos\theta/\sin^3\theta + \Omega^2\sin\theta = 0 \qquad (C.5.7)$$
$$\dot{\varphi} = h/\sin^2(\theta(t)) \qquad (C.5.6) \qquad (C.5.28)$$

With this second pair, one solves the first for θ(t) and uses that in the second to get φ(t).

In all these equations the terms are of similar size, so nothing can be neglected in an attempt to make any approximation for a "thin orbit". For this reason it is difficult to come up with an arm-waving explanation of the fact that the orbit processes due to the relation between the θ and φ behavior. It turns out that, when θ_{max} is not too large, for each period of the θ motion, the azimuth φ wraps around a little less than 2π and this causes the orbit to precess as the pendulum swings (this has nothing to do with Earth rotation which is turned off). Here we demonstrate this effect using θ(0) = 1 radian and doing mod(φ,2π) in the plot of φ:

```
inits := { theta(0)=1,phi(0)=0,D(theta)(0)=0,D(phi)(0)=0.4};
```



θ = red mod(φ,2π) = black $\dot{\varphi}$ = blue (C.5.29)

The vertical edge of the black φ curve is slipping to the left relative to the red and blue curves.

In his 1851 paper Airy (see Refs.) derived an expression for this "intrinsic apsidal precession rate" of the spherical pendulum doing thin elliptical orbits which our numerical solution above demonstrates. His formula is (now called "the Airy precession"),

$$\omega_{airy}/\omega_{swing} = T_{swing}/T_{airy} = (3/8)(ab/\ell^2) = (3/8\pi)(\pi ab/\ell^2) = (3/8\pi)(A/\ell^2) \quad . \qquad (C.5.30)$$

In this approximate formula, a and b are the semimajor and semiminor axes of the narrow ellipse which is slowly precessing, and ℓ is the length of the string (A = πab is the area of the ellipse). This formula is most accurate for small oscillations and gets less precise for larger ones, requiring correction terms. The orbit precesses in the same direction that the bob rotates around the ellipse (see Fig. (C.8.14)).

The derivation in Airy's paper is quite involved. An alternate derivation appears in the text of Synge and Griffith, pp 373-381. The result appears on p 381 in the form $\delta\varphi = (3A/4\ell^2)$ where δφ is the amount of precession during one full swing of the pendulum. Then it takes $N = \delta\varphi/2\pi$ swings to get 2π of precession and so $1/N = 2\pi/\delta\varphi = (3/8\pi)(A/\ell^2)$ in agreement with (C.5.30) above.

We shall return to this precession in the next section and later when we plot orbits in Section C.8.

(i) The Foucault Mode of the Spherical Pendulum

We now turn the Earth's rotation back on and are faced with the full equations of motion from (C.3.9),

$$\dot\theta^2 + \sin^2\theta\,\dot\varphi^2 = -(g/\ell)\cos\theta + T/(m\ell) + 2\omega\,[(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\sin\theta\,\dot\varphi + \sin\beta\sin\varphi\,\dot\theta\,]$$

$$\ddot\theta - \sin\theta\cos\theta\,\dot\varphi^2 = -(g/\ell)\sin\theta - 2\omega\sin\theta\,\dot\varphi\,(\cos\beta\cos\theta - \sin\beta\cos\varphi\sin\theta)$$

$$2\cos\theta\,\dot\theta\,\dot\varphi + \sin\theta\,\ddot\varphi = 2\omega\dot\theta\,(\cos\beta\cos\theta - \sin\beta\cos\varphi\sin\theta)\,. \qquad (C.3.9)$$

We seek a planar-like solution where ω, $\dot\varphi$ and $\ddot\varphi$ are all very small. It is still true that φ must wind roughly 2π for each θ swing, but most of the time (away from the low point) $\dot\varphi$ is very small since the assumed orbit is nearly planar. The equations then reduce to the following,

$$\dot\theta^2 = -(g/\ell)\cos\theta + T/(m\ell) + 2\omega\,[\sin\beta\sin\varphi\,\dot\theta\,]$$

$$\ddot\theta = -(g/\ell)\sin\theta$$

$$2\cos\theta\,\dot\theta\,\dot\varphi = 2\omega\dot\theta\,(\cos\beta\cos\theta - \sin\beta\cos\varphi\sin\theta)\quad . \qquad (C.5.31)$$

The second equation is the standard equation for the general-amplitude plane pendulum. This problem was exactly solved in Section C.4 for two different initial conditions. Here is one of those solutions,

$$\theta(t) = 2\sin^{-1}[\,\sin(\theta_0/2)\,\mathrm{sn}(\Omega t + K(\sin(\theta_0/2)), \sin(\theta_0/2))\,] \qquad \theta(0) = \theta_0 \qquad \dot\theta(0) = 0\ . \qquad (C.4.26)$$

Dividing the third equation of (C.5.31) by 2cosθ gives,

$$\dot\varphi = \omega\,(\cos\beta - \sin\beta\cos\varphi\tan\theta)\ . \qquad (C.5.32)$$

For small oscillations (in θ) we drop the second term to get

$$\dot\varphi = \omega\cos\beta \qquad (C.5.33)$$

In our arrangement of the spherical coordinates with $\hat{z}$ pointing down, this indicates a clockwise precession of the pendulum when it is viewed from above, and so is consistent with our initial Foucault calculation (C.5) .

(j) Interference between the Airy and Foucault precession

We have found that for small-angle motion of a spherical pendulum on the Earth's surface (magnitudes),

$\omega_F = 2\omega|\cos\beta|$ // Foucault precession, ω = rotation rate of the Earth

$\omega_I \approx \Omega(3/8\pi)(A/\ell^2)$ // Intrinsic precession, $\Omega = \sqrt{g/\ell}$ = pendulum swing frequency

where $\omega \approx \Omega = \sqrt{g/\ell}$ is the small-angle pendulum frequency. The ratio is then

$$\omega_I/\omega_F = \Omega(3/8\pi)(A/\ell^2) \ / \ (2\omega|\cos\beta|) = (3/16\pi)\,(\sqrt{g/\ell}\,/\omega)\,(A/\ell^2)\,|\sec\beta| \quad \text{// dimensionless}$$

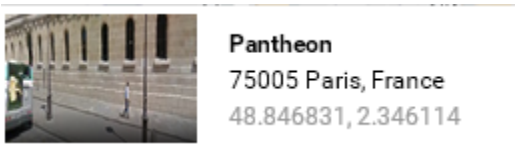
$$= (3/16\pi)\,\sqrt{g}\,A\,\omega^{-1}\,\ell^{-5/2}\,|\sec\beta| \quad . \qquad (C.5.34)$$

We would like the intrinsic precession rate to be much less than the Foucault rate so that one can then ignore the intrinsic effect. To make this ratio small, one of course wants to make the ellipse area A as small as possible, one wants to stay away from the Earth's equator where $\beta = \pi/2$ and $\sec\beta = \infty$, and one wants a large ℓ .

This subject gets a good treatment in Schumacher and Tarbet (2009), where the Airy formula appears as equation (2). These authors propose an electronic device to neutralize the intrinsic precession to allow for a much shorter string on a Foucault pendulum.

(k) The Foucault Pendulum at the Pantheon in Paris

According to Google Maps



the Pantheon is located at latitude 48.8468 degrees, so $\beta = 90\text{-}48.8468 = 41.1532$ degrees.

The length of the pendulum is purported to be 67 meters which we assume is the exact distance from the pivot point to the center of mass of the swinging weight.

Wiki https://en.wikipedia.org/wiki/Gravity_of_Earth says $g = 9.81$ m/sec^2 in Paris.

According to https://en.wikipedia.org/wiki/**Sidereal**_time the Earth's sidereal (relative to the stars) rotation period is 23.9344699 hours.

We now have Maple do a few calculations. First we compute the pendulum period (w used for ω, and the function evalf means "evaluate to a floating point number") :

```
g := 9.81;                          # Gravity at Paris
                                        g := 9.81
L := 67;                            # Length of the Pantheon pendulum (m)
                                        L := 67
wSwing := sqrt(g/L);                # swing frequency of the pendulum
                                        wSwing := .3826459335
Tswing := evalf(2*Pi/wSwing);       # time for one full swing (sec)
                                        Tswing := 16.42036347
```

(C.5.35)

The full-cycle period is therefore T_{swing} = 16.42 seconds. Stopwatch measurements for the first full swing in this video https://www.youtube.com/watch?v=59phxpjaefA were 16.37, 16.38. 16.36 which average to 16.37, pretty close. One is never sure that video speeds are reproduced in real time.

Next we compute the Foucault rotation period using $\omega_{Fouc} = \omega_{Earth}\cos\beta$ from (C.5.33) or (C.5) :

```
TEarth := 23.9344699*3600;      # Earth rotation period in sec
                                        TEarth := 86164.09164
wEarth := evalf(2*Pi/TEarth);   # Earth angular rotation frequency
                                        wEarth := .00007292115762
lat := 48.8468;                 # Pantheon latitude (deg)
                                        lat := 48.8468
colat := 90 - lat;              # Pantheon colatitude (polar angle beta in deg)
                                        colat := 41.1532
beta := evalf(colat*Pi/180);    # colatitude in radians
                                        β := .7182588379
wFouc := wEarth*cos(beta);      # Foucault angular precession frequency
                                        wFouc := .00005490618139
TFouc := evalf(2*Pi/wFouc)/3600; # Foucault full rotation period (hours)
                                        TFouc := 31.78748200
(TFouc -  31)*60;               # minutes over 31 hours
                                        47.248920
```

(C.5.36)

The period is then 31.787 hours or 31 hours and 47.2 minutes.
The site https://en.wikipedia.org/wiki/List_of_Foucault_pendulums#France claims 31 hours and 50 minutes, but that is probably a calculated and not a measured value. Says m = 28 kg so this pendulum does not blow around much in the breeze.

Next we estimate the effect of the Airy precession, assuming the swing is about 2 meter long and the ellipse width is 2 cm (hopefully it is smaller than that). The ratio_Airy is taken from (C.5.30)

```
a := 1;                    # assumed elliptical orbit semi-major axis (m)
```

$$a := 1$$

```
b := evalf(1/100);         # assumed elliptical orbit semi-minor axis 1 cm (m)
```

$$b := .01000000000$$

```
ratio_Airy := evalf((3/8)*a*b/L^2); # intrinsic precession ratio
```

$$ratio_Airy := .8353753620\ 10^{-6}$$

```
TAiry := (Tswing/ratio_Airy)/3600;  # intrinsic precession period (hours)
```

$$TAiry := 5460.074933$$

```
ratio := TAiry/TFouc;              # ratio of the Airy/Foucault precession periods
```

$$ratio := 171.7680857$$

```
TFouc_error := (TFouc/ratio)*60;    # plus/minus error in TFouc due to Airy (min)
```

$$TFouc_error := 11.10362797$$

(C.5.37)

Thus suggests that we get one Airy precession for every 171.76 Foucault precessions. The direction of the Airy precession depends on which way the ellipse path is traversed. The error induced in the Foucault period is then about ± 11 minutes, assuming the a and b values shown. One would think that with a careful "burned string" launch of the pendulum, one could get "a" down to maybe 1 mm, which would reduce the 32 hour period error to ± 1 minute.

(ℓ) General Numerical Orbits of the Spherical Pendulum

In Section C.6 below we derive the spherical pendulum equations of motion directly in Cartesian coordinates and then in Section C.8 we plot various pendulum trajectories in Cartesian space. As long as one avoids hitting the singular points θ = 0 and θ = π, one can do this directly from the angular equations of motion as we now show. This method has an advantage over that of Section C.8 in that negative values of z here are allowed, meaning motion of the pendulum bob in the upper hemisphere is permitted.

The spherical pendulum equations of motion are given in (C.5.1),

$$\begin{aligned} &\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 = -(g/\ell)\cos\theta + T/(m\ell) && \hat{\mathbf{r}} \\ &\ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = -(g/\ell)\sin\theta && \hat{\boldsymbol{\theta}} \\ &2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} = 0 \quad . && \hat{\boldsymbol{\varphi}} \end{aligned} \qquad \textit{(C.5.1)} \qquad (C.5.38)$$

We enter into Maple the last two equations as was shown in (C.5.22), and have dsolve find solutions as in (C.5.23). Here is an example where we give the pendulum a good kick at t = 0,

$$\theta(0) = 0.5 \qquad \varphi(0) = 0 \qquad \dot{\theta}(0) = 1.7 \qquad \dot{\varphi}(0) = 1.0 \qquad (C.5.39)$$

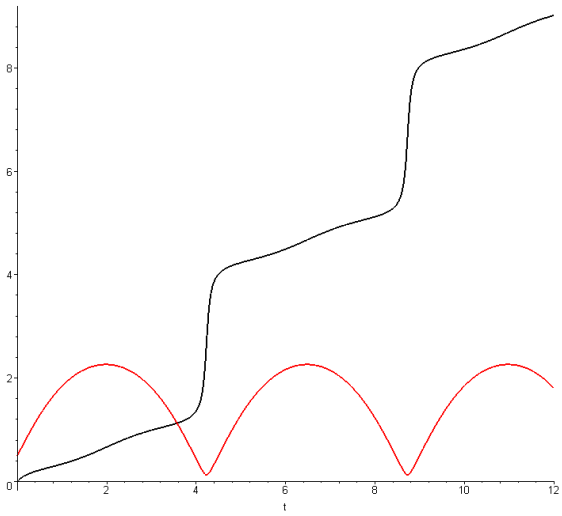
```
inits := { theta(0)=.5,phi(0)=0,D(theta)(0)=1.7,D(phi)(0)=1};
```

$$inits := \{D(\phi)(0) = 1, \phi(0) = 0, D(\theta)(0) = 1.7, \theta(0) = .5\}$$

```
f := dsolve(eqs union inits,funcs,type=numeric,method=rkf45,output=listprocedure):
```

We then extract θ(t) and φ(t) and plot them : red θ bounces and black φ winds around,

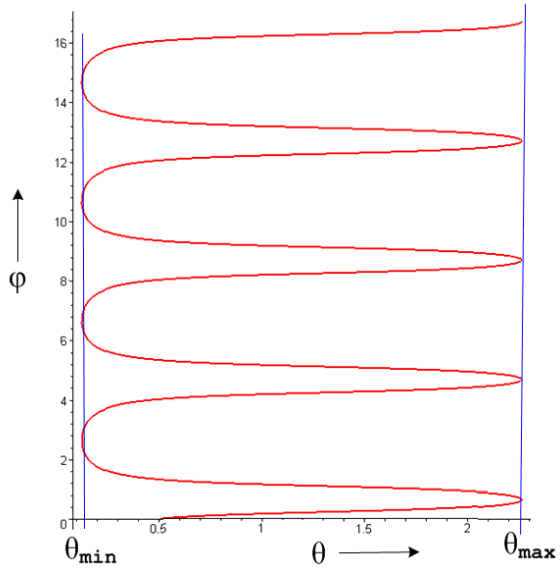
```
Theta := subs(f,theta(t)):
Phi := subs(f,phi(t)):
plot(['Theta(t)','Phi(t)'],t=0..12,numpoints = 400, color = [red,black],thickness = 2);
```



θ(t) = red φ(t) = black (C.5.40)

Alternatively, one can plot the trajectory in (θ,φ) space where θ bounces and φ winds up vertically,

```
odeplot(f,[theta(t),phi(t)],0..20,numpoints = 400,thickness=2);
```



(C.5.41)

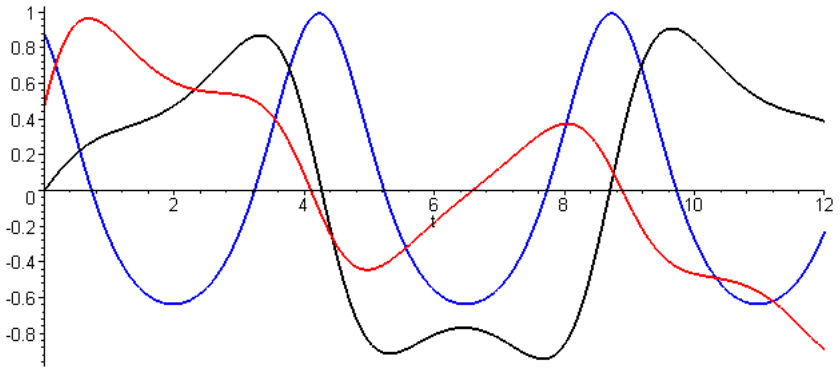
We then generate Cartesian coordinates ($\ell = 1$),

```
X := (t) -> sin(Theta(t))*cos(Phi(t)):
Y := (t) -> sin(Theta(t))*sin(Phi(t)):
Z := (t) -> cos(Theta(t)):
```

(C.5.42)

and plot the trajectory in these coordinates (recall that $\hat{\mathbf{z}}$ points down),

```
plot(['X(t)','Y(t)','Z(t)'],t=0..12,numpoints = 400, color = [red,black,blue],thickness=2);
```

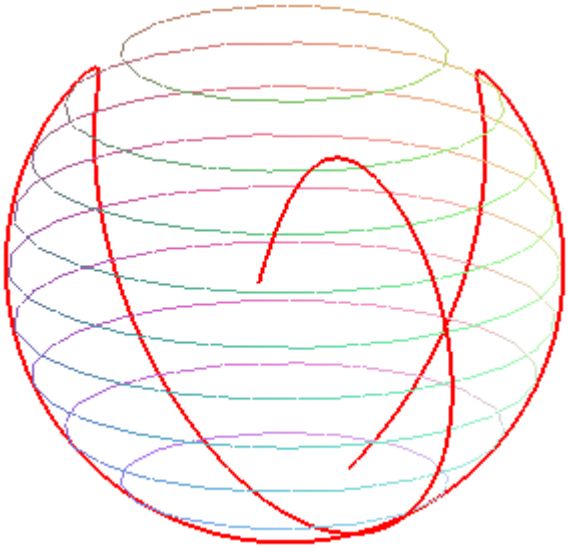


(C.5.43)

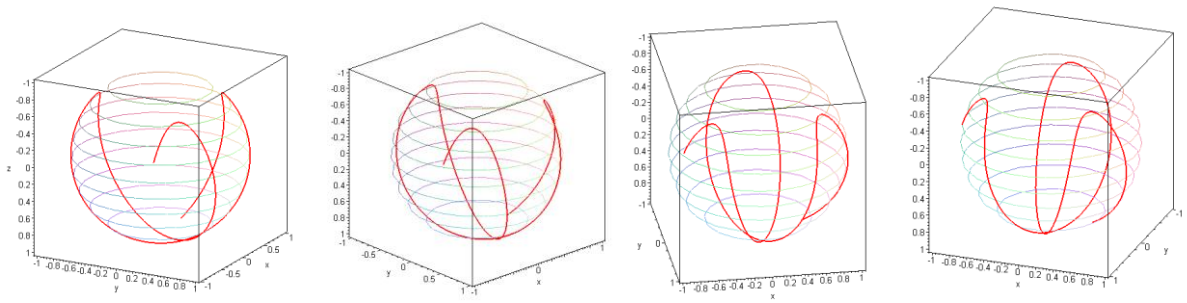
x(t) = red y(t) = black z(t) = blue

We can make a 3D plot of the trajectory as follows ($\hat{\mathbf{z}}$ axis points down, up is up)

```
s1 := spacecurve(['X(t)','Y(t)','Z(t)'],t=0..12,numpoints=500,thickness=2,color=red):
s2 := implicitplot3d(x^2+y^2+z^2 = 1,x=-1..1,y=-1..1,z=-1..1,scaling=constrained,style=contour):
display(s1,s2);
```



(C.5.44)

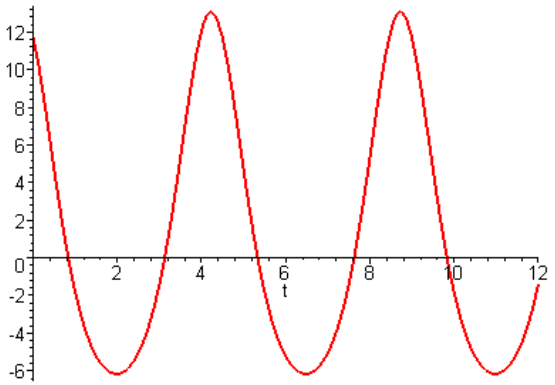


Solving the first equation in (C.5.38) for T, and using m = 1 kg, ℓ = 1 m, and g = 9.8 m/s^2, we can plot the string tension for the above trajectory,

```
Thetad := subs(f,diff(theta(t),t)):
Phid := subs(f,diff(phi(t),t)):
T := (t) ->Thetad(t)^2 + (sin(Theta(t))*Phid(t))^2 + (9.8/1)*cos(Theta(t));
```

$$T := t \to \mathrm{Thetad}(t)^2 + \sin(\Theta(t))^2\,\mathrm{Phid}(t)^2 + 9.8\cos(\Theta(t))$$

```
plot('T(t)',t=0..12,thickness=2);
```



(C.5.45)

Since the string tension goes negative at the three high spots in the motion shown in (C.5.44), the pendulum must have a massless stick in place of a string in order to realize the trajectories of this example.

C.6 Equations of Motion for the Foucault Pendulum (Cartesian Coordinates)

We start with the vector equation (C.3.3), which is Newton's Law in non-inertial Frame S,

$$m\mathbf{a} = mg\hat{\mathbf{z}} + \mathbf{T} - 2m\,\boldsymbol{\omega} \times \mathbf{v} \quad . \qquad \textit{(C.3.3)} \qquad \text{(C.6.1)}$$

where $\hat{\mathbf{z}}$ points down. From (C.1.2) $\boldsymbol{\omega}$ can be written

$$\boldsymbol{\omega} = -\omega\cos\beta\,\mathbf{e}_3 + \omega\sin\beta\,\mathbf{e}_1 = \omega[-\cos\beta\,\hat{\mathbf{z}} + \sin\beta\,\hat{\mathbf{x}}] \qquad \text{(C.6.2)}$$

so

$$-2m\,\boldsymbol{\omega} \times \mathbf{v} = -2m\omega\,[-\cos\beta\,\hat{\mathbf{z}} + \sin\beta\,\hat{\mathbf{x}}] \times [v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}]$$

$$= -2m\omega\,[-\cos\beta(v_x\hat{\mathbf{y}} - v_y\hat{\mathbf{x}}) + \sin\beta(v_y\hat{\mathbf{z}} - v_z\hat{\mathbf{y}})]$$

$$= -2m\omega\,[\cos\beta v_y\hat{\mathbf{x}} - (\cos\beta v_x + \sin\beta v_z)\,\hat{\mathbf{y}} + \sin\beta v_y\hat{\mathbf{z}}] \; . \qquad \text{(C.6.3)}$$

From (C.3.5) the string tension is,

$$\mathbf{T} = -T\,\hat{\mathbf{r}} = -(T/\ell)(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})\,. \qquad // \;\ell = \sqrt{x^2+y^2+z^2} \tag{C.6.4}$$

The three equations of motion from (C.6.1) are then

$$\begin{aligned} ma_x &= -(T/\ell)x - 2m\omega\cos\beta v_y \\ ma_y &= -(T/\ell)y + 2m\omega\,(\cos\beta v_x + \sin\beta v_z) \\ ma_z &= mg - (T/\ell)z - 2m\omega\sin\beta v_y \end{aligned} \tag{C.6.5}$$

or

$$\begin{aligned} \ddot{x} &= -(T/m\ell)x - 2\omega\cos\beta\dot{y} \\ \ddot{y} &= -(T/m\ell)y + 2\omega\,(\cos\beta\dot{x} + \sin\beta\dot{z}) \\ \ddot{z} &= g - (T/m\ell)z - 2\omega\sin\beta\dot{y}\,. \end{aligned} \tag{C.6.6}$$

Eliminate T from the first two equations :

$$\begin{aligned} y\ddot{x} &= -(T/m\ell)xy - 2\omega y\cos\beta\dot{y} \\ x\ddot{y} &= -(T/m\ell)xy + 2\omega x(\cos\beta\dot{x} + \sin\beta\dot{z}) \end{aligned}$$

$$y\ddot{x} - x\ddot{y} = -2\omega y\cos\beta\dot{y} - 2\omega x(\cos\beta\dot{x} + \sin\beta\dot{z})\,. \tag{C.6.7}$$

Do this again for the first and third equations of (C.6.6) :

$$\begin{aligned} z\ddot{x} &= -(T/m\ell)xz - 2\omega z\cos\beta\dot{y} \\ x\ddot{z} &= xg - (T/m\ell)xz - 2\omega x\sin\beta\dot{y} \end{aligned}$$

$$z\ddot{x} - x\ddot{z} = -xg - 2\omega z\cos\beta\dot{y} + 2\omega x\sin\beta\dot{y}\,. \tag{C.6.8}$$

We then have these three equations of interest

$$\begin{aligned} &1 \qquad y\ddot{x} - x\ddot{y} = -2\omega y\cos\beta\dot{y} - 2\omega x(\cos\beta\dot{x} + \sin\beta\dot{z}) \\ &2 \qquad z\ddot{x} - x\ddot{z} = -xg - 2\omega\dot{y}(z\cos\beta - x\sin\beta) \\ &3 \qquad x^2+y^2+z^2 = \ell^2 \end{aligned} \tag{C.6.9}$$

where ℓ is the length of the pendulum string. These are the **equations of motion** for the Foucault pendulum in Cartesian coordinates. We can in theory solve for x(t), y(t) and z(t) given some initial conditions. Once the three equations are solved, we can find the tension T(t) from (say) the first equation of (C.6.6) :

$$\ddot{x} = -(T/m\ell)x - 2\omega\cos\beta\dot{y} \quad \Rightarrow \quad (T/m\ell)x = -2\omega\cos\beta\dot{y} - \ddot{x} \quad \Rightarrow$$

$$T = -m\ell(2\omega\cos\beta\dot{y} + \ddot{x})/x\,. \tag{C.6.10}$$

Small oscillation limit

For very small pendulum swings one has

$$T \approx mg \qquad \Rightarrow \qquad T/m\ell \approx (g/\ell)$$
$$\dot{x} \text{ and } \dot{y} \text{ are small}$$
$$\dot{z} \approx 0 \qquad (C.6.11)$$

In zeroth order the first two equations of (C.6.6) then say ($r = \ell$ = length of string)

$$\ddot{x} + (g/\ell)x \approx 0$$
$$\ddot{y} + (g/\ell)y \approx 0 \qquad \Rightarrow \qquad \Omega = \sqrt{g/\ell} \qquad (C.6.12)$$

and the pendulum swings as $x.y \sim \sin(\Omega t)$ with $\Omega = \sqrt{g/\ell}$. To first order we add back the small velocity terms in (C.6.6) to get

$$\ddot{x} + \Omega^2 x + 2\omega\cos\beta\dot{y} \approx 0$$
$$\ddot{y} + \Omega^2 y - 2\omega\cos\beta\dot{x} \approx 0 \ . \qquad \text{// bottom view} \qquad (C.6.13)$$

In our system $\hat{z}$ points down, but if it were to point up we could take $x \rightarrow x$ and $y \rightarrow -y$ as coordinates one would use when viewing the pendulum from above. The equations are then

$$\ddot{x} + \Omega^2 x - 2\omega\cos\beta\dot{y} \approx 0$$
$$\ddot{y} + \Omega^2 y + 2\omega\cos\beta\dot{x} \approx 0 \qquad \text{// top view} \qquad (C.6.14)$$

These equations appear in (C.1).

C.7 Verification of the Cartesian equations of motion and string tension

The algebra above is quite complex so we want to be sure that our Cartesian equations of motion are correct. First, here are the last two angular equations of motion from (C.3.9) with ℓ set to r,

$$\text{eq1} \quad \ddot{\theta} - \sin\theta\cos\theta\,\dot{\varphi}^2 = -(g/r)\sin\theta - 2\omega\sin\theta\,\dot{\varphi}(\cos\beta\cos\theta - \sin\beta\cos\varphi\sin\theta) \qquad \text{// } r = \ell$$
$$\text{eq2} \quad 2\cos\theta\,\dot{\theta}\,\dot{\varphi} + \sin\theta\,\ddot{\varphi} = 2\omega\dot{\theta}(\cos\beta\cos\theta - \sin\beta\cos\varphi\sin\theta) \ . \qquad \textit{(C.2.9)} \qquad (C.7.1)$$

Meanwhile, here are the Cartesian equations of motion from (C.6.9) (in reverse order),

$$\text{eq3} \quad z\ddot{x} - x\ddot{z} = -xg - 2\omega\dot{y}(z\cos\beta - x\sin\beta)$$
$$\text{eq2} \quad y\ddot{x} - x\ddot{y} = -2\omega y\cos\beta\dot{y} - 2\omega x(\cos\beta\dot{x} + \sin\beta\dot{z}) \ . \qquad \textit{(C.6.9)} \qquad (C.7.2)$$

Below we shall show that :

Task (a): eq2 of (C.7.2) $\Rightarrow$ eq2 of (C.7.1)

Task (b): [eq3 + (cosθsinφ)*eq2] of (C.7.2) $\Rightarrow$ eq1 of (C.7.1)

That is to say, angular eq1 of (C.7.1) is a certain linear combination of eq3 and eq2 of (C.7.2). If we can show Task (a) and Task (b) above, then we have shown that (C.7.2) $\Leftrightarrow$ (C.7.1), and this then serves as verification of (C.7.2).

Maple must replace x,y,z and derivatives with r,θ,φ and derivatives. For coordinates and first derivatives,

```
x := r*sin(theta(t))*cos(phi(t));
```

$$x := r\sin(\theta(t))\cos(\phi(t))$$

```
y := r*sin(theta(t))*sin(phi(t));
```

$$y := r\sin(\theta(t))\sin(\phi(t))$$

```
z := r*cos(theta(t));
```

$$z := r\cos(\theta(t))$$

```
xd := diff(x,t);
```

$$xd := r\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\cos(\phi(t)) - r\sin(\theta(t))\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)$$

```
yd := diff(y,t);
```

$$yd := r\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t)) + r\sin(\theta(t))\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)$$

```
zd := diff(z,t);
```

$$zd := -r\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right) \qquad \text{(C.7.3)}$$

The second derivatives are messier, but Maple is happy to do the calculations,

```
> xdd := diff(x,t,t);
```

$$xdd := -r\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2\cos(\phi(t)) + r\cos(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\cos(\phi(t))$$

$$- 2\,r\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right) - r\sin(\theta(t))\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 - r\sin(\theta(t))\sin(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$

```
> ydd := diff(y,t,t);
```

$$ydd := -r\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2\sin(\phi(t)) + r\cos(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\sin(\phi(t))$$

$$+ 2\,r\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right) - r\sin(\theta(t))\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 + r\sin(\theta(t))\cos(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$

```
> zdd := diff(z,t,t);
```

$$zdd := -r\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2 - r\sin(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right) \qquad \text{(C.7.4)}$$

Task (a): Show that eq2 of (C.7.2) ⇒ eq2 of (C.7.1)

We enter eq2 of (C.7.2) and do some manipulations, suppressing the output except for the last step :

```
> cb := cos(beta):  sb := sin(beta):
> eq2 := y*xdd-x*ydd + 2*w*y*cb*yd + 2*w*x*(cb*xd+sb*zd) = 0:
> eq2a := eq2/(r^2*sin(theta(t))):
> eq2b := expand(eq2a):
> eq2c := -simplify(eq2b);
```

$$eq2c := 2\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right) + \sin(\theta(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right) - 2\,w\cos(\beta)\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)$$

$$+ 2\sin(\theta(t))\,w\cos(\phi(t))\sin(\beta)\left(\frac{\partial}{\partial t}\theta(t)\right) = 0 \qquad \text{(C.7.5)}$$

We manually transcribe the resulting equation,

$$2\cos\theta\dot\theta\dot\varphi + \sin\theta\,\ddot\varphi - 2\omega\cos\beta\cos\theta\,\dot\theta + 2\omega\sin\theta\cos\varphi\sin\beta\dot\theta = 0$$

or

$$2\cos\theta\dot\theta\dot\varphi + \sin\theta\,\ddot\varphi = 2\omega\cos\beta\cos\theta\,\dot\theta - 2\omega\sin\theta\cos\varphi\sin\beta\dot\theta$$

or

$$2\cos\theta\dot\theta\dot\varphi + \sin\theta\,\ddot\varphi = 2\omega\dot\theta(\cos\beta\cos\theta - \sin\theta\cos\varphi\sin\beta)\ . \qquad \text{(C.7.6)}$$

This last equation is a match for eq2 of (C.7.1) so we have accomplished Task (a).

Task (b): eq3 + (cosθsinφ)eq2 of (C.7.2) ⇒ eq1 of (C.7.1)

The code continues from that shown above. Equation eq2 is already entered, so we now enter eq3, form the linear combination for eq1, then process the results with a series of typical tortuous Maple steps,

```
> eq3 := z*xdd-x*zdd + x*(gor)*r + 2*w*yd*(z*cb-x*sb) = 0:    # gor = g/r
> eq3a := eq3/r^2:
> eq3b := expand(eq3a):
> eq3c := collect(eq3b,diff(phi(t),t)):
> eq1 := lhs(eq3c) + lhs(eq2c*cos(theta(t))*sin(phi(t))):expand(%):
> subs(cos(theta(t))^2=1-sin(theta(t))^2,%):expand(%/cos(phi(t))):
> collect(%,diff(phi(t),t));
```

$$-\cos(\theta(t))\sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 + (2\,w\sin(\theta(t))\cos(\theta(t))\cos(\beta) - 2\cos(\phi(t))\,w\sin(\theta(t))^2\sin(\beta))\left(\frac{\partial}{\partial t}\phi(t)\right)$$

$$+ \sin(\theta(t))\,gor + \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) \qquad \text{(C.7.7)}$$

We again manually transcribe the resulting equation,

$$-\cos\theta\sin\theta\,\dot\varphi^2 + (2\omega\sin\theta\cos\theta\cos\beta - 2\omega\sin^2\theta\cos\varphi\sin\beta)\dot\varphi + \sin\theta(g/r) + \ddot\theta = 0$$

or

$$\ddot{\theta} - \cos\theta\sin\theta\,\dot{\varphi}^2 = -\sin\theta(g/r) - 2\omega\sin\theta(\cos\theta\cos\beta - \sin\theta\cos\varphi\sin\beta)\dot{\varphi} \quad . \tag{C.7.8}$$

This is a match for eq1 of (C.7.1) so we have accomplished Task (b).

Tension equation verification

Using the angular equations of motion (C.7.1), we now show that the following two tension expressions are equivalent (the second is Cartesian (C.6.10) while the first is angular (C.3.9)), where r = ℓ ,

$$T/mr = \dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 + (g/r)\cos\theta - 2\omega\,[(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\sin\theta\,\dot{\varphi} + \sin\beta\sin\varphi\,\dot{\theta}\,]$$

$$T/mr = -(2\omega\cos\beta\dot{y} + \ddot{x})/x \quad . \tag{C.7.9}$$

Our task of showing that the above equations have equal right sides is the same as showing that

$$x\{\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2 + (g/r)\cos\theta - 2\omega\,[(\cos\beta\sin\theta + \sin\beta\cos\varphi\cos\theta)\sin\theta\,\dot{\varphi} + \sin\beta\sin\varphi\,\dot{\theta}\,]\} = -(2\omega\cos\beta\dot{y} + \ddot{x})$$

or

$$\text{LHS} = \text{RHS} \, . \tag{C.7.10}$$

We first get the complicated left hand side LHS entered:

```
> restart;
> ct := cos(theta(t)):     st := sin(theta(t)):
> cp := cos(phi(t)):       sp := sin(phi(t)):
> cb := cos(beta):         sb := sin(beta):
> td := diff(theta(t),t): pd := diff(phi(t),t):
>
> LHS := x*(td^2 +st^2*pd^2 +gor*ct-2*w*((cb*st+sb*cp*ct)*st*pd
  +sb*sp*td));
```

$$LHS := x\left(\left(\frac{\partial}{\partial t}\theta(t)\right)^2 + \sin(\theta(t))^2\left(\frac{\partial}{\partial t}\phi(t)\right)^2 + gor\cos(\theta(t))\right.$$
$$\left. - 2w\left((\cos(\beta)\sin(\theta(t)) + \sin(\beta)\cos(\phi(t))\cos(\theta(t)))\sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right) + \sin(\beta)\sin(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\right)\right) \tag{C.7.11}$$

We then compute RHS = -(2ωcosβ$\dot{y}$ + $\ddot{x}$),

```
> RHS := -(2*w*cb*yd + xdd);
```

$$RHS := -2\,w\cos(\beta)\left(r\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t)) + r\sin(\theta(t))\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)\right)$$
$$+ r\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2\cos(\phi(t)) - r\cos(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\cos(\phi(t))$$
$$+ 2\,r\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right) + r\sin(\theta(t))\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2$$
$$+ r\sin(\theta(t))\sin(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$

(C.7.12)

Notice that RHS contains second derivatives $\ddot{\theta}$ and $\ddot{\varphi}$. We shall eliminate these derivatives by manually solving the angular equations of motion (C.7.1) for Tdd = $\ddot{\theta}$ and Pdd = $\ddot{\varphi}$:

```
Tdd := st*ct*pd^2 - gor*st - 2*w*st*pd*(cb*ct-sb*cp*st);
```

$$Tdd := \sin(\theta(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 - gor\sin(\theta(t)) - 2\,w\sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)(\cos(\beta)\cos(\theta(t)) - \sin(\beta)\cos(\phi(t))\sin(\theta(t)))$$

```
Pdd := (-2*ct*td*pd+2*w*td*(cb*ct-sb*cp*st) )/st;
```

$$Pdd := \frac{-2\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right) + 2\,w\left(\frac{\partial}{\partial t}\theta(t)\right)(\cos(\beta)\cos(\theta(t)) - \sin(\beta)\cos(\phi(t))\sin(\theta(t)))}{\sin(\theta(t))}$$

(C.7.13)

To show that LHS = RHS, we define d = LHS-RHS and show that d = 0:

```
d := LHS-RHS:
d1 := subs(diff(theta(t),t,t)=Tdd,diff(phi(t),t,t)=Pdd,d):
d2 := simplify(d1);
```

$$d2 := 0$$

(C.7.14)

Thus d = 0 and LHS = RHS and the two expressions for T in (C.7.9) are equivalent.

C.8 Numerical solutions of the equations of motion (Cartesian Coordinates)

Our task is to solve the set of equations (C.6.9) (eq1 now has a new meaning) :

eq1 $x^2+y^2+z^2 = r^2$ // $r = \ell$

eq2 $y\ddot{x} - x\ddot{y} = -2\omega y\cos\beta\dot{y} - 2\omega x(\cos\beta\dot{x} + \sin\beta\dot{z})$

eq3 $z\ddot{x} - x\ddot{z} = -xg\ -2\omega\dot{y}(z\cos\beta - x\sin\beta)$. *(C.6.9)* (C.8.1)

We enter eq2 and eq3 writing derivatives for example as $\ddot{x}$ = xdd :

```
xd := diff(x(t),t):      yd := diff(y(t),t):
xdd := diff(x(t),t,t):   ydd := diff(y(t),t,t):
cb := cos(beta):         sb := sin(beta):
eq2 := y(t)*xdd - x(t)*ydd = -2*w*y(t)*cb*yd-2*w*x(t)*(cb*xd+sb*zd);
```

$$eq2 := y(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - x(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right) = -2\,w\,y(t)\cos(\beta)\left(\frac{\partial}{\partial t}y(t)\right) - 2\,w\,x(t)\left(\cos(\beta)\left(\frac{\partial}{\partial t}x(t)\right) + \sin(\beta)\,zd\right)$$

```
eq3 := z(t)* xdd - x(t)*zdd = -x(t)*g-2*w*yd*(z(t)*cb-x(t)*sb);
```

$$eq3 := z(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - x(t)\,zdd = -x(t)\,g - 2\,w\left(\frac{\partial}{\partial t}y(t)\right)(z(t)\cos(\beta) - x(t)\sin(\beta))$$

(C.8.2)

At this point zd = $\dot{z}$ and zdd = $\ddot{z}$ are unspecified. We use eq1 of (C.8.1) to compute $\dot{z}$ and $\ddot{z}$ in terms of x and y,

```
z := (t) -> sqrt(r^2-x(t)^2-y(t)^2);
```

$$z := t \to \sqrt{r^2 - x(t)^2 - y(t)^2}$$

```
zd := diff(z(t),t);
```

$$zd := \frac{1}{2}\frac{-2\,x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right)}{\sqrt{r^2 - x(t)^2 - y(t)^2}}$$

```
zdd := diff(z(t),t,t);
```

$$zdd := -\frac{1}{4}\frac{\left(-2\,x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right)\right)^2}{\left(r^2 - x(t)^2 - y(t)^2\right)^{\left(\frac{3}{2}\right)}} + \frac{1}{2}\frac{-2\left(\frac{\partial}{\partial t}x(t)\right)^2 - 2\,x(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - 2\left(\frac{\partial}{\partial t}y(t)\right)^2 - 2\,y(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right)}{\sqrt{r^2 - x(t)^2 - y(t)^2}}$$

(C.8.3)

When these expressions are installed, eq2 and eq3 becomes these formidable-looking equations which contain two unknown functions x(t) and y(t) and constants r = ℓ, β,g and ω = w :

```
> eq2;
```

$$y(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - x(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right) = -2\,w\,y(t)\cos(\beta)\left(\frac{\partial}{\partial t}y(t)\right) - 2\,w\,x(t)\left(\cos(\beta)\left(\frac{\partial}{\partial t}x(t)\right) + \frac{1}{2}\frac{\sin(\beta)\left(-2\,x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right)\right)}{\sqrt{r^2 - x(t)^2 - y(t)^2}}\right)$$

```
> eq3;
```

$$\sqrt{r^2 - x(t)^2 - y(t)^2}\left(\frac{\partial^2}{\partial t^2}x(t)\right) - x(t)\left(-\frac{1}{4}\frac{\left(-2\,x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right)\right)^2}{\left(r^2 - x(t)^2 - y(t)^2\right)^{\left(\frac{3}{2}\right)}} + \frac{1}{2}\frac{-2\left(\frac{\partial}{\partial t}x(t)\right)^2 - 2\,x(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - 2\left(\frac{\partial}{\partial t}y(t)\right)^2 - 2\,y(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right)}{\sqrt{r^2 - x(t)^2 - y(t)^2}}\right)$$
$$= -x(t)\,g - 2\,w\left(\frac{\partial}{\partial t}y(t)\right)\left(\sqrt{r^2 - x(t)^2 - y(t)^2}\cos(\beta) - x(t)\sin(\beta)\right)$$

(C.8.4)

The Foucault Pendulum at the Pantheon in Paris (revisited)

For our first plot, we again consider the Foucault pendulum set up in the Pantheon. The numbers are discussed above (C.5.35) :

```
r := 67;    # accuracy uncertain
```

$$r := 67$$

```
T := 23.93447*3600; # sidereal day
```

$$T := 86164.09200$$

```
w := evalf(2*Pi/T);
```

$$w := .000072921157309704$$

```
beta := evalf((Pi/180)*(90-48.846285));    # Pantheon colatitude
```

$$\beta := .71826782617738$$

```
g := 9.80943:    # Paris 1970 measured
```

(C.8.5)

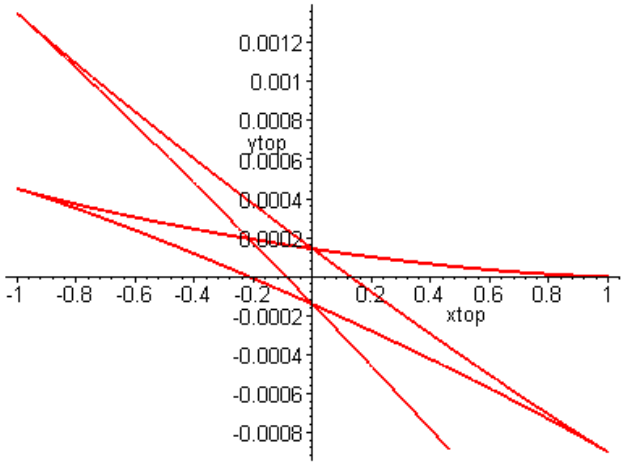
The Maple code to invoke a solution is as follows:

```
eqs := {eq2,eq3}:
funcs := {x(t),y(t)}:
inits := {x(0)=1,y(0)=0, D(x)(0)=0,D(y)(0)=0}:  Digits := 14:
f := dsolve(eqs union inits,funcs,type=numeric,method=rkf45,output=listprocedure):
```

(C.8.6)

We have initialized the pendulum at x(0) = 1m. Here is a plot of the first few swings,

```
odeplot(f,[x(t),-y(t)],0..30,numpoints=1000,axes = boxed,labels =
[xtop,ytop],thickness=2,axes=normal);
```



(C.8.7)

Notice that the vertical scale is highly magnified, so really these swings are very close to the x axis.

We want to view the orbits from above, not from below. Since the z axis points down, and since we are viewing from above, we negate the original y axis to get a new y axis appropriate for our plots viewed from the top. This negation is done in the odeplot call seen above.

As expected, and as shown earlier in (C.2.1), the Coriolis force deflects each half-swing to the right (as seen from above).

In order to plot functions like x(t) more generally, we extract our functions of interest from the Maple dsolve environment as follows,

```
dt := 1e-8:

X := subs(f,x(t)):
Xd := subs(f,diff(x(t),t)):
Xdd := (t) -> (Xd(t+dt)- Xd(t))/dt:

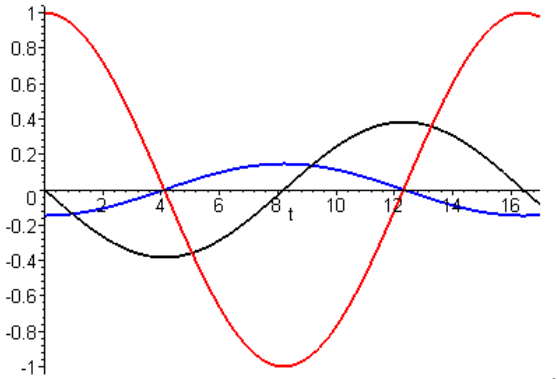
Y := subs(f,y(t)):
Yd := subs(f,diff(y(t),t)):
Ydd := (t) -> (Yd(t+dt)- Yd(t))/dt:

Z := (t) -> sqrt(r^2-X(t)^2-Y(t)^2):
Zd := (t) -> (Z(t+dt)- Z(t))/dt:
Zdd := (t) -> (Zd(t+dt)- Zd(t))/dt:
```

(C.8.8)

Here functions like X(t) are taken directly from the dsolve listprocedure output structure while unavailable derivatives such as Xdd = $\ddot{x}$ are manually approximated. For details on how this works and other information on dsolve (including a debugger's guide), see the author's Maple User Guide. We can now make a plot of the three functions x(t),$\dot{x}$(t) and $\ddot{x}$(t) for the first swing,

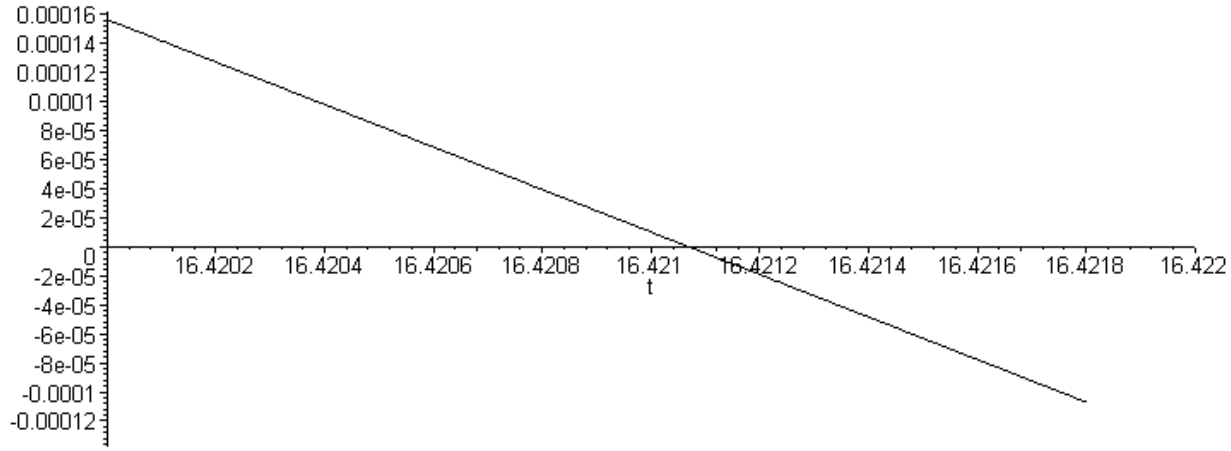
```
plot(['X(t)','Xd(t)','Xdd(t)'],t=0..17, color =
[red,black,blue],numpoints=100,thickness=2);
```



x = red $\dot{x}$ = black $\ddot{x}$ = blue (C.8.9)

The shape of red x(t) appears to be sinusoidal as predicted by the small angle model. The black curve $v_x(t) = \dot{x}(t)$ seems to cross the x axis around 16.4 suggesting that this is the period of the first swing of the pendulum, in agreement with (C.5.35). We can zoom in on the crossing to get a better view,

```
plot('Xd(t)',t=16.420..16.422, color = [black],numpoints=1000);
```



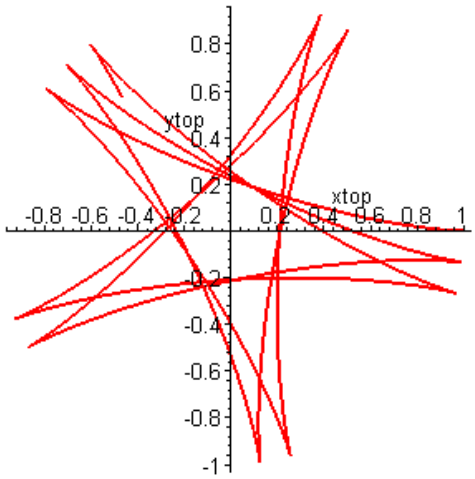
(C.8.10)

This indicates that the period is about 16.4211 sec.

Foucault Pendulum on a rotating platform

We now stop the Earth's rotation, transport the entire Pantheon to the North Pole ($\beta = 0°$) and mount it on a sturdy platform which rotates once every 77 seconds. We do this just to explore the pendulum orbit trajectories that might result. With equally-scaled axes we get this Foucault path for a duration of 90.3 seconds,

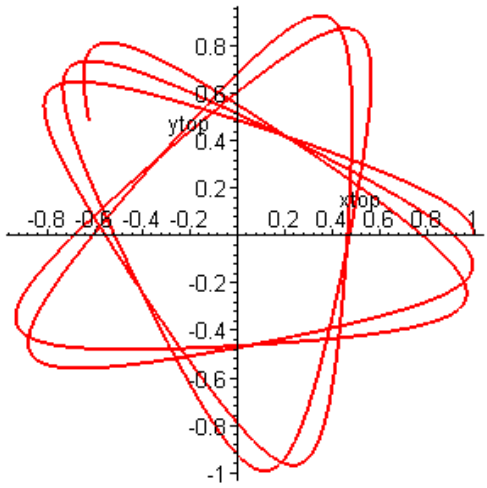
```
inits := {x(0)=1,y(0)=0, D(x)(0)=0,D(y)(0)=0}:
odeplot(f,[x(t),-y(t)],0..90.3,numpoints=1000,axes = boxed,labels =
[xtop,ytop],thickness=2,axes=normal,scaling=constrained);
```



(C.8.11)

Next, when the pendulum is released at x = 1m, it is given a small v_y velocity of -0.1 m/s, which means that v_{ytop} = +0.1 m/s :

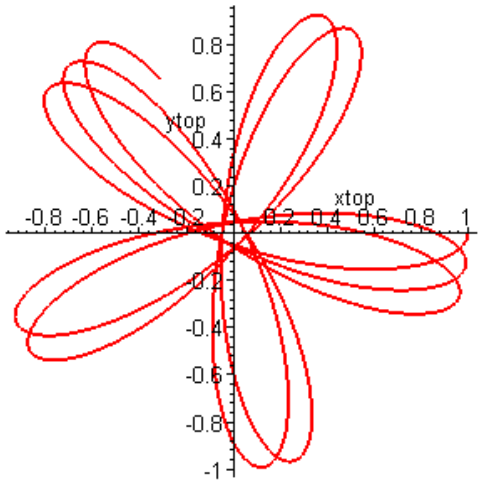
```
inits := {x(0)=1,y(0)=0, D(x)(0)=0,D(y)(0)=-0.1}:
```



(C.8.12)

The sharp cusps of the previous orbit are now smoothed out. Conversely, if we apply v_y = +0.1 m/s so that v_{ytop} = -0.1 m/s :

```
inits := {x(0)=1,y(0)=0, D(x)(0)=0,D(y)(0)=0.1}:
```

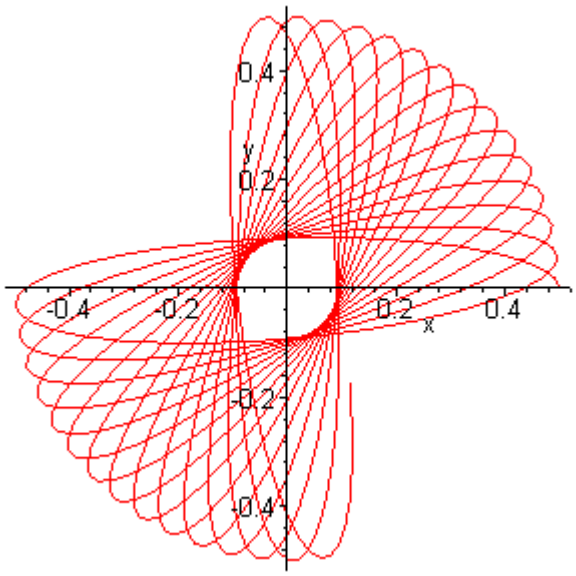


(C.8.13)

Desktop Spherical Pendulum

Here we set ℓ = 1 m and turn off the Earth's rotation completely with ω = 0, so our Foucault pendulum becomes a spherical pendulum. We start it moving with these initial conditions :

```
inits := {x(0)=0.5,y(0)=0, D(x)(0)=0,D(y)(0)=-.3}:  Digits := 16:
f := dsolve(eqs union
inits,funcs,type=numeric,method=gear,output=listprocedure):
odeplot(f,[x(t),-y(t)],0..30,numpoints=500,axes = boxed,labels =
[x,y],thickness=1,axes=normal,scaling=constrained);
```



(C.8.14)

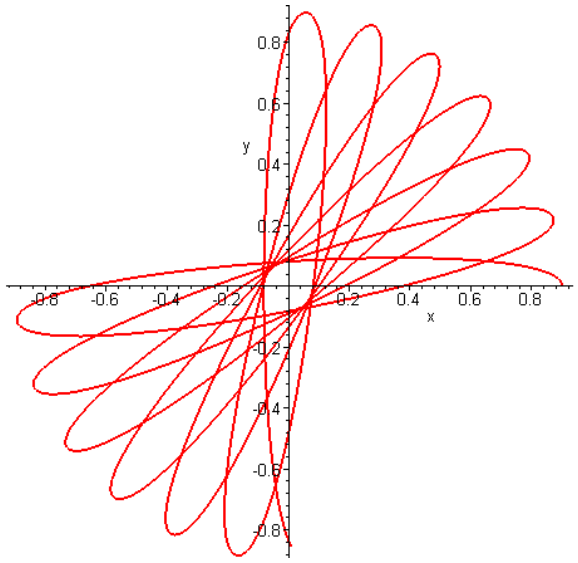
In 30 seconds the orbit precesses as shown, a phenomenon known as the intrinsic apsidal Airy precession discussed near (C.5.30). The orbit precesses in the same directional sense in which it orbits. This effect has nothing to do with the Earth's rotation, and turning ω back on makes no visible change in the above picture.

Reader Exercise:

(1) Construct the above pendulum with 1 meter of thread and a small weight hung from a ceiling lamp fixture. With x ~ 1/2m start an elliptical motion of the shape shown above. Count the number of swings it takes for the orbit to precess 90 degrees and compare to the above figure.

If we change the initial position to x(0) = 0.9 to get more swing, here is the orbit out to t = 10:

```
inits := {x(0)=0.9,y(0)=0, D(x)(0)=0,D(y)(0)=-0.3}:  Digits := 16:
f := dsolve(eqs union inits,funcs,type=numeric,method=gear,output=listprocedure)
odeplot(f,[x(t),-y(t)],0..14,numpoints=500,axes = boxed,labels =
[x,y],thickness=2,axes=normal,scaling=constrained);
```

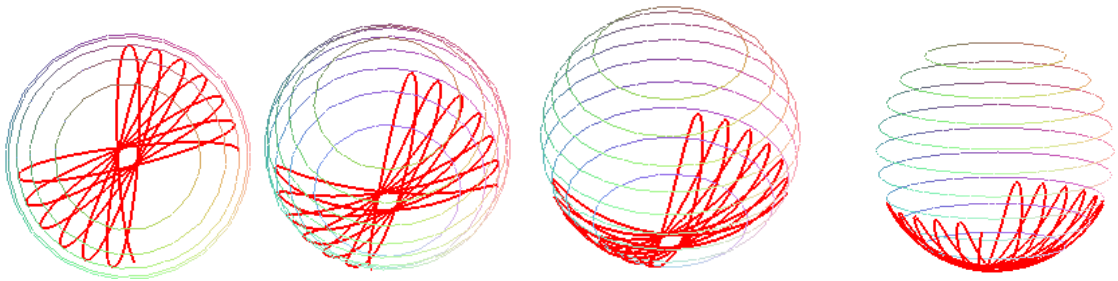


(C.8.15)

It appears that the higher starting point has increased the precession rate.

Here are some 3D plots of the above solution trajectory :

```
s1 := spacecurve(['X(t)','Y(t)','Z(t)'],t=0..14,numpoints=500,thickness=2,color=red):
s2 := implicitplot3d(x^2+y^2+z^2 =
r^2,x=-r..r,y=-r..r,z=-r..r,scaling=constrained,style=contour):
display(s1,s2);
```



(C.8.16)

In the left image, the camera is looking down from the pivot point of the pendulum so the previous 2D picture is roughly replicated. Then the camera moves down to lower viewing angles.

We already discussed in (C.5.30) Airy's 1851 formula for the precession of the elliptical orbit,

$$\omega_{airy}/\omega_{swing} = T_{swing}/T_{airy} = (3/8)(ab/\ell^2) = (3/8\pi)(\pi ab/\ell^2) = (3/8\pi)(A/\ell^2) \ . \qquad (C.5.30)$$

As noted there, in this approximate formula, a and b are the semimajor and semiminor axes of the narrow ellipse which is slowly precessing, and $\boldsymbol{\ell}$ is the length of the string ($A = \pi ab$ is the area of the ellipse). This formula is most accurate for small oscillations and gets less precise for larger ones, requiring correction terms. We can apply it to our two examples above.

A tick mark in (C.8.14) is .04m, so staring the picture we can estimate, since $\boldsymbol{\ell} = 1$ m,

a = 0.50 m
b = 2.25 ticks = 2.25 *.04 = 0.09 m
1/N = (3/8)*(0.5)*(.09) = .016875000 // Maple
N = 59.25925926
N/4 = 14.81481482 (C.8.17)

The Airy prediction for Fig (C.8.14) is that there are 14.8 ellipses per quarter turn of the precession. A count of ellipses in the figure gives about 13.5, fairly close given that these are not really small narrow ellipse oscillations.

For (C.8.15) the numbers are

a = 0.90 m
b = 2 ticks = 2 *.04 = 0.08 m
1/N = (3/8)*(0.9)*(.08) = .027000000
N = 37.03703704
N/4 = 9.259259260 (C.8.18)

The formula predicts 9.26 ellipses per quarter turn, while Fig (C.8.15) shows about 6.5. We expect a worse result since this is definitely not a small oscillation. But both results are in the ballpark, and Airy predicts that the second figure has fewer ellipses per quarter turn than the first figure.

Appendix D: Center of gravity and torque for a tethered satellite

D.1 Definition of Center of Gravity

The phrase "center of gravity" is often used as a synonym for "center of mass" which complicates searching for information about the former concept. The "center of mass" of a system of particles is well known to be

$$\mathbf{r}_{cms} = \frac{\Sigma_i m_i \mathbf{r}_i}{\Sigma_i m_i} = \frac{1}{M} \Sigma_i m_i \mathbf{r}_i \qquad M = \Sigma_i m_i \qquad \text{// discrete}$$

$$\mathbf{r}_{cms} = \frac{\int dV \rho\, \mathbf{r}}{\int dV \rho} = \frac{1}{M} \int dV\, \rho\, \mathbf{r} \qquad M = \int dV\, \rho \qquad \text{// continuous .} \tag{D.1.1}$$

We use "cms" to mean center of mass even though "com" might be more reasonable. Notice that $\mathbf{r}_{cms}$ is measured with respect to the same origin used for the $\mathbf{r}_i$ or $\mathbf{r}$. For a rigid object, the center of mass is a definite point that does not move around relative to the object. It is completely determined by the spatial mass distribution of the object.

The center of gravity is a completely different animal, though it happens to align with the center of mass for a uniform gravitational field. For that reason one never deals with a distinct center of gravity concept for human-scale engineering objects on the Earth's surface.

Consider a system of masses m_i each of which experiences some force $\mathbf{F}_i$. We define

$$\mathbf{F} = \Sigma_i \mathbf{F}_i \qquad \mathbf{N}^{(R)} = \Sigma_i\, (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i \qquad \mathbf{N}^{(0)} = \Sigma_i \mathbf{r}_i \times \mathbf{F}_i \quad . \tag{D.1.2}$$

Here $\mathbf{F}$ is the sum of the forces acting on all the masses m_i, $\mathbf{N}^{(R)}$ is the total torque on the system *with respect to* some arbitrary point $\mathbf{R}$, and $\mathbf{N}^{(0)}$ is the total torque with respect to the selected origin. An obvious theorem is that

$$\mathbf{N}^{(R)} = \Sigma_i\, (\mathbf{r}_i - \mathbf{R}) \times \mathbf{F}_i = \Sigma_i \mathbf{r}_i \times \mathbf{F}_i - \mathbf{R} \times \Sigma_i \mathbf{F}_i$$

$$= \mathbf{N}^{(0)} - \mathbf{R} \times \mathbf{F} \tag{D.1.3}$$

which shows how the two torques are related. If $\mathbf{F} = 0$, which is often the case, then the torque is the same with respect to any point.

One characteristic of a **center of gravity** point $\mathbf{r}_{cog}$ is that the total torque on a system measured *with respect to* point $\mathbf{r}_{cog}$ vanishes, so $\mathbf{N}^{(r_{cog})} = 0$. From (D.1.3) we see that this is the same as saying $\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}$, so

$$\mathbf{N}^{(r_{cog})} = 0 \quad \Leftrightarrow \quad \mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F} \qquad \mathbf{r}_{cog} = \text{"center of gravity"} \; . \tag{D.1.4}$$

The significance of $\mathbf{r}_{cog}$ is that it is a point which allows the relation between the total system torque $\mathbf{N}^{(0)}$ and the total system force $\mathbf{F}$ to have the same form as $\mathbf{N}_i = \mathbf{r}_i \times \mathbf{F}_i$ for a single point particle. It is not obvious that such a point $\mathbf{r}_{cog}$ exists for some arbitrary system of particles.

So far this notion of "center of gravity" has nothing to do specifically with gravity, but below we shall add an additional part of the definition which does bring in gravity.

Unlike the center of mass, the center of gravity (if it exists) may not be unique and it generally moves around in an object as the object changes orientation in an external force field If we dot $\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}$ with the vector $\mathbf{F}$ we find

$$\mathbf{N}^{(0)} \bullet \mathbf{F} = 0 \ . \tag{D.1.5}$$

As shown above, $\mathbf{N}^{(0)}$ and $\mathbf{F}$ are well-defined computable quantities and if $\mathbf{N}^{(0)} \bullet \mathbf{F} \neq 0$, then $\mathbf{r}_{cog}$ cannot possibly exist (because if it did exist one must have $\mathbf{N}^{(0)} \bullet \mathbf{F} = 0$). So (D.1.5) is a condition for the existence of $\mathbf{r}_{cog}$.

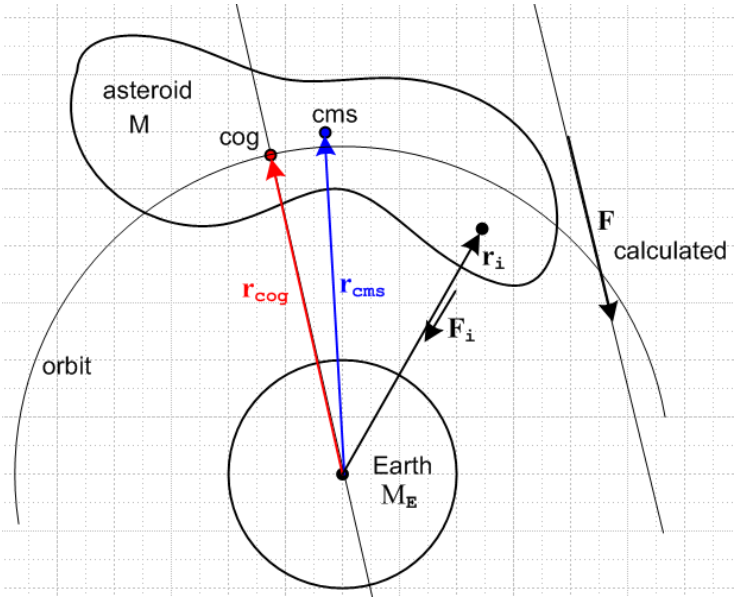
Here is a method for locating $\mathbf{r}_{cog}$, a variation of Symon p 258 which we present in the context of an asteroid of mass M near the Earth. This description includes the second characteristic of the center of gravity point which is that it is point at which gravitational action is effectively focused.

One first computes the total gravitational force $\mathbf{F} = \Sigma_i \mathbf{F}_i$ due to the Earth (summed over all points in the asteroid) and one then knows the direction of the sum vector $\mathbf{F}$. One creates a line along $\mathbf{F}$ and then translates that line parallel to $\mathbf{F}$ until the line passes through the center of the Earth. The center of gravity of the asteroid lies on that translated line a distance r_{cog} from the center of the Earth where r_{cog} is determined by,

$$GM_E M/r_{cog}^2 = F \quad \Rightarrow \quad r_{cog} = \sqrt{GM_E M/F}$$

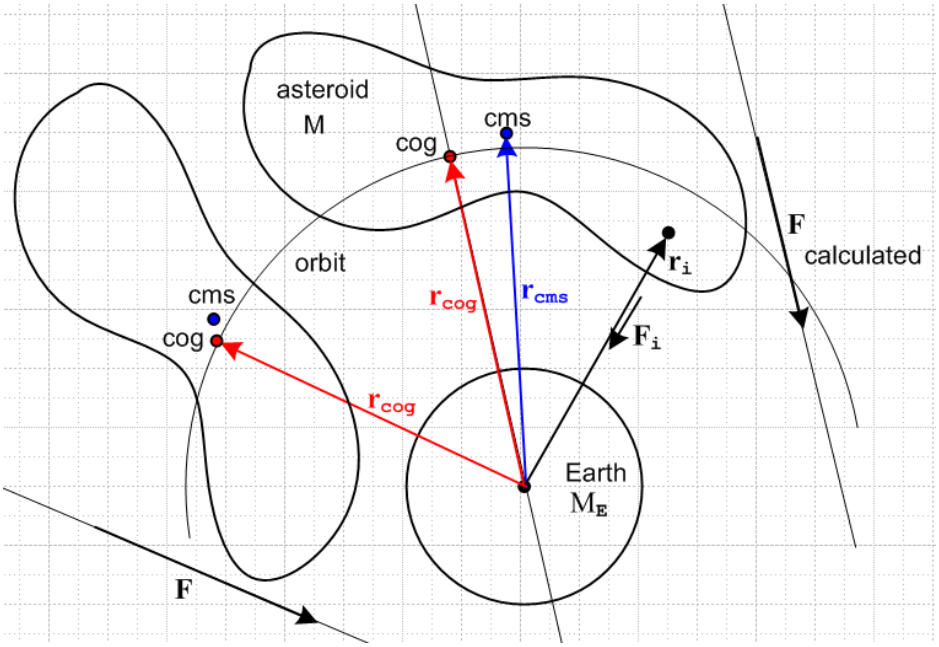
$$\mathbf{r}_{cog} = r_{cog}(-\hat{\mathbf{F}}) \qquad \mathbf{F} = F\,\hat{\mathbf{F}} \ . \tag{D.1.6}$$

Here is an illustration,



(D.1.7)

From the Earth's point of view, one could replace the entire asteroid with a point mass M at location $\mathbf{r}_{cog}$ and the Earth would feel the same gravitational pull from that point mass as it does from the asteroid. Furthermore, if the asteroid were in a circular orbit around the earth keeping its same aspect facing the earth (not very likely), then the orbiting characteristics of the asteroid would be the same as for its replacement point mass, and one would have for example $GM_EM/r_{cog}^2 = M\omega^2 r_{cog}$ as the balance between gravitational and centrifugal force. If the asteroid does not rotate or tumbles in some manner, at any point in its orbit $\mathbf{r}_{cog}$ will lie on the orbit shown, but the location of that point within the asteroid changes so that the new r_{cog} computed for a new position and orientation still equals the orbit radius. In this case the $\mathbf{r}_{cog}$ position will change relative to the asteroid, whereas the cms point always has the same position relative to the asteroid. Here we enhance the above drawing by adding another position of the tumbling asteroid in its orbit,



(D.1.8)

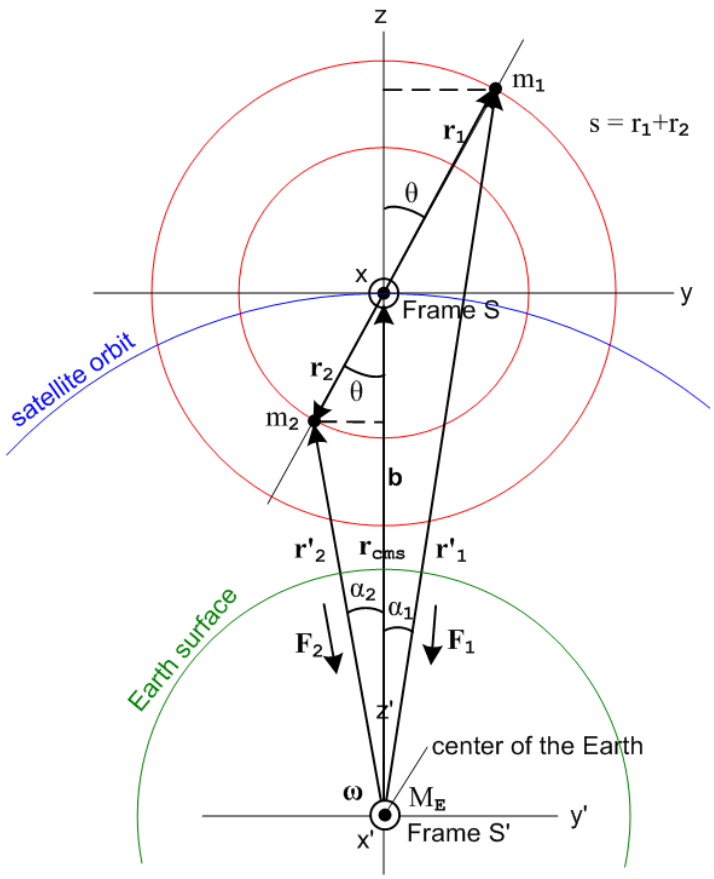
One implication is that, for any orientation of the asteroid in orbit, there exists a position of the asteroid such that the total force **F** will have the same magnitude as at any other position, since r_{cog} is the same.

One might wonder what Symon's operational prescription for computing $\mathbf{r}_{cog}$ has to do with our opening section about torque. First, for each point in the asteroid we know that $\mathbf{r}_i \times \mathbf{F}_i = 0$ ($\mathbf{r}_i$ tails are at Earth center) because each $\mathbf{r}_i$ to mass m_i is parallel to the force $\mathbf{F}_i$ on mass m_i. Thus according to (D.1.2) we have $\mathbf{N}^{(0)} = 0$. Therefore the condition (D.1.5) that $\mathbf{N}^{(0)} \bullet \mathbf{F} = 0$ is trivially satisfied. Then $\mathbf{r}_{cog}$ exists and is a vector which satisfies the center of gravity definition (D.1.4) that $\mathbf{N}^{(0)} = \mathbf{r}_{cog} \times \mathbf{F}$. This equation is satisfied by Symon's computed $\mathbf{r}_{cog}$ because $\mathbf{r}_{cog}$ is parallel to **F**, so the equation says 0 = 0. From (D.1.4) we then see that the $\mathbf{N}^{(R_{cog})} = 0$ so $\mathbf{r}_{cog}$ is a point with respect to which the total torque on the asteroid (due to gravitational force from the Earth) is 0. In this example it *happens* that any point **R** along the $\mathbf{r}_{cog}$ line satisfies $\mathbf{N}^{(0)} = \mathbf{R} \times \mathbf{F}$ and any such point **R** therefore is a point of zero total torque $\mathbf{N}^{(R)} = 0$. But only one point on this line gives the concentration point of the gravitational force such that $GM_EM/r_{cog}^2 = F$.

We shall now consider a 2-mass tethered satellite as a relatively simple example where we can compute the cog point and then watch it move relative to the cms point.

D.2 Center of Gravity for a 2-mass Dumbbell Satellite

To head off a proliferation of primes, we now switch to "swap notation" in which Frame S is the rotating frame while Frame S' is the fixed inertial frame. Rather than assume an arbitrary dumbbell orientation, we assume for this section that the dumbbell lies in the plane of paper:



(D.2.1)

Frame S' has its origin at the center of the Earth and is assumed fixed relative to the stars.
Frame S has its origin at the center of *mass* of the satellite as shown.
The orbit of the satellite lies in the plane of paper.
The vector **ω** describes the angular velocity of the satellite and has nothing to do with Earth's rotation.

Moreover, at t = 0 the axes of both frames line up: $\hat{\mathbf{x}} = \hat{\mathbf{x}}'$, $\hat{\mathbf{y}} = \hat{\mathbf{y}}'$ and $\hat{\mathbf{z}} = \hat{\mathbf{z}}'$.

The two masses m_1 and m_2 are connected by a massless rigid stick of length $s = r_1+r_2$. Since the masses are assumed to be unequal, mass m_1 is restricted to lie on a sphere of radius r_1 in Frame S', while mass m_2 lies on a different sphere of radius r_2 (to be computed below). For purposes of the drawing, we have $m_2 > m_1$ so that $r_2 < r_1$.

The polar angle in Frame S of m_1 (and the stick) is θ. The angles α_1 and α_2 are positive.

Assumption: We will show below that the center of gravity is very close to the center of mass for any angle θ. For example, if the stick is 10 m long, the distance between these two centers for a low-Earth orbit is about 3 microns. For this reason, we shall assume that it is the center of mass point $\mathbf{r}_{cms}$ which goes in a circular orbit around the Earth, even though we know from Section D.1 that it is really $\mathbf{r}_{cog}$ that does this. So as the satellite goes around the earth and θ varies in time, the point $\mathbf{r}_{cog}$ moves a very small amount relative to $\mathbf{r}_{cms}$, and we shall just ignore this tiny motion.

The satellite center of mass point then rotates around the Earth at some rate $\omega = 2\pi/T$. For a low-Earth orbit, T is on the order of 88 minutes. As the satellite moves in this orbit, Frame S rotates with the satellite while Frame S' stays fixed relative to the stars. This means that the Frame S axis z' always points away from the center of the Earth.

One can generalize the discussion for an elliptical orbit, but things are complicated enough for a circular orbit so we stick with that simplification. In Appendix F we allow the dumbbell to move out of the plane of paper, but here for simplicity we assume it is in the plane of paper for any θ.

With the above assumption, we may identify $\mathbf{r}_{cms}$ with **b**, our usual vector connecting the two Frame origins, so

$$\mathbf{b} = \mathbf{r}_{cms} \tag{D.2.2}$$

and then from the figure,

$$\begin{aligned} \mathbf{b} + \mathbf{r}_1 &= \mathbf{r'}_1 \\ \mathbf{b} + \mathbf{r}_2 &= \mathbf{r'}_2 \,. \end{aligned} \tag{D.2.3}$$

The Law of Cosines gives, for the right and left triangles,

$$\begin{aligned} r'^2_1 &= r_1^2 + b^2 - 2r_1 b\cos(\pi-\theta) = r_1^2 + b^2 + 2r_1 b\cos\theta \\ r'^2_2 &= r_2^2 + b^2 - 2r_2 b\cos\theta \quad . \end{aligned} \tag{D.2.4}$$

Center of Mass

From (D.1.1) we know that the center of mass of the satellite is given by,

$$\mathbf{r}_{cms} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2} = 0 \qquad \text{Frame S} \tag{D.2.5}$$

$$\mathbf{r'}_{cms} = \frac{m_1\mathbf{r'}_1 + m_2\mathbf{r'}_2}{m_1+m_2} = \mathbf{b} \,. \qquad \text{Frame S'} \tag{D.2.6}$$

Due to the constraint of the massless stick one has,

$$\hat{\mathbf{r}}_2 = -\,\hat{\mathbf{r}}_1 \tag{D.2.7}$$

so (D.2.5) says

$$m_1\mathbf{r}_1 = -\,m_2\mathbf{r}_2 \quad \Rightarrow \quad m_1r_1 = m_2r_2\,, \quad r_2/r_1 = m_1/m_2\,, \quad \hat{\mathbf{r}}_2 = -\,\hat{\mathbf{r}}_1\,. \tag{D.2.8}$$

Angles

Looking at the right triangles from the center of the Earth to the two dashes lines, one sees that

$$\begin{aligned} \sin\alpha_1 &= (r_1\sin\theta)\,/\,r'_1 \quad \Rightarrow \quad r'_1\sin\alpha_1 = r_1\sin\theta \\ \sin\alpha_2 &= (r_2\sin\theta)\,/\,r'_2 \quad \Rightarrow \quad r'_2\sin\alpha_2 = r_2\sin\theta\,. \end{aligned} \tag{D.2.9}$$

The same triangles reveal that

$$\begin{aligned} \cos\alpha_1 &= (b + r_1\cos\theta)/r'_1 \quad \Rightarrow \quad r'_1\cos\alpha_1 = b + r_1\cos\theta \\ \cos\alpha_2 &= (b - r_2\cos\theta)/r'_2 \quad \Rightarrow \quad r'_2\cos\alpha_2 = b - r_2\cos\theta\,. \end{aligned} \tag{D.2.10}$$

We shall need the following dot products,

$$\begin{aligned} \hat{\mathbf{r}}'_1 \bullet \hat{\mathbf{z}} &= \cos\alpha_1 \qquad \hat{\mathbf{r}}'_1 \bullet \hat{\mathbf{y}} = \cos(\pi/2-\alpha_1) = \sin\alpha_1 \\ \hat{\mathbf{r}}'_2 \bullet \hat{\mathbf{z}} &= \cos\alpha_2 \qquad \hat{\mathbf{r}}'_2 \bullet \hat{\mathbf{y}} = \cos(\pi/2+\alpha_1) = -\sin\alpha_2 \end{aligned} \tag{D.2.11}$$

$$\begin{aligned} \hat{\mathbf{r}}'_1 \bullet \hat{\mathbf{r}}'_2 &= \cos(\alpha_1+\alpha_2) = \cos\alpha_1\cos\alpha_2 - \sin\alpha_1\sin\alpha_2 \\ &= (r'_1r'_2)^{-1}[\,(r'_1\cos\alpha_1)(r'_2\cos\alpha_2) - (r'_1\sin\alpha_1)(r'_2\sin\alpha_2)\,] \\ &= (r'_1r'_2)^{-1}[\,(b + r_1\cos\theta)(b - r_2\cos\theta) - (r_1\sin\theta)(r_2\sin\theta)\,] \qquad \text{// (D.2.9) and (D.2.10)} \\ &= (r'_1r'_2)^{-1}[b^2 + b(r_1-r_2)\cos\theta - r_1r_2\cos^2\theta - r_1r_2\sin^2\theta\,] \\ &= (r'_1r'_2)^{-1}[b^2 + b(r_1-r_2)\cos\theta - r_1r_2\,]\,. \end{aligned} \tag{D.2.12}$$

Where is Center of Gravity of the Satellite?

The gravitational forces acting on the two satellite masses due to the Earth are.

$$\begin{aligned} \mathbf{F}_1 &= -\,(GM_E m_1/r'^2_1)\,\hat{\mathbf{r}}'_1 = -\,(GM_E m_1/r'^3_1)\,\mathbf{r}'_1 \\ \mathbf{F}_2 &= -\,(GM_E m_2/r'^2_2)\,\hat{\mathbf{r}}'_2 = -\,(GM_E m_2/r'^3_2)\,\mathbf{r}'_2\,. \end{aligned} \tag{D.2.13}$$

Note the direction of the "long" vectors $\mathbf{r}'_1$ and $\mathbf{r}'_2$ in the drawing.

Following the prescription of Section D.1 for finding the location of the center of gravity, we compute the total gravitational force on the satellite,

$$\mathbf{F} = -(GM_E m_1/r'^2_1)\,\hat{\mathbf{r}}'_1 - (GM_E m_2/r'^2_2)\,\hat{\mathbf{r}}'_2$$

$$= (-GM_E)\,[\,(m_1/r'^2_1)\,\hat{\mathbf{r}}'_1 + (m_2/r'^2_2)\,\hat{\mathbf{r}}'_2\,] \quad . \qquad \text{(D.2.14)}$$

Taking components:

$$F_y = \mathbf{F}\bullet\hat{\mathbf{y}} = (-GM_E)\,[\,(m_1/r'^2_1)\,\hat{\mathbf{r}}'_1\bullet\,\hat{\mathbf{y}} + (m_2/r'^2_2)\,\hat{\mathbf{r}}'_2\bullet\,\hat{\mathbf{y}}\,]$$

$$= (-GM_E)\,[\,(m_1/r'^2_1)\sin\alpha_1 - (m_2/r'^2_2)\sin\alpha_2\,] \qquad \text{// (D.2.11)}$$

$$= (-GM_E)\,[\,(m_1/r'^3_1)\,r'_1\sin\alpha_1 - (m_2/r'^3_2)\,r'_2\sin\alpha_2\,]$$

$$= (-GM_E)\,[\,(m_1/r'^3_1)\,r_1\sin\theta - (m_2/r'^3_2)\,r_2\sin\theta\,] \qquad \text{// (D.2.9)}$$

$$= (-GM_E)\,[\,(m_1/r'^3_1)\,r_1\sin\theta - (m_1/r'^3_2)\,r_1\sin\theta\,] \qquad \text{// (D.2.8)}$$

$$= (-GM_E)\,(m_1 r_1\sin\theta)\,[\,(1/r'^3_1) - (1/r'^3_2)\,]$$

$$F_z = \mathbf{F}\bullet\hat{\mathbf{z}} = (-GM_E)\,[\,(m_1/r'^2_1)\,\hat{\mathbf{r}}'_1\bullet\,\hat{\mathbf{z}} + (m_2/r'^2_2)\,\hat{\mathbf{r}}'_2\bullet\,\hat{\mathbf{z}}\,]$$

$$= (-GM_E)\,[\,(m_1/r'^2_1)\cos\alpha_1 + (m_2/r'^2_2)\cos\alpha_2\,] \qquad \text{// (D.2.11)}$$

$$= (-GM_E)\,[\,(m_1/r'^3_1)\,r'_1\cos\alpha_1 + (m_2/r'^3_2)\,r'_2\cos\alpha_2\,]$$

$$= (-GM_E)\,[\,(m_1/r'^3_1)(\,b + r_1\cos\theta) + (m_2/r'^3_2)(b - r_2\cos\theta)\,] \qquad \text{// (D.2.10)}$$

$$= (-GM_E)\,[\,b\,\{(m_1/r'^3_1) + (m_2/r'^3_2)\} + \cos\theta\,\{\,(m_1 r_1/r'^3_1) - (m_2 r_2/r'^3_2)\,\}\,]$$

$$= (-GM_E)\,[\,b\,\{(m_1/r'^3_1) + (m_2/r'^3_2)\} + \cos\theta\,\{\,(m_1 r_1/r'^3_1) - (m_1 r_1/r'^3_2)\,\}\,] \qquad \text{// (D.2.8)}$$

$$= (-GM_E)\,[\,b\,\{(m_1/r'^3_1) + (m_2/r'^3_2)\} + m_1 r_1\cos\theta\,\{\,(1/r'^3_1) - (1/r'^3_2)\,\}\,]$$

so

$$F_y = (-GM_E)\,(m_1 r_1\sin\theta)\,[\,(1/r'^3_1) - (1/r'^3_2)\,]$$

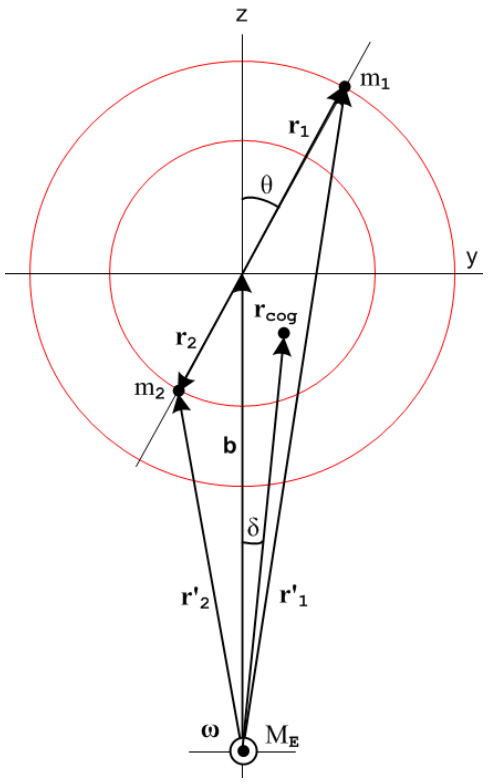
$$F_z = (-GM_E)\,[\,b\,\{(m_1/r'^3_1) + (m_2/r'^3_2)\} + m_1 r_1\cos\theta\,\{\,(1/r'^3_1) - (1/r'^3_2)\,\}\,] \quad . \qquad \text{(D.2.15)}$$

Recall now from (D.1.6) the rule for obtaining $\mathbf{r}_{cog}$:

$$GM_E M/r_{cog}^2 = F \quad \Rightarrow \quad r_{cog} = \sqrt{GM_E M/F}$$

$$\mathbf{r}_{cog} = r_{cog}(-\hat{\mathbf{F}}) \qquad \mathbf{F} = F\,\hat{\mathbf{F}}\,. \qquad \textit{(D.1.6)} \qquad \text{(D.2.16)}$$

Just to have a picture, we now add $\mathbf{r}_{cog}$ to Fig (D.2.1), picking a graphic location for the point which makes things easy to draw (we don't know yet where it actually lies),



(D.2.17)

The center of gravity point lies a distance r_{cog} (to be computed) along a line which makes some angle δ relative to the vertical axis. Since $\hat{\mathbf{r}}_{cog} = -\hat{\mathbf{F}}$, we may write using (D.2.15),

$$\tan(\delta) = -F_y/F_z = -(m_1 r_1 \sin\theta)\,\frac{(1/r'^3_1) - (1/r'^3_2)}{b\,[\,(m_1/r'^3_1) + (m_2/r'^3_2)\,] + m_1 r_1 \cos\theta\,[\,(1/r'^3_1) - (1/r'^3_2)]}$$

$$= -(m_1 r_1 \sin\theta)\,\frac{r'^3_2 - r'^3_1}{b\,[\,(m_1 r'^3_2) + (m_2 r'^3_1)\,] + m_1 r_1 \cos\theta\,[\,r'^3_2 - r'^3_1]}\,. \qquad \text{(D.2.18)}$$

If the masses are vertically aligned, meaning $\theta = 0$ or π, then $\sin\theta = 0$ and we see that $\delta = 0$, as expected.

Since for general masses and angle θ one does *not* have $r'_1 = r'_2$, one may conclude:

Fact 1: In general the center of gravity does not lie on the line between the center of the Earth and the center of mass. (D.2.19)

Earlier we conjectured that this was the case, and here we see it in our dumbbell satellite example.

Next we wish to compute the distance r_{cog} to the center of gravity. Using (D.2.14) and (D.2.12),

$$F^2 = (GM_E)^2 [(m_1/r'^2_1)^2 + (m_2/r'^2_2)^2 + (m_1/r'^2_1)(m_2/r'^2_2) \hat{\mathbf{r}}'_1 \bullet \hat{\mathbf{r}}'_2] \quad // \text{(D.2.14)}$$

$$= (GM_E)^2 [(m_1/r'^2_1)^2 + (m_2/r'^2_2)^2 + (m_1/r'^2_1)(m_2/r'^2_2)(r'_1r'_2)^{-1}(b^2 + b(r_1-r_2)\cos\theta - r_1r_2)] \quad // \text{(D.2.12)}$$

$$= (GM_E)^2 [(m_1/r'^2_1)^2 + (m_2/r'^2_2)^2 + (m_1/r'^3_1)(m_2/r'^3_2)(b^2 + b(r_1-r_2)\cos\theta - r_1r_2)]$$

so

$$F = GM_E \sqrt{(m_1/r'^2_1)^2 + (m_2/r'^2_2)^2 + 2\,(m_1/r'^3_1)\,(m_2/r'^3_2)(b^2 + b(r_1-r_2)\cos\theta - r_1r_2)} \ . \qquad \text{(D.2.20)}$$

Then, using $M = (m_1+m_2)$ and $m_i = \mu_i M$, we find from (D.2.16) that,

$$r_{cog} = \sqrt{GM_E M/F}$$

$$= \frac{1}{[(\mu_1/r'^2_1)^2 + (\mu_2/r'^2_2)^2 + 2\,(\mu_1/r'^3_1)\,(\mu_2/r'^3_2)(b^2 + b(r_1-r_2)\cos\theta - r_1r_2)]^{1/4}} \qquad \text{(D.2.21)}$$

$$r'^2_1 = r_1^2 + b^2 + 2r_1 b\cos\theta$$
$$r'^2_2 = r_2^2 + b^2 - 2r_2 b\cos\theta \qquad // \ \mu_i \equiv m_i/M \qquad \textit{(D.2.4)}$$

which is somewhat more complicated than our expression for r_{cms},

$$r_{cms} = b \ . \qquad \textit{(D.2.2)}$$

We are then led to:

Fact 2: In general the center of gravity does not lie the same distance from the center of the Earth as the center of mass. (D.2.22)

Quick check on (D.2.21) : If $m_1 = 0$, then $\mu_1 = 0$ and $\mu_2 = 1$ and the result is $r_{cog} = r'_2$ which is correct.

Reader Exercise: If the two masses are vertically aligned with $\theta = 0$, then

(a) Show that: $r_2 = b - r'_2$ $\qquad r'_1 + r'_2 = 2b$
$r_1 = r'_1 - b$ $\qquad (b^2 + b(r_1-r_2)\cos\theta - r_1r_2) = r'_1r'_2$ (D.2.23)

(b) Show that $r_{cog} = \dfrac{1}{\sqrt{(\mu_1/r'^2_1) + (\mu_2/r'^2_2)}} = \sqrt{\dfrac{m_1+m_2}{(m_1/r'^2_1) + (m_2/r'^2_2)}}$ (D.2.24)

(c) Show that $r_{cog} < r_{cms}$, where $r_{cms} = b = (r'_1+r'_2)/2$ and $\mu_1+\mu_2 = 1$

D.3 Dumbbell Satellite Center of Gravity with the Far Approximation: Numerical Examples

A practical tethered satellite is not going to have Earth-scale dimensions so we now make the obvious assumption that r'_1, r'_2 and b are much larger than r_1 and r_2. We then define smallness parameters,

$$\varepsilon_1 \equiv (r_1/b) \qquad \text{(D.2.8)}$$
$$\varepsilon_2 \equiv (r_2/b) = (r_2/r_1)\,(r_1/b) = (m_1/m_2)\varepsilon_1 = (\mu_1/\mu_2)\varepsilon_1 \ . \qquad \text{(D.3.1)}$$

Then

$$(b^2 + b(r_1 - r_2)\cos\theta - r_1 r_2) = b^2\,(1 + (\varepsilon_1 - \varepsilon_2)\cos\theta - \varepsilon_1\varepsilon_2) \qquad \text{(D.3.2)}$$

so that from (D.2.21),

$$r_{cog} = \frac{1}{[\ (\mu_1/r'^2_1)^2 + (\mu_2/r'^2_2)^2 + 2\,(\mu_1/r'^3_1)\,(\mu_2/r'^3_2)b^2(1 + (\varepsilon_1 - \varepsilon_2)\cos\theta - \varepsilon_1\varepsilon_2)\ ]^{1/4}} \ . \qquad \text{(D.3.3)}$$

From (D.2.4) we know also that

$$r'^2_1 = r_1^2 + b^2 + 2r_1 b\cos\theta = b^2\,[\,1 + 2(r_1/b)\cos\theta + (r_1/b)^2] = b^2\,[\,1 + 2\varepsilon_1\cos\theta + \varepsilon_1^2]$$

$$r'^2_2 = r_2^2 + b^2 - 2r_2 b\cos\theta = b^2\,[\,1 - 2(r_2/b)\cos\theta + (r_2/b)^2] = b^2\,[\,1 - 2\varepsilon_2\cos\theta + \varepsilon_2^2]$$

so then

$$r'_1 = b\sqrt{1 + 2\varepsilon_1\cos\theta + \varepsilon_1^2}$$
$$r'_2 = b\sqrt{1 - 2\varepsilon_2\cos\theta + \varepsilon_2^2} \ . \qquad \text{(D.3.4)}$$

So far everything is exact, but we shall now expand r_{cog} as a series in our small parameters. Since $\varepsilon_2 = (\mu_1/\mu_2)\varepsilon_1$, we replace ε_2 by this expression so there is then only one small parameter ε_1. Maple is ready to carry out this task. We first enter our expressions of interest, using rp1 for r'_1 etc :

```
restart; assume(b>0);
rcog := ((mu1^2/rp1^4) + (mu2^2/rp2^4) +
2*b^2*(mu1/rp1^3)*(mu2/rp2^3)*(1+(e1-e2)*cos(theta)- e1*e2))^(-1/4);
```

$$rcog := \frac{1}{\left(\frac{\mu 1^2}{rp1^4} + \frac{\mu 2^2}{rp2^4} + 2\,\frac{b^2\,\mu 1\,\mu 2\,(1+(e1-e2)\cos(\theta) - e1\,e2)}{rp1^3\,rp2^3}\right)^{\left(\frac{1}{4}\right)}}$$

```
rp1 := b*sqrt(1+2*e1*cos(theta)+e1^2);
```

$$rp1 := b\sqrt{1+2\,e1\cos(\theta)+e1^2}$$

```
rp2 := b*sqrt(1-2*e2*cos(theta)+e2^2);
```

$$rp2 := b\sqrt{1-2\,e2\cos(\theta)+e2^2}$$

```
e2 := (mu1/mu2)*e1;
```

$$e2 := \frac{\mu 1\,e1}{\mu 2}$$

```
mu2 := 1-mu1;
```

$$\mu 2 := 1-\mu 1 \qquad \text{(D.3.5)}$$

We then instruct Maple to expand rcog in a power series around $\varepsilon_1 = 0$ and we ask for the first four terms of the expansion,

```
series(rcog,e1=0,4): simplify(%);
```

$$b + \frac{3}{4}\,\frac{\mu 1\,(3\cos(\theta)^2-1)\,b}{-1+\mu 1}\,e1^2 - \frac{\mu 1\cos(\theta)\,(-5\cos(\theta)^2+10\cos(\theta)^2\,\mu 1+3-6\,\mu 1)\,b}{1-2\,\mu 1+\mu 1^2}\,e1^3 + O(e1^4) \qquad \text{(D.3.6)}$$

A perhaps unexpected result is that there is no term linear in ε_1, regardless of the masses. We leave it to the energetic reader to concoct a theoretical explanation of this fact (not all odd terms vanish, just the first odd term). Our conclusion then is, to second order in smallness parameter ε_1,

$$r_{cog} \approx b\left[1+\frac{3}{4}\frac{\mu_1}{1-\mu_1}(-3\cos^2\theta+1)\,\varepsilon_1^2\right] = b\left[1+\frac{3}{4}\frac{m_1}{m_2}(-3\cos^2\theta+1)\,\varepsilon_1^2\right]$$

$$= b\left[1+\frac{3}{4}\frac{m_1}{m_2}\left(1-3\cos^2\theta\right)\frac{r_1^2}{b^2}\right] = b\left[1+\frac{3}{4}\frac{r_2}{r_1}\left(1-3\cos^2\theta\right)\frac{r_1^2}{b^2}\right] \qquad \text{// (D.2.8)}$$

$$= b\left[1+\frac{3}{4}\left(1-3\cos^2\theta\right)\frac{r_1 r_2}{b^2}\right] . \qquad \text{(D.3.7)}$$

Then

$$r_{cog} - r_{cms} = r_{cog} - b \approx \left[\frac{3}{4}\left(1-3\cos^2\theta\right)\frac{r_1 r_2}{b^2}\right] b \qquad \text{(D.3.8)}$$

and

$$\frac{r_{cog} - r_{cms}}{r_{cms}} \approx \frac{3}{4}(1 - 3\cos^2\theta) \frac{r_1 r_2}{b^2} \ . \qquad (D.3.9)$$

Fact 3: If $\theta = \pm 54.7°$, we get $r_{cog} = r_{cms}$ through order ε_1^2, since this angle has $\cos\theta = 1/\sqrt{3}$. But the line to the center of gravity does not in general have $\delta = 0$ since in general $r'_1 \neq r'_2$ at this angle. (D.3.10)

Fact 4: If the masses are vertically aligned, meaning $\theta = 0$ or π, then $\cos^2\theta = 1$ and we find that

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{3}{4}(1 - 3) \frac{r_1 r_2}{b^2} = -\frac{3}{2} \frac{r_1 r_2}{b^2} \ . \qquad (D.3.11)$$

Since the right side is negative, we have $r_{cog} < r_{cms}$ and the center of gravity in this case is *closer* to the Earth than the center of mass and of course lies on the line to the center of mass.

Fact 5: If the masses are horizontally aligned, meaning $\theta = \pm\pi/2$, then $\cos\theta = 0$ and we find that

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{3}{4}(1 - 0) \frac{r_1 r_2}{b^2} = + \frac{3}{4} \frac{r_1 r_2}{b^2} \ . \qquad (D.3.12)$$

The center of gravity in this case is *farther* from the Earth than the center of mass. The line to the center of gravity does not have $\delta = 0$ unless $m_1 = m_2$.

The above cases show the extremes of the factor $(1 - 3 \cos^2\theta)$ and hence of $r_{cog} - r_{cms}$. For a general angle θ the result lies between the two limiting cases.

Numerical Examples

From (8.8.6) one has $R_E =$ 6371 km. If we put our satellite in orbit 200 km above the Earth's surface, then b = 6371+200 = 6571 km. A higher orbit of course gives a larger b and a smaller offset between r_{cms} and r_{cog}. We take the maximum displacement between r_{cog} and r_{cms} from Fact 5 to obtain

$$\left| \frac{r_{cog} - r_{cms}}{r_{cms}} \right|_{max} = \frac{3}{4} \frac{r_1 r_2}{b^2} \ . \qquad (D.3.13)$$

For a given mass separation (tether length) $s = r_1 + r_2$ the product $r_1 r_2$ is maximized when $r_1 = r_2$ (easy to show) and this in turn means $m_1 = m_2$, so to get the worst case we set $r_1 = r_2 = s/2$ to get

$$\left| \frac{r_{cog} - r_{cms}}{r_{cms}} \right|_{max} = \frac{3}{4} \frac{1}{4} (s/b)^2 = (3/16)(s/b)^2 \qquad s = \text{stick length} = \text{tether length} \qquad (D.3.14)$$

Note the quadratic dependence on the ratio s/b. Regarding the left side of this equation as dr/b, we have Maple compute dr for three cases:

```
dr := b*(3/16)*(s/b)^2;
```

$$dr := \frac{3}{16}\frac{s^2}{b}$$

```
b := RE + h;
```

$$b := RE + h$$

```
h := 200e3*m;
```

$$h := 200000.\ m$$

```
RE := 6371e3*m;
```

$$RE := .6371\ 10^{7}\ m$$

(D.3.15)

```
s := 10*m;
```

$$s := 10\ m$$

```
dr/b;
```

$$.4342485107\ 10^{-12}$$

```
dr;
```

$$.2853446964\ 10^{-5}\ m$$

```
s := 1000*m;    # 1 km
```

$$s := 1000\ m$$

```
dr/b;
```

$$.4342485107\ 10^{-8}$$

```
dr;
```

$$.02853446964\ m$$

```
s := 50000*m;   # 50 km
dr/b;
```

$$s := 50000\ m$$

$$.00001085621277$$

```
dr;
```

$$71.33617412\ m$$

For a s = 10 m tether length, the max offset is 2.8 microns which we feel pretty comfortable ignoring.
For a s = 1 km tether length, the max offset is 2.8 cm, still pretty small.
For a s = 50 km tether length, the max offset is about 70 m. (D.3.16)

The quantities $\left|\frac{r_{cog} - r_{cms}}{r_{cms}}\right|_{max}$ for each case are shown as dr/b : 10^{-12}, 10^{-8} and 10^{-5}.

This justifies our approximation that it is essentially the center of mass point which orbits the Earth, and the variation between $\mathbf{r}_{cog}$ and $\mathbf{r}_{cms}$ is generally relatively small. However, in the last case if one really deployed a 50 km tether from a small spacecraft, the center of gravity point would lie 70 meters out onto the tether. (The YES2 satellite in 2007 deployed a 30 km tether, see Chen et al.).

D.4 Dumbbell Satellite Center of Gravity for equal masses and no approximation

(a) Equal Masses and general θ

Recall the general no-approximation results (D.2.18) and (D.2.21),

$$\tan(\delta) = -(m_1 r_1 \sin\theta)\ \frac{(1/r'_1{}^3) - (1/r'_2{}^3)}{b\left[(m_1/r'_1{}^3) + (m_2/r'_2{}^3)\right] + m_1 r_1 \cos\theta\left[(1/r'_1{}^3) - (1/r'_2{}^3)\right]} \qquad (D.2.18)$$

$$r_{cog} = \frac{1}{\left[(\mu_1/r'_1{}^2)^2 + (\mu_2/r'_2{}^2)^2 + 2(\mu_1/r'_1{}^3)(\mu_2/r'_2{}^3)(b^2 + b(r_1 - r_2)\cos\theta - r_1 r_2)\right]^{1/4}} \qquad (D.2.21)$$

where

$$r'_1{}^2 = r_1{}^2 + b^2 + 2r_1 b\cos\theta$$
$$r'_2{}^2 = r_2{}^2 + b^2 - 2r_2 b\cos\theta \ . \qquad \textit{(D.2.4)} \qquad \text{(D.4.1)}$$

Setting $m_1 = m_2$ (which implies $r_1 = r_2 = s/2$ and also $\mu_1 = \mu_2 = 1/2$) gives these simpler forms,

$$\tan(\delta) = -(r_1\sin\theta)\ \frac{r'_2{}^3 - r'_1{}^3}{b\,(r'_2{}^3 + r'_1{}^3) + r_1\cos\theta\,(r'_2{}^3 - r'_1{}^3)}$$

$$r_{cog} = \frac{\sqrt{2}\,r'_1 r'_2}{[\ r'_2{}^4 + r'_1{}^4 + 2r'_1 r'_2(b^2 - r_1{}^2)\ ]^{1/4}}\ . \qquad \text{(D.4.2)}$$

(b) Equal Masses and $\theta = 0$ (vertically aligned)

If we further specify that $\theta = 0$ we get $\delta = 0$ and moreover,

$$r'_1 = (b+r_1) \qquad r'_2 = (b - r_1) \qquad \text{(D.4.3)}$$

$$b^2 = r'_1{}^2 + r_1{}^2 - 2r'_1 r_1 \quad \Rightarrow \quad b^2 - r_1{}^2 = r'_1(r'_1 - 2r_1) \qquad \text{(D.4.4)}$$

$$r_{cog} = \frac{\sqrt{2}\,r'_1 r'_2}{[\ r'_2{}^4 + r'_1{}^4 + 2r'_1 r'_2(b^2 - r_1{}^2)\ ]^{1/4}} = \frac{\sqrt{2}\,(b+r_1)(b-r_1)}{[\ (b-r_1)^4 + (b+r_1)^4 + 2(b+r_1)(b-r_1)(b^2 - r_1{}^2)\ ]^{1/4}}$$

$$= \frac{\sqrt{2}\,(b+r_1)(b-r_1)}{[\ 4(b^2 + r_1{}^2)^2\ ]^{1/4}} = \frac{\sqrt{2}\,(b+r_1)(b-r_1)}{\sqrt{2}\,\sqrt{b^2 + r_1{}^2}} = \frac{(b^2 - r_1{}^2)}{\sqrt{b^2 + r_1{}^2}} = \frac{b^2}{b}\,\frac{(1 - (r_1/b)^2)}{\sqrt{1 + (r_1/b)^2}}$$

$$= b\,\frac{1 - (r_1/b)^2}{\sqrt{1 + (r_1/b)^2}}\ . \qquad \text{(D.4.5)}$$

The exact results are then (equal masses and $\theta = 0$):

$$\tan(\delta) = 0 \quad \Rightarrow \quad \delta = 0 \qquad \text{(D.4.6)}$$

$$r_{cog} = b\,\frac{1 - (r_1/b)^2}{\sqrt{1 + (r_1/b)^2}}\ . \qquad // \ m_1 = m_2 \text{ and } \theta = 0 \qquad \text{(D.4.7)}$$

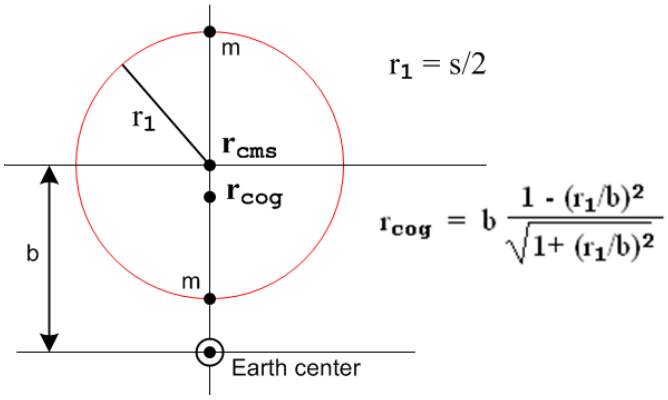
We then find that

$$r_{cog} - r_{cms} = b\left[\frac{1 - (r_1/b)^2}{\sqrt{1 + (r_1/b)^2}} - 1\right]$$

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{1 - (r_1/b)^2}{\sqrt{1 + (r_1/b)^2}} - 1\ . \qquad \text{(D.4.8)}$$

The first term is less than 1, so the right side is negative and thus $r_{cog} < r_{cms}$. So in this vertically aligned case the center of gravity is *closer* to the Earth than the center of mass.

This figure summarizes our result (with $r_{cms} - r_{cog}$ exaggerated)



(D.4.9)

Looking back at the force equation (D.2.14), we can confirm result (D.4.7) fairly quickly :

$$\mathbf{F} = -(GM_E m_1/r'^2_1)\,\hat{\mathbf{r}}'_1 - (GM_E m_2/r'^2_2)\,\hat{\mathbf{r}}'_2 = -[(GM_E m_1/r'^2_1) + (GM_E m_2/r'^2_2)]\,\hat{\mathbf{z}} \qquad // \theta = 0$$

$$F = GM_E m_1[(1/r'^2_1) + (1/r'^2_2)] = GM_E m_1[r'^2_2 + r'^2_1] / [r'_1 r'_2]^2 \qquad // m_1 = m_2$$

$$= GM_E m_1[(b-r_1)^2 + (b+r_1)^2] / [(b+r_1)(b-r_1)]^2 =$$

$$= GM_E m_1[2b^2 + 2r_1^2] / (b^2-r_1^2)^2 = 2\,GM_E m_1\,(b^2 + r_1^2) / (b^2-r_1^2)^2$$

$$= 2\,GM_E m_1\,b^{-2}\,(1 + (r_1/b)^2) / (1-(r_1/b)^2)^2$$

so from (D.2.16),

$$r_{cog} = \sqrt{GM_E(2m_1)/F} = b\,(1-(r_1/b)^2) / \sqrt{1 + (r_1/b)^2}\,, \qquad \text{(D.4.10)}$$

in agreement with (D.4.7).

If we now <u>further assume</u> $(r_1/b) << 1$ then (D.4.7) becomes

$$r_{cog} \approx b\,(1 - (r_1/b)^2)\,(1 - (1/2)\,(r_1/b)^2) = b\,[1 - (3/2)(r_1/b)^2]\,.$$

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{r_{cog} - b}{b} = -(3/2)(r_1/b)^2 \qquad \text{(D.4.11)}$$

in agreement with (D.3.11).

(c) Equal Masses and $\theta = \pi/2$ (horizontally aligned)

We start again with (D.4.2) for equal masses,

$$\tan(\delta) = -(r_1 \sin\theta)\ \frac{r'^3_2 - r'^3_1}{b\,(r'^3_2 + r'^3_1) + r_1 \cos\theta\,(r'^3_2 - r'^3_1)}$$

$$r_{cog} = \frac{\sqrt{2}\, r'_1 r'_2}{[\, r'^4_2 + r'^4_1 + 2r'_1 r'_2 (b^2 - r_1^2)\,]^{1/4}} \qquad (D.4.2)$$

$$r'^2_1 = r_1^2 + b^2 + 2r_1 b\cos\theta \qquad r'^2_2 = r_2^2 + b^2 - 2r_2 b\cos\theta\ . \qquad (D.4.1)$$

When $\theta = \pi/2$ we not only have $r_1 = r_2$ but also $r'_1 = r'_2$ along with $\cos\theta = 0$. This at once implies that $\tan(\delta) = 0$, and for r_{cog} we find

$$r'^2_1 = r_1^2 + b^2 \quad \Rightarrow \quad -r_1^2 = b^2 - r'^2_1 \quad \Rightarrow \quad [b^2 - r_1^2] = 2b^2 - r'^2_1$$

$$r_{cog} = \frac{\sqrt{2}\, r'^2_1}{[\, 2r'^4_1 + 2r'^2_1 (b^2 - r_1^2)\,]^{1/4}} = \frac{\sqrt{2}\, r'^2_1}{[\, 2r'^4_1 + 2r'^2_1 (2b^2 - r'^2_1)\,]^{1/4}} = \frac{\sqrt{2}\, r'^2_1}{\{\, 4r'^2_1 b^2\,\}^{1/4}}$$

$$= \left[\frac{r'^6_1}{b^2}\right]^{1/4} = \left[\frac{r'^3_1}{b}\right]^{1/2} = \sqrt{\frac{r'^3_1}{b}} \qquad (D.4.12)$$

$$= \left[\frac{r'^6_1}{b^2}\right]^{1/4} = \left[\frac{(b^2 + r_1^2)^3}{b^2}\right]^{1/4} = b\ [1 + (r_1/b)^2\,]^{3/4}\ . \qquad (D.4.13)$$

The exact results are then (equal masses and $\theta = \pi/2$)

$$\tan(\delta) = 0$$

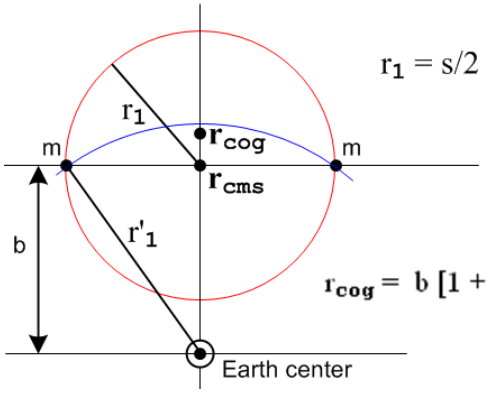
$$r_{cog} = \sqrt{\frac{r'^3_1}{b}} = b\,[1 + (r_1/b)^2\,]^{3/4}$$

$$r_{cog} - r_{cms} = b\,\{\,[1 + (r_1/b)^2\,]^{3/4} - 1\,\}$$

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = [1 + (r_1/b)^2\,]^{3/4} - 1 \qquad (D.4.14)$$

Since the first term on the right is greater than 1, we find $r_{cog} > r_{cms}$ so for the horizontally-aligned equal-mass satellite the center of gravity is *farther* from the center of the Earth than the center of mass, in agreement with our earlier conclusion based on approximation.

This figure summarizes our result (with $r_{cog} - r_{cms}$ exaggerated),



$$r_{cog} = b\,[1 + (r_1/b)^2\,]^{3/4} = \sqrt{\frac{r'^3_1}{b}} \qquad \text{(D.4.15)}$$

We have added in blue a portion of a circle centered at Earth center which passes through the masses. The point r_{cog} lies below this circle, as we now verify,

$$r_{cog} < r'_1 \qquad ?$$

$$\sqrt{\frac{r'^3_1}{b}} < r'_1 \qquad ?$$

$$r'^3_1 < r'^2_1\, b \qquad ?$$

$$r'_1 < b \qquad ? \quad \text{yes} \quad . \qquad \text{(D.4.16)}$$

Looking back at the force equations (D.2.15), we can confirm result (D.4.12) very quickly,

$$F_z = -2GM_E m_1 (b/r'^3_1) \qquad F_y = 0$$

$$r_{cog} = \sqrt{GM_E(2m_1)/F} = \sqrt{\frac{r'^3_1}{b}} \quad . \qquad \text{(D.4.17)}$$

If we now <u>further assume</u> $(r_1/b) << 1$ the result (D.4.13) becomes

$$r_{cog} = b\,[1 + (r_1/b)^2\,]^{3/4} \approx b\,[1 + (3/4)\ (r_1/b)^2\,]$$

so

$$\frac{r_{cog} - r_{cms}}{r_{cms}} = \frac{r_{cog} - b}{b} = +\,(3/4)(r_1/b)^2 \qquad \text{(D.4.18)}$$

in agreement with (D.3.12).

D.5 Center of Gravity for a single-sphere satellite

For the vertically aligned dumbbell satellite, we found that $r_{cog} < r_{cms}$. People argue that this is so because gravity acts more strongly on the mass closer to the Earth. This argument does not help much, however, for the horizontally aligned satellite where instead one has $r_{cog} > r_{cms}$ and neither mass is closer than the other to the Earth.

One wonders if one can assemble a rigid satellite from a finite number of masses such that these two opposite effects cancel out, resulting in $\mathbf{r}_{cog} = \mathbf{r}_{cms}$, at least for some orientation of the masses. Could such a solution be found that works for any orientation of the rigid satellite? We leave these questions to the reader and return instead to the above "argument".

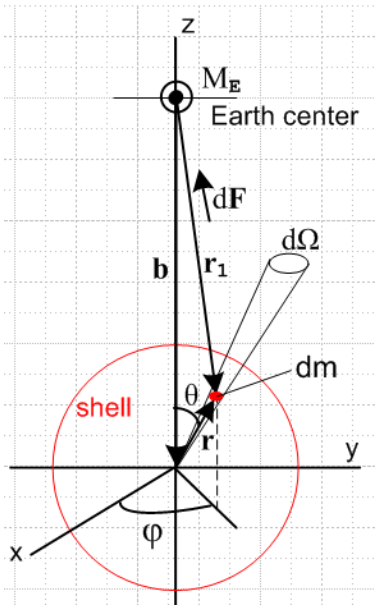
The argument applied to a sphere gives a wrong answer. One would argue for a spherical satellite that the near half of the sphere is closer to the Earth (where gravity is stronger) than the far half, so the center of gravity should be offset toward the Earth from the center of mass. As the reader no doubt knows, a uniform sphere or spherical shell is in fact a "rigid assembly of masses" for which $\mathbf{r}_{cog} = \mathbf{r}_{cms}$. So in such an object, the two effects found for the vertically and horizontally aligned satellites do in fact exactly cancel out.

Here we demonstrate that $\mathbf{r}_{cog} = \mathbf{r}_{cms}$ for a uniform thin shell by two basic methods. Once this is shown, the result then applies for any symmetric assembly of shells such as a sphere or a thick spherical shell.

Method A. In electrostatics one of Maxwell's equations says $\text{div}\mathbf{E} = 4\pi\rho$ (cgs units). One applies the integral form of this law $\int \mathbf{E} \bullet d\mathbf{A} = 4\pi \int \rho \, dV = 4\pi Q$ to show that the electric field outside a uniform spherical shell of charge is independent of the radius of the shell and thus is the same as if the charge were all concentrated at the center of the shell. The argument is that, due to rotational invariance (or "symmetry"), the direction of the electric field can only be radial, so then $\int \mathbf{E} \bullet d\mathbf{A} = E * 4\pi R^2$ and one then finds that $E = Q/R^2$ which is indeed the electric field of a point charge Q at distance R.

The same argument can be applied to a uniform shell of mass so that the gravitational field must be in the radial direction and is independent of the shell radius R, so the shell acts as a point mass at its center. If this shell is a satellite then the Earth cannot determine from gravity alone the radius of that satellite, so one can replace the satellite by a point mass at its center. The Earth never knows, and nothing changes in the orbit. But for the point mass certainly $r_{cog} = r_{cms}$, so this fact applies as well to any uniform spherical shell or sphere satellite.

Method B (the ever-popular brute force method). This method is more direct, and is found in high-school physics texts, though the integral we obtain below is not. The drawing below is upside-down relative to our earlier drawings, to be more compatible with usual spherical coordinate notation. So the center of the Earth is now at the top and the center of *mass* of the satellite is located a distance b from Earth center, as earlier. This particular satellite is a spherical shell of radius r and very thin thickness dr. The mass of the shell is then $M_{shell} = [(4\pi r^2)dr]\rho$ where ρ is the shell's uniform mass density.



(D.5.1)

Quantity dm is a tiny chunk of mass on the spherical shell of radius r with

$$dm = \rho(dA)dr = \rho(r^2 d\Omega)dr = \rho\,(r^2 \sin\theta d\theta d\varphi)dr\,. \tag{D.5.2}$$

The force experienced by this mass dm due to mass M_E is given by

$$d\mathbf{F} = -(GM_E dm/r_1^2)\hat{\mathbf{r}}_1 = -(GM_E/r_1^3)\, dm\, \mathbf{r}_1\,. \tag{D.5.3}$$

As usual in spherical coordinates, and as shown in (E.2.6),

$$\mathbf{r} = r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}} \tag{D.5.4}$$

so then

$$\mathbf{r}_1 = \mathbf{b} + \mathbf{r} = -b\hat{\mathbf{z}} + r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}}$$

$$= r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + (-b + r\cos\theta)\hat{\mathbf{z}}\,. \tag{D.5.5}$$

Therefore,

$$d\mathbf{F} = -(GM_E/r_1^3)\, dm\, \mathbf{r}_1$$

$$= -(GM_E/r_1^3)\, dm\, [r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + (-b + r\cos\theta)\hat{\mathbf{z}}\,]\,. \tag{D.5.6}$$

Then,

$$dF_x = d\mathbf{F} \bullet \hat{\mathbf{x}} = -(GM_E/r_1^3)\,\rho\,(r^2\sin\theta d\theta d\varphi)dr\,(r\sin\theta\cos\varphi)$$
$$dF_y = d\mathbf{F} \bullet \hat{\mathbf{y}} = -(GM_E/r_1^3)\,\rho\,(r^2\sin\theta d\theta d\varphi)dr\,(r\sin\theta\sin\varphi)$$
$$dF_z = d\mathbf{F} \bullet \hat{\mathbf{z}} = -(GM_E/r_1^3)\,\rho\,(r^2\sin\theta d\theta d\varphi)dr\,(-b+r\cos\theta)\ . \qquad \text{(D.5.7)}$$

Now integrate dφ from 0 to 2π to get the force on a *ring* of mass of angular width dθ on the shell at fixed θ. But sinφ has zero integral over this range and so does cosφ, so the first two terms integrate to nothing, while the integral of dφ in the last line gives 2π. We are left with this resulting force, all in the z direction,

$$dF_{ring} = (GM_E/r_1^3)\ \rho\,(r^2\sin\theta d\theta\, 2\pi)\,dr\ (b-r\cos\theta)\ . \qquad \text{(D.5.8)}$$

Now integrate over all rings of the shell to get a total force F experienced by the spherical shell satellite,

$$F = \int_0^\pi d\theta\ (GM_E/r_1^3)\ \rho\,(r^2\sin\theta d\theta\, 2\pi)\,dr\ (b-r\cos\theta)$$

$$= 2\pi\rho GM_E\, r^2 dr \int_0^\pi d\theta\, \frac{\sin\theta\,(b-r\cos\theta)}{(r^2+b^2-2rb\cos\theta)^{3/2}}\ . \qquad \text{(D.5.9)}$$

This is a famous discontinuous integral that we evaluate below, but for now we use Maple. Assuming that r < b which means the Earth center lies outside the shell, Maple says,

```
restart; assume(b>0,r>0,r<b);
Int(sin(theta)*(b-r*cos(theta))/ (b^2+r^2-2*b*r*cos(theta))^(3/2),theta=0..Pi);
```

$$\int_0^{\pi} \frac{\sin(\theta)\,(b - r\cos(\theta))}{\left(b^2 + r^2 - 2\,b\,r\cos(\theta)\right)^{\left(\frac{3}{2}\right)}}\,d\theta$$

```
value(%);
```

$$2\,\frac{1}{b^2}$$

(D.5.10)

where $2/b^2$ is the value of the integral. Thus we have shown that

$$F = 2\pi\rho GM_E\, r^2 dr \int_0^\pi d\theta\, \frac{\sin\theta\,(b-r\cos\theta)}{(r^2+b^2-2rb\cos\theta)^{3/2}} = 2\pi\rho GM_E\, r^2 dr\,(2/b^2)$$

$$= GM_E\ (4\pi r^2 dr)\rho\,/b^2 = GM_E M_{shell}\,/\,b^2\ . \qquad \text{(D.5.11)}$$

We now use the center of gravity definition from (D.1.6) to find,

$$r_{cog} = \sqrt{GM_E M_{shell}/F} = b \qquad \text{(D.5.12)}$$

Since $r_{cms} = b$ as well, we conclude that for a spherical shell satellite one has $r_{cog} = r_{cms}$.

If the mass center M_E were inside the shell so b < r, Maple gives a different answer ,

```
restart; assume(b>0,r>0,r>b);
Int(sin(theta)*(b-r*cos(theta))/ (b^2+r^2-2*b*r*cos(theta))^(3/2),theta=0..Pi);
```

$$\int_0^{\pi} \frac{\sin(\theta)\,(b - r\cos(\theta))}{\left(b^2 + r^2 - 2\,b\,r\cos(\theta)\right)^{\left(\frac{3}{2}\right)}}\,d\theta$$

```
value(%);
```

$$0$$

(D.5.13)

showing that the force experienced by the uniform shell due to a mass M_E inside the shell is zero.

It seems unsportsmanlike not to actual *do* the integral to see why it is discontinuous at r = b. Here is a brief tour:

1. Change variables from θ to r_1 and look at the endpoints in the new variable,

$$r_1^2(\theta=0) = r^2 + b^2 - 2rb(1) = (b-r)^2 \quad \Rightarrow \quad r_1(\theta=0) = |b-r| \equiv \alpha$$
$$r_1^2(\theta=\pi) = r^2 + b^2 - 2rb(-1) = (b+r)^2 \quad \Rightarrow \quad r_1(\theta=\pi) = b+r \equiv \beta$$

It is the absolute value that causes the discontinuous behavior at b = r, as we shall see.

2. Since $r_1^2 = r^2 + b^2 - 2rb\cos\theta$ one has $\sin\theta d\theta = r_1 dr_1/(rb)$. (D.5.14)

3. The factor $(b - r\cos\theta) = (b^2 - r^2 + r_1^2)/(2b)$, while the denominator of the integral is just r_1^3.

4. The integral reduces to two elementary power integrals to give

$$I(r) = \int_0^{\pi} d\theta \frac{\sin\theta\,(b - r\cos\theta)}{(r^2+b^2-2rb\cos\theta)^{3/2}} = [\,(b^2-r^2)\int_{\alpha}^{\beta} dr_1/r_1^2 + \int_{\alpha}^{\beta} dr_1\,1\,]\,/\,(2rb^2)$$

$$= [\;(b-r)\,\{\frac{(b+r) - |b-r|}{|b-r|}\} + (b+r) - |b-r|\;]\;/\;(2rb^2)\;. \qquad \text{(D.5.15)}$$

For b > r one has |b-r| = b-r and the value comes out

$$I(r) = [\;2r + 2r\;]\,/\,(2rb^2) = 2/b^2\;. \qquad \text{(D.5.16)}$$

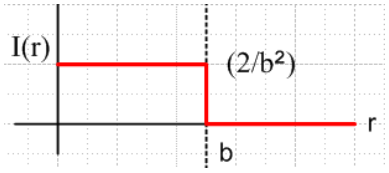
For b < r one has |b-r| = r-b and the value is instead

$$I(r) = [\ -2b \quad + 2b\ ] / (2rb^2) = 0\ . \tag{D.5.17}$$

The full result is then

$$I(r) = (2/b^2)H(b-r) \qquad \text{// Heaviside function} \tag{D.5.18}$$

which has this appearance,



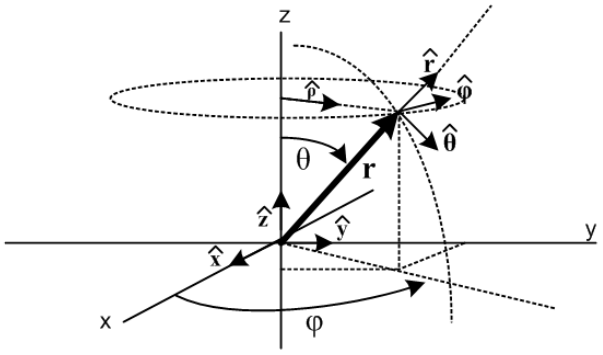
(D.5.19)

On can argue that $I(b) = (1/b^2)$ which is the average of the values at the discontinuity.

Appendix E: Spherical Coordinate Unit Vectors and Particle Kinematics

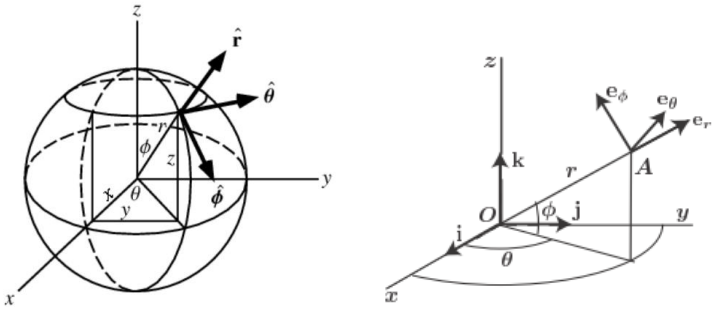
E.1 Angle Conventions

Here is our "physics" convention for spherical coordinate angles θ and φ :



(E.1.1)

The reader is advised that the results below are often quoted in the literature with θ↔φ which is the "math" convention (eg, Wolfram, left below). Sometimes after doing θ↔φ some sources replace polar angle φ with latitude π/2 - φ (right below) :



Two spherical angle conventions we do not use.

In the right picture, φ and θ might be longitude and latitude, or right ascension and declination.

If you are in charge of getting a spacecraft to Vulcan, please pay attention to these conventions.

E.2 Matrix Approach

The usual matrices for actively rotating a vector about the x, y, or z axis are these:

$$R_{\mathbf{x}}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad R_{\mathbf{y}}(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_{\mathbf{z}}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(E.2.1)

The Cartesian unit vectors $\hat{\mathbf{x}}, \hat{\mathbf{y}}. \hat{\mathbf{z}}$ can be rotated into the spherical unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ as follows

$$\hat{\boldsymbol{\theta}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{x}} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi \\ \cos\theta\sin\varphi \\ -\sin\theta \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{x}} \\ \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}}\bullet\hat{\mathbf{y}} \end{pmatrix}$$

$$\hat{\boldsymbol{\varphi}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{y}} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\hat{\mathbf{r}} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta)\, \hat{\mathbf{z}} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} . \qquad \text{(E.2.2)}$$

One can interpret the elements of the vectors on the right as dot products as we show on the first line. Notice that these three vectors on the right are just the columns of the matrix shown, so one can say

$$(\,\hat{\boldsymbol{\theta}}\ \ \hat{\boldsymbol{\varphi}}\ \ \hat{\mathbf{r}}\,) = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta) \equiv \boldsymbol{\mathcal{R}} . \qquad \text{(E.2.3)}$$

It is then obvious that

$$\begin{pmatrix} \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{x}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{x}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{y}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{y}}\bullet\hat{\mathbf{r}} \\ \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\theta}} & \hat{\mathbf{z}}\bullet\hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}}\bullet\hat{\mathbf{r}} \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_{\mathbf{z}}(\varphi)\, R_{\mathbf{y}}(\theta) = \boldsymbol{\mathcal{R}} \qquad \text{(E.2.4)}$$

from which one can read off any desired dot product. As verification, consider an example,

Example: $\hat{\mathbf{y}} \bullet \hat{\boldsymbol{\theta}} = \hat{\mathbf{y}} \bullet (\boldsymbol{\mathcal{R}}\hat{\mathbf{x}}) = \Sigma_{\mathtt{i}}\ (\hat{\mathbf{y}})_{\mathtt{i}}(\boldsymbol{\mathcal{R}}\hat{\mathbf{x}})_{\mathtt{i}} = \Sigma_{\mathtt{ij}}(\hat{\mathbf{y}})_{\mathtt{i}}\boldsymbol{\mathcal{R}}_{\mathtt{ij}}(\hat{\mathbf{x}})_{\mathtt{j}} = \Sigma_{\mathtt{ij}}\delta_{2\mathtt{i}}\boldsymbol{\mathcal{R}}_{\mathtt{ij}}\delta_{1\mathtt{j}} = \boldsymbol{\mathcal{R}}_{21}$.

Looking at the last equation of (E.2.2) multiplied by r, since $\mathbf{r} = r\,\hat{\mathbf{r}}$ one sees that

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \qquad \text{(E.2.5)}$$

which is the "inverse transformation" associated with spherical coordinates. See Section E.4 below.

The three equations (E.2.2) can be trivially written out as follows (changing to a standard r,θ,φ order)

$$\hat{\mathbf{r}} = \sin\theta\cos\varphi\ \hat{\mathbf{x}} + \sin\theta\sin\varphi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}$$
$$\hat{\boldsymbol{\theta}} = \cos\theta\cos\varphi\ \hat{\mathbf{x}} + \cos\theta\sin\varphi\ \hat{\mathbf{y}} - \sin\theta\ \hat{\mathbf{z}}$$

$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \tag{E.2.6}$$

which can be inverted to give

$$\begin{aligned}
\hat{\mathbf{x}} &= \sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}} \\
\hat{\mathbf{y}} &= \sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}} \\
\hat{\mathbf{z}} &= \cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\ .
\end{aligned} \tag{E.2.7}$$

For those not wanting to invert (transpose) the 3x3 matrix $\boldsymbol{\mathcal{R}}$, (E.2.7) can be quickly verified by looking at the unit vector dot products in (E.2.4). Eq. (E.2.6) can be similarly verified .

The spherical unit vectors are orthogonal due to their construction in (E.2.3).

Example: $\hat{\mathbf{r}} \bullet \hat{\boldsymbol{\theta}} = (\boldsymbol{\mathcal{R}}\,\hat{\mathbf{z}}) \bullet (\boldsymbol{\mathcal{R}}\,\hat{\mathbf{x}}) = \hat{\mathbf{z}} \bullet (\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\hat{\mathbf{x}}) = \hat{\mathbf{z}} \bullet \hat{\mathbf{x}} = 0$

Example: $\hat{\mathbf{r}} \bullet \hat{\mathbf{r}} = (\boldsymbol{\mathcal{R}}\,\hat{\mathbf{z}}) \bullet (\boldsymbol{\mathcal{R}}\,\hat{\mathbf{z}}) = \hat{\mathbf{z}} \bullet (\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\hat{\mathbf{z}}) = \hat{\mathbf{z}} \bullet \hat{\mathbf{z}} = 1$

Here $\boldsymbol{\mathcal{R}} \equiv R_{\mathbf{z}}(\varphi)\,R_{\mathbf{y}}(\theta)$ has the property $\boldsymbol{\mathcal{R}}^{\mathrm{T}}\boldsymbol{\mathcal{R}} = 1$ since it is a rotation. So

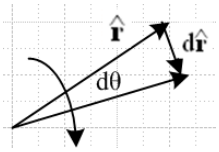
$$\hat{\mathbf{r}} \bullet \hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \bullet \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\theta}} \bullet \hat{\boldsymbol{\varphi}} = 0\ . \tag{E.2.8}$$

The spatial derivatives of the spherical unit vectors are easy to compute ($\partial_{\boldsymbol{\theta}}\hat{\mathbf{r}}$ means $\partial\hat{\mathbf{r}}/\partial\theta$),

$$\begin{array}{lll}
\partial_{\mathbf{r}}\hat{\mathbf{r}} = 0 & \partial_{\boldsymbol{\theta}}\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} & \partial_{\boldsymbol{\varphi}}\hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\varphi}} \\
\partial_{\mathbf{r}}\hat{\boldsymbol{\theta}} = 0 & \partial_{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} = -\,\hat{\mathbf{r}} & \partial_{\boldsymbol{\varphi}}\hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\varphi}} \\
\partial_{\mathbf{r}}\hat{\boldsymbol{\varphi}} = 0 & \partial_{\boldsymbol{\theta}}\hat{\boldsymbol{\varphi}} = 0 & \partial_{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}} = -\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}\ .
\end{array} \tag{E.2.9}$$

Example: $\partial_{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} = \partial_{\boldsymbol{\theta}}\begin{pmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{pmatrix} = \begin{pmatrix}-\sin\theta\cos\varphi\\ -\sin\theta\sin\varphi\\ -\cos\theta\end{pmatrix} = -\,\hat{\mathbf{r}}.$

In Fig (E.1.1) one can see for example that $\partial_{\boldsymbol{\theta}}\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$. Often one draws a little triangle to verify an equation like this:



(E.2.10)

$$d\hat{\mathbf{r}} = dr\,\hat{\boldsymbol{\theta}} \approx 1{*}d\theta\,\hat{\boldsymbol{\theta}} = d\theta\,\hat{\boldsymbol{\theta}} \quad \Rightarrow \quad d\hat{\mathbf{r}} = d\theta\,\hat{\boldsymbol{\theta}} \quad \Rightarrow \quad \hat{\boldsymbol{\theta}} = d\hat{\mathbf{r}}/d\theta = \partial_{\boldsymbol{\theta}}\hat{\mathbf{r}}\ .$$

Time derivatives of the unit vectors are then obtained by the chain rule ($\partial_{\mathbf{t}}\hat{\mathbf{r}}$ means $d\hat{\mathbf{r}}/dt$),

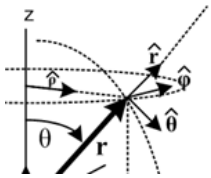
$$\partial_t \hat{\mathbf{r}} = \dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$$
$$\partial_t \hat{\boldsymbol{\theta}} = -\dot{\theta}\,\hat{\mathbf{r}} + \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\varphi}}$$
$$\partial_t \hat{\boldsymbol{\varphi}} = -\dot{\varphi}\sin\theta\,\hat{\mathbf{r}} - \dot{\varphi}\cos\theta\,\hat{\boldsymbol{\theta}}\,. \qquad \text{(E.2.11)}$$

Example: $\partial_t \hat{\mathbf{r}} = (\partial_r \hat{\mathbf{r}})\,\dot{r} + (\partial_\theta \hat{\mathbf{r}})\,\dot{\theta} + (\partial_\varphi \hat{\mathbf{r}})\,\dot{\varphi} = 0 + \hat{\boldsymbol{\theta}}\dot{\theta} + \sin\theta\,\hat{\boldsymbol{\varphi}}\dot{\varphi}$.

Note: If we regard r,θ,φ as coordinates of Frame S, then $\partial_t \hat{\mathbf{r}} = (d\hat{\mathbf{r}}/dt)_S = \partial_S \hat{\mathbf{r}}$ as in Section 1.7 and 1.8.

The cross products of the spherical unit vectors follow the right hand rule, so looking at Fig (E.1.1):

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}} \qquad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{r}} \qquad \hat{\boldsymbol{\varphi}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \qquad \text{(E.2.12)}$$



The first cross product has ordering r,θ,φ and the last two are cyclic permutations. Here are some other cross products:

$$\hat{\mathbf{x}} \times \hat{\mathbf{r}} = [\sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\mathbf{r}} = -\cos\theta\cos\varphi\,\hat{\boldsymbol{\varphi}} - \sin\varphi\,\hat{\boldsymbol{\theta}}$$
$$\hat{\mathbf{x}} \times \hat{\boldsymbol{\theta}} = [\sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\boldsymbol{\theta}} = \sin\theta\cos\varphi\,\hat{\boldsymbol{\varphi}} + \sin\varphi\,\hat{\mathbf{r}}$$
$$\hat{\mathbf{x}} \times \hat{\boldsymbol{\varphi}} = [\sin\theta\cos\varphi\,\hat{\mathbf{r}} + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\boldsymbol{\varphi}} = -\sin\theta\cos\varphi\,\hat{\boldsymbol{\theta}} + \cos\theta\cos\varphi\,\hat{\mathbf{r}} \qquad \text{(E.2.13)}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{r}} = [\sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\mathbf{r}} = -\cos\theta\sin\varphi\,\hat{\boldsymbol{\varphi}} + \cos\varphi\,\hat{\boldsymbol{\theta}}$$
$$\hat{\mathbf{y}} \times \hat{\boldsymbol{\theta}} = [\sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\boldsymbol{\theta}} = \sin\theta\sin\varphi\,\hat{\boldsymbol{\varphi}} - \cos\varphi\,\hat{\mathbf{r}}$$
$$\hat{\mathbf{y}} \times \hat{\boldsymbol{\varphi}} = [\sin\theta\sin\varphi\,\hat{\mathbf{r}} + \cos\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \cos\varphi\,\hat{\boldsymbol{\varphi}}] \times \hat{\boldsymbol{\varphi}} = -\sin\theta\sin\varphi\,\hat{\boldsymbol{\theta}} + \sin\theta\sin\varphi\,\hat{\mathbf{r}} \qquad \text{(E.2.14)}$$

$$\hat{\mathbf{z}} \times \hat{\mathbf{r}} = [\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] \times \hat{\mathbf{r}} = \sin\theta\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} \times \hat{\boldsymbol{\theta}} = [\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] \times \hat{\boldsymbol{\theta}} = \cos\theta\,\hat{\boldsymbol{\varphi}}$$
$$\hat{\mathbf{z}} \times \hat{\boldsymbol{\varphi}} = [\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] \times \hat{\boldsymbol{\varphi}} = -\cos\theta\,\hat{\boldsymbol{\theta}} - \sin\theta\,\hat{\mathbf{r}} \qquad \text{(E.2.15)}$$

E.3 The motion of a particle in spherical coordinates

The equations below are easily derived from (E.2.11):

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad \text{// position} \qquad \text{(E.3.1)}$$

$$\mathbf{v} = v_r\,\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\boldsymbol{\varphi}} \qquad \text{// velocity}$$

$$v_r = \dot{r}$$

$$v_\theta = r\,\dot{\theta}$$

$$v_\varphi = r\,\dot{\varphi}\sin\theta \qquad \text{(E.3.2)}$$

$$\mathbf{a} = a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}} \qquad \text{// acceleration}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta$$

$$a_\theta = 2\,\dot{r}\,\dot{\theta} + r\ddot{\theta} - r\,\dot{\varphi}^2\sin\theta\cos\theta$$

$$a_\varphi = 2\,\dot{r}\,\dot{\varphi}\sin\theta + 2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta\ . \qquad \text{(E.3.3)}$$

Example: $\mathbf{v} = \dot{\mathbf{r}} = \partial_t(r\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r(\partial_t\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r[\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}\,] = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + r\dot{\varphi}\sin\theta\hat{\boldsymbol{\varphi}}$.

Notice that $\ddot{r} = \dot{v}_r \neq a_r$ and similarly for other components.

If for some reason particle motion is restricted to a spherical surface of radius r (such as in our dumbbell satellite), we can set $\dot{r} = \ddot{r} = 0$ in the above equations to get

$$\mathbf{r} = r\,\hat{\mathbf{r}} \qquad \text{// position} \qquad \text{(E.3.4)}$$

$$\mathbf{v} = v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}} \qquad \text{// velocity is tangent to the sphere : } \mathbf{r}\bullet\mathbf{v} = 0$$

$$v_\theta = r\,\dot{\theta}$$

$$v_\varphi = r\,\dot{\varphi}\sin\theta \qquad \text{(E.3.5)}$$

$$\mathbf{a} = a_r\,\hat{\mathbf{r}} + a_\theta\hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\boldsymbol{\varphi}} \qquad \text{// acceleration}$$

$$a_r = -r\dot{\theta}^2 - r\,\dot{\varphi}^2\sin^2\theta$$

$$a_\theta = r\ddot{\theta} - r\,\dot{\varphi}^2\sin\theta\cos\theta$$

$$a_\varphi = 2\,r\,\dot{\theta}\,\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta\ . \qquad \text{(E.3.6)}$$

From these equations many other useful results can be obtained, for example,

$$\hat{\mathbf{r}} \times \mathbf{v} = v_\theta\,\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = v_\theta\,\hat{\boldsymbol{\varphi}} - v_\varphi\,\hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{r}} \times \mathbf{a} = a_\theta\,\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} + a_\varphi\,\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = a_\theta\,\hat{\boldsymbol{\varphi}} - a_\varphi\,\hat{\boldsymbol{\theta}} \qquad \text{(E.3.7)}$$

E.4 Curvilinear coordinates approach

Spherical coordinates are defined by the "inverse transformation" on the left below:

inverse transformation	transformation			
$x = r\sin\theta\cos\varphi$	$r = +\sqrt{x^2+y^2+z^2}$	$0 \le r < \infty$		
$y = r\sin\theta\sin\varphi$	$\cos\theta = z/r$	$0 \le \theta \le \pi$	$-1 \le \cos\theta \le 1$	
$z = r\cos\theta$	$\sin\theta = +\sqrt{1-(z/r)^2}$		$0 \le \sin\theta \le 1$	
	$\sin\varphi = y/(r\sin\theta)$		$-1 \le \sin\varphi \le 1$	
	$\cos\varphi = x/(r\sin\theta)$	$0 \le \varphi < 2\pi$	$-1 \le \cos\varphi \le 1$	(E.4.1)

We quote now a set of results from our *Tensor* document which won't be needed but which we include for completeness. Equation numbers $(...)_T$ refer to that document.

Spherical coordinates are just an example of curvilinear coordinates which fit into a general formalism:

$$\mathbf{x} = (x_1, x_2, x_3) = (x,y,z) \qquad \text{// x-space coordinates}$$
$$\mathbf{x}' = (x_1', x_2', x_3') = (r,\theta,\varphi) \qquad \text{// x'-space coordinates}$$

$$\mathbf{x} = F^{-1}(\mathbf{x}') \;\leftrightarrow\; \begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \;. \end{aligned} \qquad \text{// a non-linear inverse transformation} \qquad (1.6)_T$$

The linearized transformation local to a point defines certain R and S matrices (the "differentials") :

$$d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x} \qquad R_{ik}(\mathbf{x}) \equiv (\partial x'_i/\partial x_k) \qquad R = S^{-1} \qquad \text{// } dx'_i = R_{ij}\, dx_j$$
$$d\mathbf{x} = S(\mathbf{x}')\, d\mathbf{x}' \qquad S_{ik}(\mathbf{x}') \equiv (\partial x_i/\partial x'_k) \qquad S = R^{-1} \qquad \text{// } dx_i = S_{ij}\, dx'_j \;. \qquad (2.1.6)_T$$

$$S = \begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} \quad \text{// compute from above definition of } S_{ij} \qquad (3.4.4)_T$$

$$\bar{g}' = S^T S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}, \quad \det(\bar{g}') = r^4\sin^2\theta \quad \text{// metric tensor and its determinant} \qquad (5.13.14)_T$$

$$J(r,\theta,\varphi) = \det(S) = \sqrt{\det(\bar{g}')} = r^2\sin\theta \;. \qquad \text{// Jacobian} \qquad (5.13.16)_T$$

$$(ds)^2 = \Sigma_{ij}\bar{g}'_{ij}\, dx'_i\, dx_j = (dr)^2 + r^2(d\theta)^2 + r^2\sin^2(d\varphi)^2 \qquad \text{// distance} \qquad (5.13.18)_T$$

Because the metric tensor $\bar{g}'$ is diagonal, the coordinates r,θ.φ are orthogonal.

Footnote: Above we use the temporary developmental notation of *Tensor* where all indices are down and covariant objects get an overbar. Some rules for conversion to the Standard Notation are these:

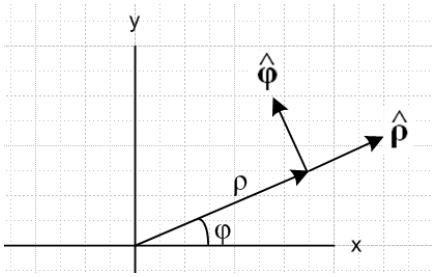
$\bar{g}_{ij} \rightarrow g_{ij}$ covariant metric tensor $\qquad \bar{v}_i \rightarrow v_i$ covariant vector

$g_{ij} \rightarrow g^{ij}$ contravariant metric tensor $\qquad v_i \rightarrow v^i$ contravariant vector $(dx_i \rightarrow dx^i)$

$dx'_i = R_{ij}\, dx_j \;\rightarrow\; dx'^i = R^i{}_j\, dx^j$

E.5 Polar Coordinates

Here we use a notation common for two of the cylindrical coordinates (ρ,φ,z),



(E.5.1)

$$x = \rho\cos\varphi$$
$$y = \rho\sin\varphi \qquad \text{(E.5.2)}$$

$$\hat{\boldsymbol{\rho}} = \cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}} \qquad \hat{\boldsymbol{\rho}} = R_z(\varphi)\,\hat{\mathbf{x}}$$
$$\hat{\boldsymbol{\varphi}} = -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \qquad \hat{\boldsymbol{\varphi}} = R_z(\varphi)\,\hat{\mathbf{y}} \qquad \text{(E.5.3)}$$

$$\hat{\mathbf{x}} = \cos\varphi\,\hat{\boldsymbol{\rho}} - \sin\varphi\,\hat{\boldsymbol{\varphi}} \qquad \hat{\mathbf{x}} = R_z(-\varphi)\,\hat{\boldsymbol{\rho}}$$
$$\hat{\mathbf{y}} = \sin\varphi\,\hat{\boldsymbol{\rho}} + \cos\varphi\,\hat{\boldsymbol{\varphi}} \qquad \hat{\mathbf{y}} = R_z(-\varphi)\,\hat{\boldsymbol{\theta}} \qquad \text{(E.5.4)}$$

$$\hat{\boldsymbol{\rho}} \bullet \hat{\mathbf{x}} = \cos\varphi \qquad \hat{\boldsymbol{\rho}} \bullet \hat{\mathbf{y}} = \sin\varphi$$
$$\hat{\boldsymbol{\varphi}} \bullet \hat{\mathbf{x}} = -\sin\varphi \qquad \hat{\boldsymbol{\varphi}} \bullet \hat{\mathbf{y}} = \cos\varphi \qquad \text{(E.5.5)}$$

$$\dot{\hat{\boldsymbol{\rho}}} = \dot{\varphi}\,\hat{\boldsymbol{\varphi}}$$
$$\dot{\hat{\boldsymbol{\varphi}}} = -\dot{\varphi}\,\hat{\boldsymbol{\rho}} \qquad \text{(E.5.6)}$$

Proof of (E.5.6) :

$$\dot{\hat{\boldsymbol{\rho}}} = d\hat{\boldsymbol{\rho}}/dt = d/dt(\cos\varphi\,\hat{\mathbf{x}} + \sin\varphi\,\hat{\mathbf{y}}) = -\sin\varphi\,\dot{\varphi}\,\hat{\mathbf{x}} + \cos\varphi\,\dot{\varphi}\,\hat{\mathbf{y}}$$
$$= -\sin\varphi\,\dot{\varphi}\,[\cos\varphi\,\hat{\boldsymbol{\rho}} - \sin\varphi\,\hat{\boldsymbol{\theta}}] + \cos\varphi\,\dot{\varphi}\,[\sin\varphi\,\hat{\boldsymbol{\rho}} + \cos\varphi\,\hat{\boldsymbol{\theta}}]$$
$$= \dot{\varphi}\,\hat{\boldsymbol{\varphi}}$$

$$\dot{\hat{\boldsymbol{\varphi}}} = d\hat{\boldsymbol{\theta}}/dt = d/dt(-\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}}) = -\cos\varphi\,\dot{\varphi}\,\hat{\mathbf{x}} - \sin\varphi\,\dot{\varphi}\,\hat{\mathbf{y}}$$
$$= -\cos\varphi\,\dot{\varphi}\,[\cos\varphi\,\hat{\boldsymbol{\rho}} - \sin\varphi\,\hat{\boldsymbol{\theta}}] - \sin\varphi\,\dot{\varphi}\,[\sin\varphi\,\hat{\boldsymbol{\rho}} + \cos\varphi\,\hat{\boldsymbol{\theta}}]$$
$$= -\dot{\varphi}\,\hat{\boldsymbol{\rho}}$$

E.6 The Affine Connection

Recall from above the claim that

$$\begin{array}{lll} \partial_r \hat{\mathbf{r}} = 0 & \partial_\theta \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} & \partial_\varphi \hat{\mathbf{r}} = \sin\theta\, \hat{\boldsymbol{\varphi}} \\ \partial_r \hat{\boldsymbol{\theta}} = 0 & \partial_\theta \hat{\boldsymbol{\theta}} = -\hat{\mathbf{r}} & \partial_\varphi \hat{\boldsymbol{\theta}} = \cos\theta\, \hat{\boldsymbol{\varphi}} \\ \partial_r \hat{\boldsymbol{\varphi}} = 0 & \partial_\theta \hat{\boldsymbol{\varphi}} = 0 & \partial_\varphi \hat{\boldsymbol{\varphi}} = -\sin\theta\, \hat{\mathbf{r}} - \cos\theta\, \hat{\boldsymbol{\theta}} \quad . \end{array} \qquad (E.2.9) \qquad (E.6.1)$$

We wish to put these equations into a more general framework which is an extension of the discussion of Section E.4.

In the notation of our *Tensor* document, for an *arbitrary* curvilinear coordinate system $\mathbf{x}'$ the derivatives of the tangent base vectors (called $\mathbf{e}_n$ in that document) are written

$$\partial'_j \mathbf{e}_n = \Sigma_i \Gamma'^{i}_{jn}\, \mathbf{e}_i \quad . \qquad (E.6.2)$$

This just says that the change in a basis vector obtained by moving a small amount in some direction is (and of course must be) some linear combination of the basis vectors. The coefficients of the linear combination are known as the **affine connection** (or **Levi-Civita connection**) $\Gamma'^{c}_{ab} = \Gamma'^{c}_{ba}$. The reason for the primes is that Cartesian coordinates are thought of as $\mathbf{x}$, while curvilinear ones of some particular type are $\mathbf{x}'$. In Cartesian x-space one has $\Gamma^{c}_{ab} = 0$ because basis vectors don't vary with position in that space. Then Γ'^{c}_{ab} is the affine connection in x'-space. For example, as noted in Section E.4, in spherical coordinates one has

$$\begin{aligned} \mathbf{x} &= (x_1, x_2, x_3) = (x, y, z) \\ \mathbf{x}' &= (x'_1, x'_2, x'_3) = (r, \theta, \varphi) \end{aligned} \qquad (E.6.3)$$

and the inverse transformation (E.2.5) is $\mathbf{x} = F^{-1}(\mathbf{x}')$. As shown below, $\mathbf{e}_1(\mathbf{x}') = \mathbf{e}_1(r,\theta,\varphi) = \hat{\mathbf{r}}$ and then as an example of (E.6.2) we write (for j=2 and n=1, and the linear combination has only one non-zero term):

$$\partial'_2 \mathbf{e}_1 = \partial_{x'_2} \mathbf{e}_1 = \partial_\theta \mathbf{e}_1(r,\theta,\varphi) = \partial_\theta \hat{\mathbf{r}}(r,\theta,\varphi) = \hat{\boldsymbol{\theta}} = (1/r)\mathbf{e}_\theta = (1/r)\mathbf{e}_2 = \Gamma'^{2}_{21}\, \mathbf{e}_2 = \Gamma^{\theta}_{\theta r}\, \mathbf{e}_\theta \quad . \qquad (E.6.4)$$

The curvilinear *unit* basis vectors $\hat{\mathbf{e}}_n$ are related to the tangent base vectors by $\hat{\mathbf{e}}_n = (1/h'_n)\, \mathbf{e}_n$ where the $h'_n = |\mathbf{e}_n|$ are the so-called scale factors. The derivatives of the $\hat{\mathbf{e}}_n$ are then given by :

$$\begin{aligned} \partial'_j \hat{\mathbf{e}}_n &= \partial'_j (h'^{-1}_n \mathbf{e}_n) = (\partial'_j h'^{-1}_n)\mathbf{e}_n + h'^{-1}_n (\partial'_j \mathbf{e}_n) \\ &= -h'^{-2}_n (\partial'_j h'_n)\, \mathbf{e}_n + h'^{-1}_n (\Sigma_i \Gamma'^{i}_{jn}\, \mathbf{e}_i) \\ &= -h'^{-1}_n (\partial'_j h'_n)\, \hat{\mathbf{e}}_n + h'^{-1}_n (\Sigma_i \Gamma'^{i}_{jn}\, h'_i\, \hat{\mathbf{e}}_i) \end{aligned}$$

$$= (1/h'_n)\,[\Sigma_i\ h'_i\,\Gamma'^{i}{}_{jn}\hat{\mathbf{e}}_i\ - (\partial'_j h'_n)\,\hat{\mathbf{e}}_n]\ . \qquad \text{(E.6.5)}$$

In *Tensor* the Cartesian basis vectors of x-space are called $\mathbf{u}_i$, but in this document they are called $\mathbf{e}_i$ so we need a different symbol $\boldsymbol{e}_i$ for the tangent base vectors. In Section 14 we use these $\boldsymbol{e}_i$ with $\hat{\boldsymbol{e}}_i$ as the unit tangent base vectors, $\boldsymbol{\xi} = (\xi_1,\xi_2,\xi_3)$ in place of $\mathbf{x}' = (x'_1,x'_2,x'_3)$, and $\Gamma' = \Gamma$ with no prime since $\boldsymbol{\xi}$ has no prime (trying not to confuse $\Gamma = 0$ of x-space with $\Gamma \neq 0$ of ξ-space). In this notation (E.6.5) would appear as

$$(\partial\hat{\boldsymbol{e}}_n/\partial\xi_j)\ = (1/h_n)\,[\Sigma_i(h_i\Gamma^{i}{}_{jn}\hat{\boldsymbol{e}}_i)\ - (\partial h_n/\partial\xi_j)\hat{\boldsymbol{e}}_n]\ . \qquad \text{(E.6.6)}$$

Going back to the *Tensor* notation, the affine connection for a coordinate system is related to the system's metric tensor g according to

$$\Gamma^{d}{}_{ab} = (1/2)\ g_{dc}\ [\ \partial_a\bar{g}_{bc} + \partial_b\bar{g}_{ca} - \partial_c\bar{g}_{ab}] = 0\ \text{ since } \bar{g}_{ij} = \delta_{ij} \qquad \text{x-space (Cartesian)}$$

$$\Gamma'^{d}{}_{ab} = (1/2)\ g'_{dc}\ [\ \partial'_a\bar{g}'_{bc} + \partial'_b\bar{g}'_{ca} - \partial'_c\bar{g}'_{ab}]\ . \qquad \text{x'-space} \qquad \text{(E.6.7)}$$

For spherical coordinates, one has (using r,θ,φ = 1,2,3 where θ = polar, φ = azimuth)

1	2	3	
$h_r = 1$	$h_\theta = r$	$h_\varphi = r\sin\theta$	// scale factors ($h'_1 = h_r$)
$\mathbf{e}_r = \hat{\mathbf{r}}$	$\mathbf{e}_\theta = r\,\hat{\boldsymbol{\theta}}$	$\mathbf{e}_\varphi = r\sin\theta\,\hat{\boldsymbol{\varphi}}$	// tangent base vectors ($\mathbf{e}_1 = \mathbf{e}_r$)
$\hat{\mathbf{e}}_r = \hat{\mathbf{r}}$	$\hat{e}_\theta = \hat{\boldsymbol{\theta}}$	$\hat{e}_\varphi = \hat{\boldsymbol{\varphi}}$	// curvilinear unit basis vectors

$$\bar{g}'_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad g'_{ab} = \text{inverse}(\bar{g}'_{ab}) \qquad \text{// metric tensor} \qquad \text{(E.6.8)}$$

where for example $\bar{g}'_{33} = g_{\varphi\varphi} = r^2\sin^2\theta$. For spherical coordinates only 9 of the 27 elements of $\Gamma'^{d}{}_{ab}$ are non-zero (computed from (E.6.7)) :

$$\Gamma'^{1}{}_{22} = -\,r \qquad \Gamma'^{2}{}_{12} = \Gamma'^{2}{}_{21} = 1/r \qquad \text{// notation example: } \Gamma'^{1}{}_{22} = \Gamma^{r}{}_{\theta\theta}$$
$$\Gamma'^{1}{}_{33} = -\,r\sin^2\theta \qquad \Gamma'^{3}{}_{13} = \Gamma'^{3}{}_{31} = 1/r$$
$$\Gamma'^{2}{}_{33} = -\cos\theta\sin\theta \qquad \Gamma'^{3}{}_{23} = \Gamma'^{3}{}_{32} = \cot\theta\ . \qquad \text{(E.6.9)}$$

Example done in Section 14 notation with (E.6.6) :

$$(\partial\hat{\boldsymbol{e}}_n/\partial\xi_j)\ = (1/h_n)\,[\Sigma_i(h_i\Gamma^{i}{}_{jn}\hat{\boldsymbol{e}}_i)\ - (\partial h_n/\partial\xi_j)\hat{\boldsymbol{e}}_n]$$

$$(\partial\hat{\boldsymbol{e}}_3/\partial\xi_2)\ = (1/h_3)\,[\Sigma_i(h_i\Gamma^{i}{}_{23}\hat{\boldsymbol{e}}_i)\ - (\partial h_3/\partial\xi_2)\hat{\boldsymbol{e}}_3]$$

$$(\partial\hat{\boldsymbol{\varphi}}/\partial\theta)\ = (1/r\sin\theta)\,[(h_3\Gamma^{3}{}_{23}\hat{\boldsymbol{e}}_3)\ - (\partial[r\sin\theta]/\partial\theta)\hat{\boldsymbol{e}}_3]$$

$$= (1/r\sin\theta)\,[(r\sin\theta * \cot\theta * \hat{\boldsymbol{\varphi}}) - r\cos\theta * \hat{\boldsymbol{\varphi}}]$$

$$= (1/r\sin\theta)\,[r\cos\theta\,\hat{\boldsymbol{\varphi}} - r\cos\theta\,\hat{\boldsymbol{\varphi}}]$$

$$= 0 \qquad \text{(E.6.10)}$$

and with some effort we have verified that $\partial_{\boldsymbol{\theta}}\hat{\boldsymbol{\varphi}} = 0$ as appears in (E.6.1). This fact is of course obvious just looking at Fig (E.1.1), but things can be less obvious in obscure coordinate systems.

Footnote : To be consistent with the Footnote at the end Section E.4 we really should use $\bar{\partial}_{\bar{\jmath}}$ to indicate the derivative in (E.6.2) since it transforms as a covariant vector, but it just adds confusion to do so.

Appendix F : The Dumbbell (Tethered) satellite as an example of rotating frame analysis

This is a very long appendix so we provide an overview:

Section F.1 lays out kinematic details of the coordinates we use and discusses simple geometric facts of the satellite. We use the "swap notation" wherein Frame S' is the inertial frame and Frame S is the rotating frame. The masses m_1 and m_2 can be equal or different.

Section F.2 computes the angular momentum $\mathbf{L}^{(0)}$ of the satellite and its time derivative $\dot{\mathbf{L}}^{(0)}$ in rotating Frame S where the origin of Frame S is used as a reference point.

Section F.3 computes the *true* torque $\mathbf{N}'^{(b)}$ on the satellite due to the Earth's gravitational attraction acting on the two masses. The resulting torque is computed in inertial Frame S' and has only a $\hat{\boldsymbol{\varphi}}$ component as shown in (F.3.7). This computation is done in two extra ways to verify the result. In the far approximation, the expression for $\mathbf{N}'^{(b)}$ simplifies to that shown in (F.3.13).

Section F.4 then computes the *fictitious* torque which appears in rotating Frame S. This torque $\mathbf{N}^{(0)}{}_{\mathrm{fict}}$ is stated in (F.4.11) and has $\hat{\boldsymbol{\varphi}}$ and $\hat{\boldsymbol{\theta}}$ components. An interpretation is provided.

Section F.5 then uses "Newton's Angular Law" $\dot{\mathbf{L}}^{(0)} = \mathbf{N}'^{(b)} + \mathbf{N}^{(0)}{}_{\mathrm{fict}}$ to obtain the angular equations of motion for the satellite as stated in (F.5.7) using the far approximation. Certain special-case solutions are extracted which demonstrate the notion of in-plane and out-of-plane small-θ libration of the satellite with frequencies given in (F.5.10) and (F.5.13).

Section F.6 basically starts over working only with forces, not torques. The two angular equations of motion obtained in Section F.5 are again obtained, and a third equation determines the tension T in the dumbbell stick. One can interpret this tension T as a tidal force acting on mass m_1 which is distance r_1 from the Frame S center of mass. An equal and opposite tidal force acts on mass m_2 at the other end of the stick. When the satellite is vertically aligned, one finds $T = 3m_1\,\omega^2 r_1$ in (F.6.30). The tension T does not appear in the angular analysis of Section F.5 since T makes no contribution to torque about the Frame S origin, being aligned with the stick. Tension T is part of the radial force equation (F.6.19) of Section F.6, but there is no radial equation in Section F.5. There the stick is regarded just as a constraint, and it is typical for the forces of constraint not to be determined in the simplest analysis.

Section F.7 provides some Maple numerical solutions of the far-approximation θ,φ equations of motion of the satellite stated in (F.5.7). The two libration modes are verified, and a more general case is examined.

Section F.8 reworks the satellite force analysis entirely in Cartesian coordinates. The equations of motion for the variables x,y,z are shown in (F.8.15) with tension T provided by (F.8.20).

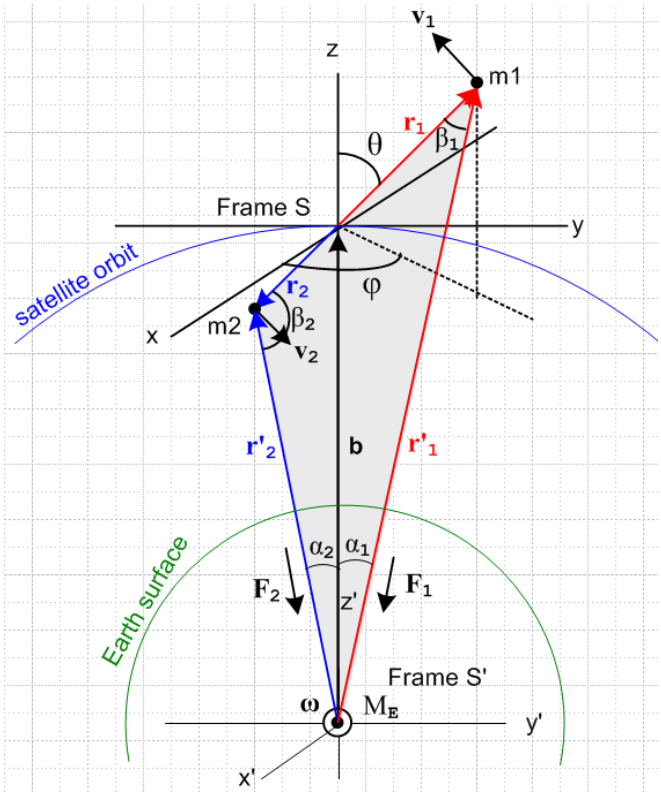
Section F.9 shows that the x,y,z equations of motion of Section F.8 are entirely equivalent to the θ,φ equations of motion of Section F.5.

Section F.10 examines various numerical solutions for the dumbbell satellite, working now in Cartesian coordinates. The libration modes are again examined, and a more general solution is studied. There is no "conical solution".

F.1 Kinematics of the satellite in rotating Frame S

As with Appendix D, this section uses the "swap notation" described in the Summary for Section 1 where the rotating frame is Frame S and the non-rotating frame is Frame S'. This is done to reduce the number of primes since most activity will be in the rotating satellite Frame S.

We now place the dumbbell satellite in a more general orientation than it was in Appendix D :



(F.1.1)

Description of the Figure

Half the battle is having a clear picture of what is going on and we shall expend many words to describe the above drawing. It shows the dumbbell satellite in orbit in a completely arbitrary orientation. The two masses m_1 and m_2 are connected by a massless **stick** (not shown) of length $r_1+r_2 = s$. If we find that this stick is always in tension for some situation, we can replace it in that situation with a non-stretching massless **tether**. If the stick were to go into compression, the replacement tether would lose its linear shape and we don't want to deal with that problem. As shown much later in Section F.10, the stick is always in tension for normal situations.

The gray-filled triangle is a part of the plane φ = constant which has normal vector $\hat{\boldsymbol{\varphi}}$. This plane is *not* in the plane of paper. The fill region contains two non-right triangles shown in blue and red. The red

triangle is on the viewer's side of paper, while the blue one lies behind the paper. Each of these triangles contains the vector **b** as an edge.

Frame S' is an inertial frame whose center is the center of the Earth and which is assumed fixed with respect to the stars.

Frame S is a non-inertial frame whose center is located at the satellite center of mass point. We make the assumption discussed above (D.2.2) that we can ignore the tiny offset between the center of mass and center of gravity of the satellite, so we then regard the Frame S origin as travelling in a circular orbit around the Earth. Recall from (D.3.16) that this offset is about 3 microns for s = 10 meters.

At time t = 0 shown in the figure, the axes of both Frames align with each other. At any time, axes y,z and y',z' are in the plane of paper. The unit vector $\hat{\mathbf{z}}$ always points from the center of the Earth to the origin of Frame S. so $\mathbf{b} = b\hat{\mathbf{z}}$ at any time. The axes x,x' always point directly out of the plane of paper so $\hat{\mathbf{x}} = \hat{\mathbf{x}}'$.

The orbital rotation vector $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$ also points out of the plane of paper and recall that $\omega = 2\pi/T$ where T is about 88 minutes for a low-Earth orbit. We have in mind an orbit at any altitude, but we do assume a circular orbit.

The circled dot at the bottom is the center of the Earth (mass M_E) and we have drawn the Earth surface in green. The blue circle is the orbit of the center of mass of the satellite (the Frame S origin) and it lies in the plane of paper. Note that ω is for the orbit and has nothing whatsoever to do with the rotation of the Earth. The above picture is the same whether or not the Earth rotates at its 24 hour ω_E rate about some obscure $\hat{\boldsymbol{\omega}}_E$ axis (not shown).

One could try to treat the satellite as a reduced-mass single-particle system as is done for planetary orbits, but even if possible this would obscure details we want to be visible. We really have here a 3-body problem where the three bodies are point masses, but there is a constraint (the stick), so perhaps it is a 2 1/2-body problem. If the dumbbell were treated as a rigid object, it is then a 2-body problem where one body is not a point mass.

Naming of coordinates

In Frame S mass m_1 has spherical coordinates (r_1,θ_1,φ_1) while mass m_2 has coordinates (r_2,θ_2,φ_2). The reader is now forewarned about our upcoming slipshod notation. We *define* $(\theta,\varphi) \equiv (\theta_1,\varphi_1)$. Thus Fig (F.1.1) shows θ,φ and not θ_1,φ_1. The three unit vectors of spherical coordinates for mass m_1 will always be called $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ and never $\hat{\mathbf{r}}_1$, $\hat{\boldsymbol{\theta}}_1$, $\hat{\boldsymbol{\varphi}}_1$.

These spherical coordinates are in the "physics" convention: angle θ is the polar angle down from the "vertical" z axis, φ is the azimuthal angle measured from the x axis toward the y axis. See Appendix E.1 regarding conventions.

Because $\mathbf{r}_1$ and $\mathbf{r}_2$ are collinear, we know that $\theta_2 = \pi - \theta$ and $\varphi_2 = \pi + \varphi$. Since the Frame S origin is the center of mass, we also know that $r_2 = (m_1/m_2)r_1$. Our strategy is to avoid the subscript-2 coordinates whenever possible and express everything in terms of the mass m_1 coordinates (r_1,θ,φ).

The mass m_2 unit vectors are related to the m_1 unit vectors by $\hat{\mathbf{r}}_2 = -\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}_2 = +\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\varphi}}_2 = -\hat{\boldsymbol{\varphi}}$. Each unit vector points toward increasing parameter value for its associated mass. To see these last relations, it helps to stare at Fig (F.1.1) and think of the gray triangle as being in the plane of paper. We shall only make use of $\hat{\boldsymbol{\varphi}}_2 = -\hat{\boldsymbol{\varphi}}$ below. See Appendix E regarding the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\varphi}}$.

To summarize :

$\mathbf{r}_1 = [x_1,y_1,z_1] = (r_1,\theta,\varphi)$ = coordinates of mass m_1 velocity = $\mathbf{v}_1$
$\mathbf{r}_2 = [x_2,y_2,z_2] = (r_2,\theta_2,\varphi_2)$ = coordinates of mass m_2 velocity = $\mathbf{v}_2$

$\theta_2 = \pi-\theta$ $\varphi_2 = \varphi+\pi$ $r_2 = (m_1/m_2)\, r_1$ $\hat{\mathbf{r}}_2 = -\hat{\mathbf{r}}_1 = -\hat{\mathbf{r}}$. (F.1.2)

In Frame S mass m_1 is constrained to lie on a sphere of radius r_1, while mass m_2 is constrained to lie on a sphere of radius r_2. The picture assumes $m_1 < m_2$ so $r_1 > r_2$. If m_1 lies at a point on its sphere, m_2 lies on the inverse point but on its sphere, as the spherical coordinates above show. If the masses are the same, the two spheres coincide.

Due to these constraints on the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, the corresponding velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the two masses must be tangential to their respective spheres. Thus, for example, for mass m_1 we can write

$\mathbf{v}_1 = v_{1\theta}\,\hat{\boldsymbol{\theta}} + v_{1\varphi}\,\hat{\boldsymbol{\varphi}}$, whereas $\mathbf{r}_1 = r_1\,\hat{\mathbf{r}}$. (F.1.3)

Some Basic Kinematic Facts

The internal angles of the red triangle are β_1, α_1 and $\pi-\theta$ at the Frame S origin. Thus we know from a Law of Cosines that

$$r'^2_1 = b^2 + r_1^2 - 2br_1\cos(\pi-\theta) = b^2 + r_1^2 + 2br_1\cos\theta \ .$$

The internal angles of the blue triangle are β_2, α_2 and θ at the Frame S origin. Thus,

$$r'^2_2 = b^2 + r_2^2 - 2br_2\cos\theta \ .$$

The three Laws of Sines for the two triangles tells us that

$$\frac{\sin\beta_1}{b} = \frac{\sin\alpha_1}{r_1} = \frac{\sin(\pi-\theta)}{r_1'} = \frac{\sin\theta}{r_1'} \qquad \text{// red triangle}$$

$$\frac{\sin\beta_2}{b} = \frac{\sin\alpha_2}{r_2} = \frac{\sin\theta}{r'_2} \ . \qquad \text{// blue triangle}$$

Looking at the drawing it is clear that $\mathbf{b} + \mathbf{r}_1 = \mathbf{r}'_1$ and $\mathbf{b} + \mathbf{r}_2 = \mathbf{r}_2'$, so one can write

$$\mathbf{r}'_2 - \mathbf{r}_2 = \mathbf{r}'_1 - \mathbf{r}_1 = \mathbf{b} \ .$$

Recall the center of mass condition from (D.2.8) that

$m_1\mathbf{r}_1 = -\,m_2\mathbf{r}_2 \quad \Rightarrow \quad m_1r_1 = m_2r_2$, $r_2/r_1 = m_1/m_2$, $\hat{\mathbf{r}}_2 = -\,\hat{\mathbf{r}}$. *(D.2.8)*

Applying the Frame S time derivative ∂_S gives similar results for $\mathbf{r} \to \mathbf{v}$ and $\mathbf{r} \to \mathbf{a}$, see below (velocity and acceleration).

Summary:

$$r'_1{}^2 = b^2 + r_1{}^2 + 2br_1\cos\theta \qquad \text{(F.1.4)}$$

$$r'_2{}^2 = b^2 + r_2{}^2 - 2br_2\cos\theta \qquad \text{(F.1.5)}$$

$$\frac{\sin\beta_1}{b} = \frac{\sin\alpha_1}{r_1} = \frac{\sin\theta}{r_1'} \qquad \text{// red triangle} \qquad \text{(F.1.6)}$$

$$\frac{\sin\beta_2}{b} = \frac{\sin\alpha_2}{r_2} = \frac{\sin\theta}{r'_2} \qquad \text{// blue triangle} \qquad \text{(F.1.7)}$$

$$\mathbf{r}'_2 - \mathbf{r}_2 = \mathbf{r}'_1 - \mathbf{r}_1 = \mathbf{b} \qquad \mathbf{r}'_1 = \mathbf{b} + \mathbf{r}_1 \qquad \mathbf{r}'_2 = \mathbf{b} + \mathbf{r}_2 \qquad \text{(F.1.8)}$$

$$m_1\mathbf{r}_1 = -\, m_2\mathbf{r}_2 \quad \Rightarrow \quad m_1r_1 = m_2r_2\,, \quad r_2/r_1 = m_1/m_2\,, \quad \hat{\mathbf{r}}_2 = -\,\hat{\mathbf{r}}_1 \qquad \text{(F.1.9)}$$

$$m_1\mathbf{v}_1 = -\, m_2\mathbf{v}_2 \quad \Rightarrow \quad m_1v_1 = m_2v_2\,, \quad v_2/v_1 = m_1/m_2\,, \quad \hat{\mathbf{v}}_2 = -\,\hat{\mathbf{v}}_1 \qquad \text{(F.1.10)}$$

$$m_1\mathbf{a}_1 = -\, m_2\mathbf{a}_2 \quad \Rightarrow \quad m_1a_1 = m_2a_2\,, \quad a_2/a_1 = m_1/m_2\,, \quad \hat{\mathbf{a}}_2 = -\,\hat{\mathbf{a}}_1 \qquad \text{(F.1.11)}$$

F.2 Angular momentum of the satellite and its time derivative in Frame S

Our upcoming path is to obtain the equations of motion for the satellite in terms of θ, φ coordinates. As a demonstration of the method of Section 11.3 we shall do this first using Newton's rotational second law with fictitious torques. Later in Section F.6 we will do this again using Newton's linear second law with fictitious forces as a demonstration of the method of Section 8.1.

For angular momentum (and later torque) we take our reference point to be $\mathbf{c} = 0$ in Frame S (the origin) and thus $\mathbf{c}' = \mathbf{b}$ in Frame S'. Then (we show all detail in this first calculation),

$$\mathbf{L}^{(0)} = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = m_1\ \mathbf{r}_1 \times \mathbf{v}_1 + m_2\ \mathbf{r}_2 \times \mathbf{v}_2 \qquad \text{// dim} = ML^2/T$$

$$= m_1\mathbf{r}_1 \times \mathbf{v}_1 + m_2\mathbf{[-(}m_1/m_2\mathbf{)}\mathbf{r}_1\mathbf{]} \times \mathbf{[-(}m_1/m_2\mathbf{)}\mathbf{v}_1\mathbf{]} \qquad \text{// (F.1.9) and (F.110)}$$

$$= m_1\mathbf{r}_1 \times \mathbf{v}_1 + \mathbf{[}m_1\mathbf{r}_1\mathbf{]} \times \mathbf{[(}m_1/m_2\mathbf{)}\mathbf{v}_1\mathbf{]} = m_1\{\, \mathbf{r}_1 \times \mathbf{v}_1 + \mathbf{(}m_1/m_2\mathbf{)}\ \mathbf{r}_1 \times \mathbf{v}_1 \,\}$$

$$= m_1[\ 1 + \mathbf{(}m_1/m_2\mathbf{)}\ ]\ \mathbf{r}_1 \times \mathbf{v}_1 = m_1[\ (m_2+m_1)/m_2\ ]\ \mathbf{r}_1 \times \mathbf{v}_1$$

$$= (m_1/m_2)\ M\ \mathbf{r}_1 \times \mathbf{v}_1 = (m_1/m_2)\ Mr_1\ \hat{\mathbf{r}} \times \mathbf{v}_1 \qquad \text{// } M \equiv m_1+m_2 \qquad \text{(F.2.1)}$$

$$= (m_1/m_2)\ Mr_1\ \hat{\mathbf{r}} \times (v_{1\theta}\hat{\boldsymbol{\theta}} + v_{1\varphi}\hat{\boldsymbol{\varphi}}) = (m_1/m_2)\ Mr_1\ [\ v_{1\theta}\ \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} + v_{1\varphi}\ \hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}}\ ] \qquad \text{// (F.1.3)}$$

$$= (m_1/m_2)\, M\, r_1\, (v_{1\theta}\, \hat{\boldsymbol{\varphi}} - v_{1\varphi}\, \hat{\boldsymbol{\theta}}) \qquad \text{// (E.2.12)}$$

$$= (m_1/m_2)\, M\, r_1^2\, (\dot{\theta}\, \hat{\boldsymbol{\varphi}} - \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\theta}}) \ . \qquad \text{// (E.3.5)} \qquad \dim = ML^2/T \ \checkmark$$

The dumbbell has no angular momentum around the $\hat{\mathbf{r}}$ axis which seems very reasonable since it consists of two point masses aligned with $\hat{\mathbf{r}}$. (A real tethered satellite has non-point masses and one could imagine undesired torsion oscillation being a problem.)

Taking a ∂_S time derivative gives the rate of change of angular momentum in Frame S (all variables here are the "natural" ones in Frame S),

$$\dot{\mathbf{L}}^{(0)} = \partial_S \{ (m_1/m_2)\, M\ \mathbf{r}_1 \times \mathbf{v}_1 \} \qquad \text{// (F.2.1)}$$

$$= (m_1/m_2)\, M\, (\mathbf{v}_1 \times \mathbf{v}_1 + \mathbf{r}_1 \times \mathbf{a}_1)$$

$$= (m_1/m_2)\, Mr_1\, (\hat{\mathbf{r}} \times \mathbf{a}_1) = (m_1/m_2)\, Mr_1\, \hat{\mathbf{r}} \times [\, a_r\, \hat{\mathbf{r}} + a_\theta \hat{\boldsymbol{\theta}} + a_\varphi\, \hat{\boldsymbol{\varphi}}\,] \qquad \text{// (E.3.6)}$$

$$= (m_1/m_2)\, Mr_1\, [\, a_\theta\, \hat{\boldsymbol{\varphi}} - a_\varphi\, \hat{\boldsymbol{\theta}}\,] \qquad \text{// (E.2.12) and then (E.3.6) for next line}$$

$$= (m_1/m_2)\, Mr_1^2\ [\, (\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta)\, \hat{\boldsymbol{\varphi}} - (2\dot{\theta}\, \dot{\varphi} \cos\theta + \ddot{\varphi} \sin\theta)\, \hat{\boldsymbol{\theta}}\,] \ .$$

Here are the conclusions so far:

$$\mathbf{L}^{(0)} = (m_1/m_2)\, M\, r_1^2\, (\dot{\theta}\, \hat{\boldsymbol{\varphi}} - \dot{\varphi} \sin\theta\, \hat{\boldsymbol{\theta}}) \qquad \text{(F.2.2)}$$

$$\dot{\mathbf{L}}^{(0)} = (m_1/m_2)\, Mr_1^2\ [\, (\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta)\, \hat{\boldsymbol{\varphi}} - (2\dot{\theta}\, \dot{\varphi} \cos\theta + \ddot{\varphi} \sin\theta)\, \hat{\boldsymbol{\theta}}\,] \ . \qquad \text{(F.2.3)}$$

One obvious statement can be made looking at these equations: there is no angular momentum about the $\hat{\mathbf{r}}$ axis and this vanishing angular momentum never changes.

The equation of motion for the satellite within Frame S is given by (11.3.4), but converted to swap notation,

$$\mathbf{N}^{(0)}_{\mathbf{eff}} = \dot{\mathbf{L}}^{(0)} \ . \qquad \textit{(11.3.4)}_S \qquad \text{(F.2.4)}$$

Our next task then is to compute the total effective torque on the satellite in Frame S which from (11.3.5) is (again converted to swap notation),

$$\mathbf{N}^{(0)}_{\mathbf{eff}} = \mathbf{N}'^{(b)} + \mathbf{N}^{(0)}_{\mathbf{fict}} \ . \qquad \textit{(11.3.5)}_S \qquad \text{(F.2.5)}$$

Here $\mathbf{N'^{(b)}}$ is the Frame S' torque on the satellite relative to point **b** in Frame S', and $\mathbf{N^{(0)}_{fict}}$ is the fictitious torque that arises because Frame S is a rotating frame of reference.

F.3 The torque on the Dumbbell Satellite in Frame S'

The torque (in Frame S') of the Earth on the satellite (*relative to* the Frame S origin) is given by

$$\mathbf{N'^{(b)}} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 \qquad \text{// dim} = L^3\, M\, /T \qquad \text{(F.3.1)}$$

where all four of these vectors are shown in Fig (F.1.1). Recall from (D.2.13) that

$$\mathbf{F}_1 = -(GM_E m_1/r'^2_1)\,\hat{\mathbf{r}}'_1 = -(GM_E m_1/r'^3_1)\,\mathbf{r}'_1 \qquad F_1 \equiv |\mathbf{F}_1| = (GM_E m_1/r'^2_1)$$
$$\mathbf{F}_2 = -(GM_E m_2/r'^2_2)\,\hat{\mathbf{r}}'_2 = -(GM_E m_2/r'^3_2)\,\mathbf{r}'_2 \qquad F_2 \equiv |\mathbf{F}_2| = (GM_E m_2/r'^2_2)\,. \qquad \text{(F.3.2)}$$

// The stick tension **T** exerts no torque on the satellite masses since **T** is collinear with $\mathbf{r}_1$ and $\mathbf{r}_2$.

Therefore,

$$\mathbf{r}_1 \times \mathbf{F}_1 = \mathbf{r}_1 \times [\,-(GM_E m_1/r'^3_1)\,\mathbf{r}'_1\,] = -(GM_E m_1/r'^3_1)\,\mathbf{r}_1 \times \mathbf{r}'_1$$

$$\mathbf{r}_2 \times \mathbf{F}_2 = \mathbf{r}_2 \times [\,-(GM_E m_2/r'^3_2)\,\mathbf{r}'_2\,] = -(GM_E m_2/r'^3_2)\,\mathbf{r}_2 \times \mathbf{r}'_2\,. \qquad \text{(F.3.3)}$$

Using (F.1.8) we evaluate the cross products making use of the first line of (E.2.15) for $\hat{\mathbf{r}} \times \hat{\mathbf{z}}$ **,**

$$\mathbf{r}_1 \times \mathbf{r}'_1 = \mathbf{r}_1 \times (\mathbf{b} + \mathbf{r}_1) = \mathbf{r}_1 \times \mathbf{b} = r_1 b\,\hat{\mathbf{r}} \times \hat{\mathbf{z}} = r_1 b\,(-\sin\theta\,\hat{\boldsymbol{\varphi}}) = -r_1 b \sin\theta\,\hat{\boldsymbol{\varphi}}$$

$$\mathbf{r}_2 \times \mathbf{r}'_2 = \mathbf{r}_2 \times (\mathbf{b} + \mathbf{r}_2) = \mathbf{r}_2 \times \mathbf{b} = [-r_2\hat{\mathbf{r}}] \times [b\hat{\mathbf{z}}] = -r_2 b\,\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -r_2 b(-\sin\theta\,\hat{\boldsymbol{\varphi}}) = r_2 b \sin\theta\,\hat{\boldsymbol{\varphi}}\,. \qquad \text{(F.3.4)}$$

Then inserting (F.3.4) into (F.3.3),

$$\mathbf{r}_1 \times \mathbf{F}_1 = (GM_E m_1/r'^3_1)\, r_1 b \sin\theta\,\hat{\boldsymbol{\varphi}} \qquad \text{(F.3.5)}$$

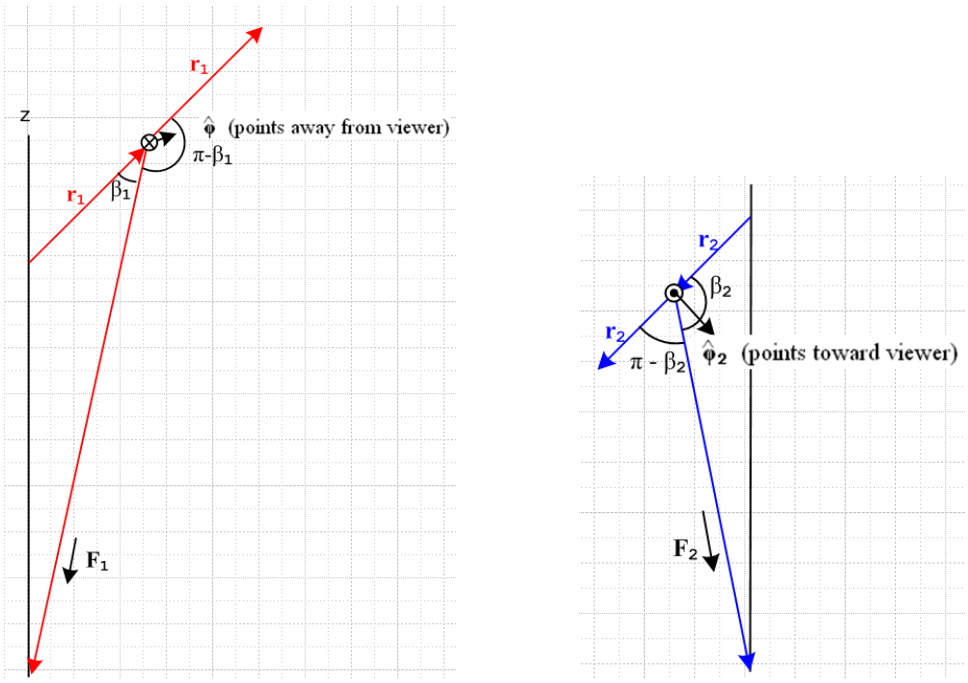
$$\mathbf{r}_2 \times \mathbf{F}_2 = -(GM_E m_2/r'^3_2)\, r_2 b \sin\theta\,\hat{\boldsymbol{\varphi}}$$
$$= -(GM_E m_1/r'^3_2)\, r_1 b \sin\theta\,\hat{\boldsymbol{\varphi}}. \qquad \text{// (F.1.9)} \qquad \text{(F.3.6)}$$

Thus the Earth's torque on the satellite in Frame S' is,

$$\mathbf{N'^{(b)}} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2 = (GM_E m_1/r'^3_1)\, r_1 b \sin\theta\,\hat{\boldsymbol{\varphi}} - (GM_E m_1/r'^3_2)\, r_1 b \sin\theta\,\hat{\boldsymbol{\varphi}}$$
$$= (GM_E m_1 b r_1 \sin\theta)\,[1/r'^3_1 - 1/r'^3_2]\,\hat{\boldsymbol{\varphi}}\,. \qquad \text{(F.3.7)}$$

Note that this equation uses $\mathbf{r}_1$ and $\mathbf{r}_2$ and not $\mathbf{r'}_1$ and $\mathbf{r'}_2$ because the torque reference point is point **b**.

We can confirm the two torque contributions by computing them geometrically from Fig (F.1.1) using these right-hand-rule helper drawings:



(F.3.8)

Then:

$\mathbf{r}_1 \times \mathbf{F}_1 = r_1F_1 \sin(\pi-\beta_1)\,\hat{\boldsymbol{\varphi}} = r_1F_1\sin\beta_1\hat{\boldsymbol{\varphi}} = r_1F_1(b\sin\theta/r'_1)\,\hat{\boldsymbol{\varphi}}$ // (F.1.6) then (F.3.2)

$= r_1(GM_Em_1/r'^2_1)(b\sin\theta/r'_1)\,\hat{\boldsymbol{\varphi}} = (GM_Em_1/r'^3_1)\,r_1b\sin\theta\,\hat{\boldsymbol{\varphi}}$ // agrees with (F.3.5)

$\mathbf{r}_2 \times \mathbf{F}_2 = r_2F_2 \sin(\pi-\beta_2)\,\hat{\boldsymbol{\varphi}}_2 = r_2F_2\sin\beta_2\,\hat{\boldsymbol{\varphi}}_2 = r_2F_2\sin\beta_2\,[\,-\hat{\boldsymbol{\varphi}}\,]$

$= r_2F_2\,(b\sin\theta/r'_2)\,[\,-\hat{\boldsymbol{\varphi}}\,] = r_2(GM_Em_2/r'^2_2)\,(b\sin\theta/r'_2)\,[\,-\hat{\boldsymbol{\varphi}}\,]$ // (F.1.7) and (F.3.2)

$= r_1(GM_Em_1/r'^2_2)\,(b\sin\theta/r'_2)\,[\,-\hat{\boldsymbol{\varphi}}\,]$ // (F.1.9)

$= -(GM_Em_1/r'^3_2)\,r_1b\sin\theta\,\hat{\boldsymbol{\varphi}}$ // agrees with (F.3.6)

where we have used the fact that $\hat{\boldsymbol{\varphi}}_2 = -\hat{\boldsymbol{\varphi}}_1 = -\hat{\boldsymbol{\varphi}}$.

There is a third way to compute the above torque, based on the torque theorem (D.1.3) which says that $\mathbf{N}^{(R)} = \mathbf{N}^{(0)} - \mathbf{R} \times \mathbf{F}$. In our current context of working in Frame S' this reads

$$\mathbf{N'}^{(b)} = \mathbf{N'}^{(0)} - \mathbf{b} \times \mathbf{F} \,. \tag{F.3.9}$$

The torque $\mathbf{N'}^{(0)}$ relative to the Frame S' origin is exactly 0

$$\mathbf{N'}^{(0)} = \mathbf{r'} \times \mathbf{F_1} + \mathbf{r'_2} \times \mathbf{F_2} = 0 + 0 = 0$$

since $\mathbf{r'_1}$ is collinear with $\mathbf{F_1}$ and $\mathbf{r'_2}$ is collinear with $\mathbf{F_2}$ as shown in Fig (F.1.1). We should include the stick tension/compression **T**, but $(\mathbf{r'_1} - \mathbf{r'_2}) \times \mathbf{T} = (\mathbf{r_1} - \mathbf{r_2}) \times \mathbf{T} = 0 - 0 = 0$ so **T** can be ignored. Thus we find from (F.3.9) that

$$\mathbf{N'}^{(b)} = -\mathbf{b} \times (\mathbf{F_1} + \mathbf{F_2}) = -\mathbf{b} \times [\, -(GM_E m_1/r'^3_1)\,\mathbf{r'_1} - (GM_E m_2/r'^3_2)\,\mathbf{r'_2}\,]$$

$$= (GM_E m_1/r'^3_1)\,\mathbf{b} \times \mathbf{r'_1} + (GM_E m_2/r'^3_2)\,\mathbf{b} \times \mathbf{r'_2}$$

$$= (GM_E m_1/r'^3_1)\,\mathbf{(r'_1 - r_1)} \times \mathbf{r'_1} + (GM_E m_2/r'^3_2)\,\mathbf{(r_2' - r_2)} \times \mathbf{r'_2} \quad \text{// (F.1.8)}$$

$$= -(GM_E m_1/r'^3_1)\,\mathbf{r_1} \times \mathbf{r'_1} - (GM_E m_2/r'^3_2)\,\mathbf{r_2} \times \mathbf{r'_2}$$

$$= -(GM_E m_1/r'^3_1)[-r_1 b \sin\theta\,\hat{\boldsymbol{\varphi}}] - (GM_E m_2/r'^3_2)\,[\, r_2 b \sin\theta\,\hat{\boldsymbol{\varphi}}\,] \quad \text{// (F.3.4) then (F.1.9)}$$

$$= (GM_E m_1 b r_1 \sin\theta)\,[1/r'^3_1 - 1/r'^3_2]\,\hat{\boldsymbol{\varphi}} \tag{F.3.10}$$

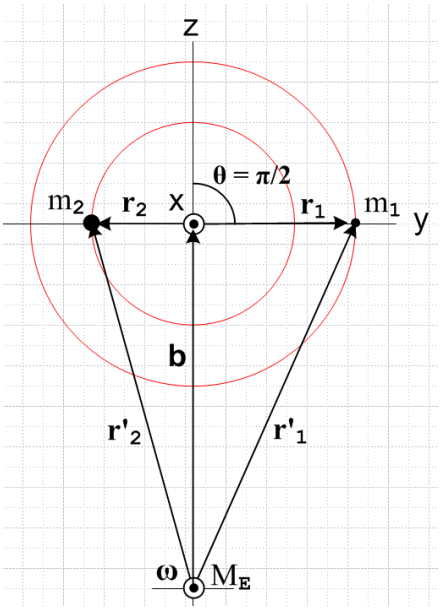
in agreement with (F.3.7).

Here are some quick checks on (F.3.10) :

- If the satellite is vertically aligned, $\theta = 0, \pi$, then $\sin\theta = 0$ and $\mathbf{N'}^{(b)} = 0$ as expected (no moment arms).

- If the satellite is horizontally aligned *and* $m_1 = m_2$, then $r'_1 = r'_2$ so $\mathbf{N'}^{(b)} = 0$ (balanced moment arms).

- If $m_2 > m_1$, then $r_2 < r_1$ so for horizontal alignment one has $r'_2 < r'_1$ so $[\,(1/r'^3_1) - (1/r'^3_2)\,] < 0$. Then if m_1 is on the right, we have $\theta = \pi/2$ and $\sin\theta = 1$ and then (F.3.10) has $\mathbf{N}^{(b)} = -(\text{positive})\hat{\boldsymbol{\varphi}}$. But for this orientation $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}_1 = -\hat{\mathbf{x}}$ so $\mathbf{N}^{(b)} = (\text{positive})\hat{\mathbf{x}}$:



(F.3.11)

In this case the lever arms balance in the sense that $m_2r_2 = m_1r_1$, but m_2 is closer to Earth center so it feels the stronger force and so we expect $\mathbf{N}'^{(b)} = (\text{positive})\hat{\mathbf{x}}$.

Far Approximation

If we now assume as before that $r_1', r_2', b >> r_1, r_2$ we can approximate $[\ (1/r_1'^3) - (1/r_2'^3)\ ]$ by adding on to the Maple code shown in (D.3.5) to get

```
f := (1/rp1)^3 - (1/rp2)^3:
series(f,e1=0,3): simplify(%);
```

$$3\frac{\cos(\theta)}{b^3(-1+\mu 1)}e1 - \frac{3}{2}\frac{1-2\,\mu 1-5\cos(\theta)^2+10\cos(\theta)^2\mu 1}{b^3(1-2\,\mu 1+\mu 1^2)}e1^2+O(e1^3)$$

where recall $e1 = \varepsilon_1 \equiv (r_1/b)$. This time there is a leading linear term in ε_1 so

$$(1/r'^3_1) - (1/r'^3_2) \approx 3\frac{\cos\theta}{b^3(-\mu_2)}(r_1/b) = -3r_1\cos\theta/(\mu_2 b^4)\ . \qquad (F.3.12)$$

Installing this result into (F.3.7) gives

$$\mathbf{N}'^{(b)} = GM_E m_1 b\, r_1\sin\theta\ [\ (1/r'^3_1) - (1/r'^3_2)]\ \hat{\boldsymbol{\varphi}} \qquad (F.3.7)$$

$$\approx -GM_E m_1 b\, r_1\sin\theta\ (\ 3r_1\cos\theta/(\mu_2 b^4)\ )\ \hat{\boldsymbol{\varphi}}$$

$$= -3GM_E(m_1/\mu_2)\, b^{-3} r_1^2 \sin\theta\cos\theta\ \hat{\boldsymbol{\varphi}}\ . \qquad (F.3.13)$$

To this order of approximation, the torque $\mathbf{N}^{(b)}$ vanishes when the satellite is horizontally aligned as well as when it is vertically aligned, and this is due to $r_1' \approx r'_2$ in the horizontal case.

F.4 The fictitious torque on the satellite in Frame S

Recall the general fictitious torque expression given in (11.3.10), acting on a single particle of mass m,

$$\mathbf{N}'^{(c')}_{fict} = -(\mathbf{r}'-\mathbf{c}') \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \,]$$
$$+ m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v}' \quad . \qquad (11.3.10)$$

Converted to swap notation this says,

$$\mathbf{N}^{(c)}_{fict} = -(\mathbf{r}-\mathbf{c}) \times [\, m\ddot{\mathbf{b}}_{S'} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \,]$$
$$+ m(\dot{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c} + \dot{\mathbf{b}}_{S'}) \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}) - m\dot{\mathbf{c}} \times \mathbf{v} \quad . \qquad (11.3.10)_S \qquad (F.4.1)$$

Here ω is the angular rotation rate of the satellite about the Earth.

Our application has the torque center at $\mathbf{c} = 0$ (and $\mathbf{c}' = \mathbf{b}$) so this simplifies somewhat to

$$\mathbf{N}^{(0)}_{fict} = -m\mathbf{r} \times [\, \ddot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\,\boldsymbol{\omega} \times \mathbf{v} + \dot{\boldsymbol{\omega}} \times \mathbf{r}] + \dot{\mathbf{b}}_{S'} \times (m\mathbf{v} + \boldsymbol{\omega} \times [m\mathbf{r}]) \quad . \qquad (F.4.2)$$
frame centrifugal Coriolis Euler

Recall from (8.1.8) that the square bracket in the above is $-\mathbf{F}_{fict}/m$ so we can trace the origin of the terms.

Now write (F.4.2) separately for each of the two masses of the satellite :

$$\mathbf{N}^{(0)}_{f,1} = -m_1\mathbf{r}_1 \times [\, \ddot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) + 2\boldsymbol{\omega} \times \mathbf{v}_1 + \dot{\boldsymbol{\omega}} \times \mathbf{r}_1] + \dot{\mathbf{b}}_{S'} \times (m_1\mathbf{v}_1 + \boldsymbol{\omega} \times [m_1\mathbf{r}_1])$$
$$\mathbf{N}^{(0)}_{f,2} = -m_2\mathbf{r}_2 \times [\, \ddot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_2) + 2\boldsymbol{\omega} \times \mathbf{v}_2 + \dot{\boldsymbol{\omega}} \times \mathbf{r}_2] + \dot{\mathbf{b}}_{S'} \times (m_2\mathbf{v}_2 + \boldsymbol{\omega} \times [m_2\mathbf{r}_2]) \quad . \qquad (F.4.3)$$

In the second line, use (F.1.9) to replace $m_2\mathbf{r}_2 = -m_1\mathbf{r}_1$ and (F.1.10) to replace $m_2\mathbf{v}_2 = -m_1\mathbf{v}_1$:

$$\mathbf{N}^{(0)}_{f,2} = +m_1\mathbf{r}_1 \times [\, \ddot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_2) + 2\boldsymbol{\omega} \times \mathbf{v}_2 + \dot{\boldsymbol{\omega}} \times \mathbf{r}_2] + \dot{\mathbf{b}}_{S'} \times (-m_1\mathbf{v}_1 + \boldsymbol{\omega} \times [-m_1\mathbf{r}_1]) \quad . \qquad (F.4.4)$$

Next, add the two torques to get the total fictitious torque on the satellite seen in Frame S,

$$\mathbf{N}^{(0)}{}_{\mathtt{fict}} = \mathbf{N}^{(0)}{}_{\mathtt{f,1}} + \mathbf{N}^{(0)}{}_{\mathtt{f,2}}$$

$$= -\, m_1\mathbf{r}_1 \times [\, \ddot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) + 2\boldsymbol{\omega} \times \mathbf{v}_1 + \dot{\boldsymbol{\omega}} \times \mathbf{r}_1] \;+\; \dot{\mathbf{b}}_{S'} \times (\; m_1\mathbf{v}_1 + \boldsymbol{\omega} \times [m_1\mathbf{r}_1])$$
$$+ m_1\mathbf{r}_1 \times [\, \ddot{\mathbf{b}}_{S'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_2) + 2\boldsymbol{\omega} \times \mathbf{v}_2 + \dot{\boldsymbol{\omega}} \times \mathbf{r}_2] \;+\; \dot{\mathbf{b}}_{S'} \times (\; -m_1\mathbf{v}_1 + \boldsymbol{\omega} \times [-m_1\mathbf{r}_1])$$

$$= -\, m_1\mathbf{r} \;\times \{\; \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}_1 - \mathbf{r}_2]) + \; 2\boldsymbol{\omega} \times [\mathbf{v}_1 - \mathbf{v}_2] + \;\; \dot{\boldsymbol{\omega}} \times [\mathbf{r}_1 - \mathbf{r}_2] \;\}$$
centrifugal Coriolis Euler

$$= -\, m_1\mathbf{r} \;\times \{\; \boldsymbol{\omega} \times (\boldsymbol{\omega} \times [\mathbf{r}_1 + \frac{m_1}{m_2}\mathbf{r}_1]) + \; 2\boldsymbol{\omega} \times [\mathbf{v}_1 + \frac{m_1}{m_2}\mathbf{v}_1] + \;\; \dot{\boldsymbol{\omega}} \times [\mathbf{r}_1 + \frac{m_1}{m_2}\mathbf{r}_1] \;\}$$

$$= -\, m_1 (1+\frac{m_1}{m_2})\, \mathbf{r}_1 \;\times \{\; \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) + \; 2\boldsymbol{\omega} \times \mathbf{v}_1 + \;\; \dot{\boldsymbol{\omega}} \times \mathbf{r}_1 \;\}$$

$$= -\, \frac{m_1}{m_2} M\, \mathbf{r}_1 \;\times \{\; \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) + \; 2\boldsymbol{\omega} \times \mathbf{v}_1 + \;\; \dot{\boldsymbol{\omega}} \times \mathbf{r}_1 \;\} \,. \qquad \text{(F.4.5)}$$

All terms involving $\dot{\mathbf{b}}_{S'}$ and $\ddot{\mathbf{b}}_{S'}$ have cancelled out. For the Earth orbit we have $\dot{\boldsymbol{\omega}} = 0$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$.

Now consider this vector identity:

$$\mathbf{C} \times [\mathbf{A} \times (\mathbf{A} \times \mathbf{C})] = \mathbf{C} \times [\, (\mathbf{A}\bullet\mathbf{C})\mathbf{A} - A^2\mathbf{C} \,] = (\mathbf{A}\bullet\mathbf{C})\, \mathbf{C} \times \mathbf{A} = -\; (\mathbf{A}\bullet\mathbf{C})\, \mathbf{A} \times \mathbf{C} \,. \qquad \text{(F.4.6)}$$

Then

$$\mathbf{r}_1 \times [\; \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) \;] = -\; (\boldsymbol{\omega}\bullet\mathbf{r}_1)\, (\boldsymbol{\omega} \times \mathbf{r}_1)$$

$$= -\, \omega^2 r_1^2 \,(\, \hat{\mathbf{x}} \bullet \hat{\mathbf{r}})(\, \hat{\mathbf{x}} \times \hat{\mathbf{r}}) \qquad \text{// use (E.2.4) for } \hat{\mathbf{x}} \bullet \hat{\mathbf{r}} \text{ and (E.2.13) for } \hat{\mathbf{x}} \times \hat{\mathbf{r}}$$

$$= -\, \omega^2 r_1^2 \,(\, \sin\theta\cos\varphi\,)(-\, \cos\theta\cos\varphi\, \hat{\boldsymbol{\varphi}} - \sin\varphi\, \hat{\boldsymbol{\theta}}\,)$$

$$= \omega^2 r_1^2 \sin\theta\cos\varphi\, (\cos\theta\cos\varphi\, \hat{\boldsymbol{\varphi}} + \sin\varphi\, \hat{\boldsymbol{\theta}}\,) \;. \qquad \text{(F.4.7)}$$

Next, the Coriolis term in (F.4.5) involves,

$$\mathbf{r}_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1) \;=\; (\mathbf{r}_1\bullet\mathbf{v}_1)\boldsymbol{\omega} - (\mathbf{r}_1\bullet\boldsymbol{\omega})\mathbf{v}_1 \qquad \text{// } \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C} \,. \qquad \text{(F.4.8)}$$

But $\mathbf{v}_1$ is tangent to the radius-r_1 sphere to which m_1 is constrained, so $(\mathbf{r}_1\bullet\mathbf{v}_1) = 0$. Then

$$\mathbf{r}_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1) = -\, (\mathbf{r}_1\bullet\boldsymbol{\omega})\mathbf{v}_1 = -\, r_1\omega\, (\hat{\mathbf{r}}\bullet\hat{\mathbf{x}})\mathbf{v}_1 \;= -\, r_1\omega \sin\theta\cos\varphi\, \mathbf{v}_1 \;. \qquad \text{(F.4.9)}$$

We now have this somewhat complicated expression for $\mathbf{N}^{(0)}{}_{\mathtt{fict}}$:

$$\mathbf{N}^{(0)}{}_{fict} = -(m_1/m_2)M\,\mathbf{r}_1 \times [\ \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) + 2\boldsymbol{\omega} \times \mathbf{v}_1\ ]$$

$$= -(m_1/m_2)M\ \{\ \mathbf{r}_1 \times [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)]\ + 2\,\mathbf{r}_1 \times (\boldsymbol{\omega} \times \mathbf{v}_1)\ \}$$

$$= -(m_1/m_2)M\ \{\ \omega^2 r_1^2 \sin\theta\cos\varphi\ [\ \cos\theta\cos\varphi\ \hat{\boldsymbol{\varphi}} + \sin\varphi\ \hat{\boldsymbol{\theta}}\ ]\ \ - 2r_1\omega\sin\theta\cos\varphi\ \mathbf{v}_1\ \} \qquad \text{(F.4.10)}$$

centrifugal Coriolis

Now using (E.3.5),

$$\mathbf{v}_1 = v_{1\theta}\hat{\boldsymbol{\theta}} + v_{1\varphi}\hat{\boldsymbol{\varphi}} = r_1\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r_1\sin\theta\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}} \qquad \textit{(E.3.5)}$$

we then have,

centrifugal Coriolis

$$\mathbf{N}^{(0)}{}_{fict} = -(m_1/m_2)M\{\omega^2 r_1^2\sin\theta\cos\varphi\ [\cos\theta\cos\varphi\ \hat{\boldsymbol{\varphi}} + \sin\varphi\ \hat{\boldsymbol{\theta}}] - 2r_1\omega\sin\theta\cos\varphi\ (r_1\dot{\theta}\,\hat{\boldsymbol{\theta}} + r_1\sin\theta\,\dot{\varphi}\,\hat{\boldsymbol{\varphi}}\ )\ \}$$

$$= -(m_1/m_2)M\ \{\ (\omega^2 r_1^2\sin\theta\cos\varphi\sin\varphi - 2r_1^2\omega\sin\theta\cos\varphi\dot{\theta})\ \hat{\boldsymbol{\theta}}$$
$$+ (\omega^2 r_1^2\sin\theta\cos\varphi\cos\theta\cos\varphi\ - 2r_1^2\omega\sin\theta\cos\varphi\sin\theta\dot{\varphi}\ )\ \hat{\boldsymbol{\varphi}}\ \ \}$$

$$= -(m_1/m_2)M\ \{\ \omega r_1^2\sin\theta\cos\varphi\ (\omega\sin\varphi - 2\dot{\theta})\ \hat{\boldsymbol{\theta}}\ + \omega r_1^2\sin\theta\cos\varphi\ (\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi})\ \hat{\boldsymbol{\varphi}}\ \ \}$$

$$= -(m_1/m_2)M\omega r_1^2\sin\theta\cos\varphi\ [\ (\omega\sin\varphi - 2\dot{\theta})\ \hat{\boldsymbol{\theta}}\ +\ (\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi})\ \hat{\boldsymbol{\varphi}}\ \ ]\ . \qquad \text{(F.4.11)}$$

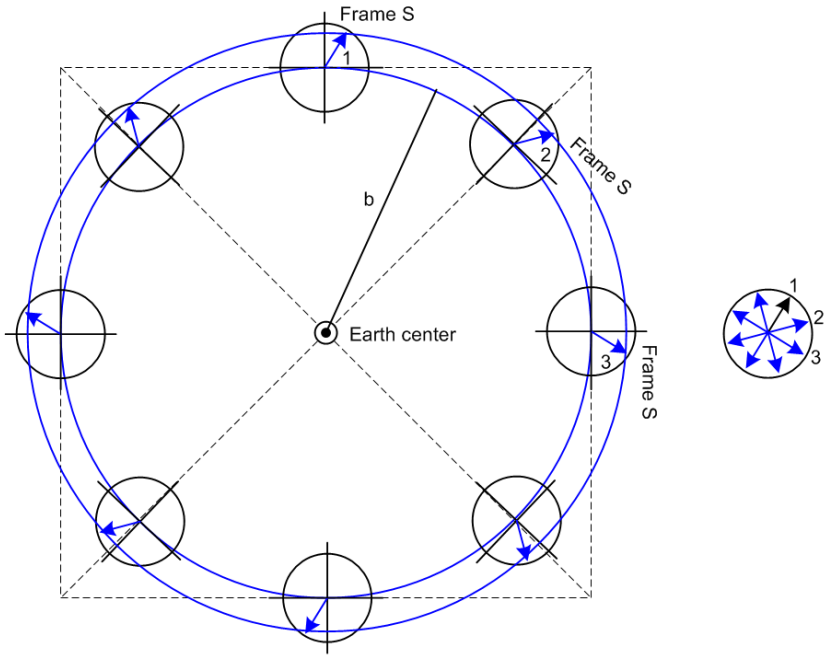
<u>How might one interpret this simple result?</u>

We examine the pieces of $\mathbf{N}^{(0)}{}_{fict} = 0$ as they appear in (F.4.11). Terms involving velocities $\dot{\theta}$ and $\dot{\varphi}$ arise from the Coriolis term, the other terms come from the centrifugal term.

(a) the "frame effects" due to the motion of **b** ($\dot{\mathbf{b}}_{S'}$ and $\ddot{\mathbf{b}}_{S'}$) are equal and opposite for the two masses because the origin of Frame S is at the center of mass causing $m_2\mathbf{r}_2 = -m_1\mathbf{r}_1$, as shown in (F.1.9).

(b) within Frame S, the (F.4.5) centrifugal acceleration term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) = -\omega^2\mathbf{r}_1$ tries to push m_1 to a larger radius. But $\mathbf{r}_1$ is constrained to lie on a sphere of radius r_1 so m_1 cannot *go* to a larger radius. This centrifugal acceleration is neutralized by part of the tension in the stick which we avoided talking about.

This centrifugal term, by the way, involves "the short vector" $\mathbf{r}_1$ and not the long vector $\mathbf{r}'_1$. We discussed this situation in Section 8.2 for an Earth-based Frame S'. In our current context, the picture that corresponds to Fig (8.2.10) is the following



(F.4.12)

The arrow in each location during the orbit represents the position vector $\mathbf{r}_1$ of mass m_1 where we assume that other effects are turned off so $\mathbf{r}_1$ stays fixed in Frame S. When these arrows are transferred to the picture on the right with common tails, we see that the tip of $\mathbf{r}_1$ does in fact go around in a circle of radius r_1 and that is why the corresponding centrifugal force acting on m_1 is $-\omega^2\mathbf{r}_1$.

(c) Within Frame S, the Coriolis force - $2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1$ tries to deflect mass m_1 "to the right" in Fig (F.1.1). But again, $\mathbf{r}_1$ is constrained to lie on a sphere of radius r_1 so m_1 cannot *deflect* to a different radius. This Coriolis force is neutralized by the rest of the tension in the stick.

F.5 Equations of Motion for the satellite in Frame S (Spherical Coordinates)

After much effort, we have arrived at this set of results for the satellite :

$$\mathbf{L}^{(0)} = (m_1/m_2)\, M\, r_1^2\, (\dot{\theta}\,\hat{\boldsymbol{\varphi}} - \dot{\varphi}\sin\theta\,\hat{\boldsymbol{\theta}}) \qquad (F.2.2)$$

$$\dot{\mathbf{L}}^{(0)} = (m_1/m_2)\, Mr_1^2\, [\,(\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta)\,\hat{\boldsymbol{\varphi}} - (2\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta)\,\hat{\boldsymbol{\theta}}\,] \quad . \qquad (F.2.3)$$

$$\mathbf{N}^{(b)} = (GM_E m_1 b r_1 \sin\theta)\,(1/r'^3_1 - 1/r'^3_2)\,\hat{\boldsymbol{\varphi}} \qquad (F.3.7)$$

$$r'^2_1 = b^2 + r_1^2 + 2br_1\cos\theta \qquad (F.1.4)$$

$$r'^2_2 = b^2 + r_2^2 - 2br_2\cos\theta \qquad (F.1.5)$$

$$\mathbf{N}^{(0)}{}_{fict} = -\,(m_1/m_2)M\omega r_1^2\sin\theta\cos\varphi\,[\,(\omega\sin\varphi - 2\dot{\theta})\,\hat{\boldsymbol{\theta}} + (\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi})\,\hat{\boldsymbol{\varphi}}\,] \,. \qquad (F.4.11)$$

Using the orbit equation (8.6.7) applied to the satellite,

$$GM_E = \omega^2 b^3 , \qquad \text{// larger b means smaller } \omega \tag{F.5.1}$$

we can rewrite the true torque above as

$$\mathbf{N}'^{(b)} = (\omega^2 b^4 m_1 r_1 \sin\theta)\,(1/r'^3_1 - 1/r'^3_2)\,\hat{\boldsymbol{\varphi}} . \tag{F.5.2}$$

Writing $\dot{\mathbf{L}}^{(0)} = \mathbf{N}'^{(b)} + \mathbf{N}^{(0)}{}_{fict}$ then gives the satellite vector equation of motion,

$$\begin{aligned}(m_1/m_2)\,Mr_1^2\ &[\,(\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta)\,\hat{\boldsymbol{\varphi}} - (2\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta)\,\hat{\boldsymbol{\theta}}\,] \\ &= (\omega^2 b^4 m_1 r_1 \sin\theta)\,(1/r'^3_1 - 1/r'^3_2)\,\hat{\boldsymbol{\varphi}} \\ &- (m_1/m_2)M\omega r_1^2 \sin\theta\cos\varphi\,[\,(\omega\sin\varphi - 2\dot{\theta})\,\hat{\boldsymbol{\theta}} + (\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi})\,\hat{\boldsymbol{\varphi}}\,] \quad .\end{aligned} \tag{F.5.3}$$

Divide all three terms by the factor $(m_1/m_2)Mr_1^2$. The coefficient of the first term on the right becomes

$$(\omega^2 b^4 m_1 r_1 \sin\theta) \,/\, [\,(m_1/m_2)Mr_1^2] = (1/r_1)\,(\omega^2 b^4 (m_2/M)\sin\theta)$$

so the vector equation of motion is then,

$$\begin{aligned}&[\,(\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta)\,\hat{\boldsymbol{\varphi}} - (2\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta)\,\hat{\boldsymbol{\theta}}\,] \\ &= (1/r_1)(\omega^2 b^4 \mu_2 \sin\theta)\,(1/r'^3_1 - 1/r'^3_2)\,\hat{\boldsymbol{\varphi}} \\ &- \omega\sin\theta\cos\varphi\,[\,(\omega\sin\varphi - 2\dot{\theta})\,\hat{\boldsymbol{\theta}} + (\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi})\,\hat{\boldsymbol{\varphi}}\,] \quad .\end{aligned} \tag{F.5.4}$$

Using now the far approximation (F.3.12) that $(1/r'^3_1) - (1/r'^3_2) = -3r_1\cos\theta/(\mu_2 b^4)$, we get

$$\begin{aligned}&[\,(\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta)\,\hat{\boldsymbol{\varphi}} - (2\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta)\,\hat{\boldsymbol{\theta}}\,] \\ &= -3\omega^2 \sin\theta\cos\theta\,\hat{\boldsymbol{\varphi}} \\ &- \omega\sin\theta\cos\varphi\,[\,(\omega\sin\varphi - 2\dot{\theta})\,\hat{\boldsymbol{\theta}} + (\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi})\,\hat{\boldsymbol{\varphi}}\,]\end{aligned} \tag{F.5.5}$$

As a reminder, the first line above is $\dot{\mathbf{L}}^{(0)}$, the second line is true torque $\mathbf{N}'^{(b)}$, and the last line is the fictitious torque $\mathbf{N}^{(0)}{}_{fict}$ created by the fact that Frame S is a rotating frame of reference, and we display the fictitious contributions in **blue** to keep track of them for a while below.

Comment: Notice that the equation of motion is independent of m_1 and m_2 and hence of r_1 and r_2. If we were to vary the ratio m_1/m_2, we just "slide the stick" in Fig (F.1.1) so the Frame S origin remains at the center of mass point. The equation does depend on b through ω, since $\omega^2 = GM_E/b^3$.

Moving all terms to the left side, (F.5.5) becomes

$$(\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta)\,\hat{\boldsymbol{\varphi}} - (2\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta)\,\hat{\boldsymbol{\theta}} \qquad \text{(F.5.6)}$$
$$+3\omega^2\sin\theta\cos\theta\,\hat{\boldsymbol{\varphi}} + \omega\sin\theta\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta})\,\hat{\boldsymbol{\theta}} + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi})\,\hat{\boldsymbol{\varphi}} = 0\,.$$

The **component equations** are then

$$\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta + 3\omega^2\sin\theta\cos\theta + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$
$$-(2\dot{\theta}\,\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta) + \omega\sin\theta\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}}$$

where the fictitious torque terms are shown in blue. Changing the sign of the second equation and making a few adjustments we get,

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$
$$\ddot{\varphi} + 2\dot{\theta}\,\dot{\varphi}\cot\theta - \omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}} \qquad \text{(F.5.7)}$$

After much effort using the fictitious torque method we have finally arrived at the **spherical equations of motion** for the dumbbell satellite!

These are two ordinary differential equations with time t as the variable. The equations are 2nd order in both θ(t) and φ(t) and they are non-linear due to terms like $\dot{\theta}\dot{\varphi}$ and $\dot{\varphi}^2$ and sinθ. Finally, the equations are coupled, so they form a system of two 2nd order, coupled, non-linear ODE's. This system of two non-linear 2nd order ODE's can be trivially replaced with an equivalent system of four non-linear 1st order ODE's as follows,

$$\dot{v}_\theta + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$
$$\dot{v}_\varphi + 2\cot\theta\,\dot{\theta}\,\dot{\varphi} - \omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}}$$
$$\dot{\theta} = v_\theta$$
$$\dot{\varphi} = v_\varphi \qquad \text{(F.5.8)}$$

We mention this only because this is the first step Maple takes (internally) when it sets about solving a pair of 2nd order ODE's with its numerical dsolve command (coming soon).

Right now, we make an anzats that (F.5.7) has a solution for which $\varphi = \pi/2$ (so $\cos\varphi = 0$) and φ does not change, so that both $\dot{\varphi}$ and $\ddot{\varphi} = 0$ at all times. For such a solution, the dumbbell lies *in the plane* of paper

of Fig (F.1.1) (in the plane of the orbit) and the only Frame S motion is in the θ degree of freedom. In this case the two equations in (F.5.7) simplify to

$$\ddot{\theta} + 3\omega^2 \sin\theta\cos\theta = 0 \qquad // \hat{\boldsymbol{\varphi}}$$
$$0 = 0 . \qquad // \text{in-plane} \qquad // \hat{\boldsymbol{\theta}} \qquad \text{(F.5.9)}$$

For small θ, the first equation becomes

$$\ddot{\theta} + 3\omega^2 \theta = 0$$

which indicates sinusoidal oscillation in θ of the dumbbell about θ = 0 with frequency $\omega_{osc}^2 = 3\omega^2$ so

$$\omega_{osc1} = \sqrt{3}\,\omega \qquad T_{osc1} = (1/\sqrt{3})\,T \approx 0.58\,T \qquad // \text{in-plane libration} \qquad \text{(F.5.10)}$$

where recall that ω is the orbital rotation rate of the satellite. For a low-Earth orbit with T = 88 minutes, the dumbbell initialized to φ = π/2 and a small angle θ would have an oscillation period of 88*.58 = 51 minutes.

Next we look for a solution with $\varphi = \dot{\varphi} = 0$ where the satellite at t = 0 is in a *plane perpendicular* to the plane of paper in Fig (F.1.1) (perp to the orbit plane). In this case the equations (F.5.7) become,

$$\ddot{\theta} + 3\omega^2 \sin\theta\cos\theta + \omega\sin\theta(\omega\cos\theta) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$
$$\sin\theta\,\ddot{\varphi} - \omega\sin\theta\,(-2\dot{\theta}) = 0 \qquad // \hat{\boldsymbol{\theta}}$$

or

$$\ddot{\theta} + (2\omega)^2 \sin\theta\cos\theta = 0 \qquad // \hat{\boldsymbol{\varphi}}$$
$$\ddot{\varphi} + 2\omega\dot{\theta} = 0 . \qquad // \text{out-of-plane} \qquad // \hat{\boldsymbol{\theta}} \qquad \text{(F.5.11)}$$

For small angles θ we have then

$$\ddot{\theta} + (2\omega)^2 \theta = 0 \qquad // \hat{\boldsymbol{\varphi}}$$
$$\ddot{\varphi} = -2\omega\dot{\theta} . \qquad // \hat{\boldsymbol{\theta}} \qquad \text{(F.5.12)}$$

The first equation implies θ oscillation at frequency

$$\omega_{osc2} = 2\omega \qquad T_{osc2} = 0.5T \qquad // \text{out-of-plane libration} \qquad \text{(F.5.13)}$$

while the second equation shows that the dumbbell cannot remain very long at φ = 0 since $\ddot{\varphi} \neq 0$, so this oscillation solution is only a temporary solution and the dumbbell is not stable in the plane φ = 0.

Both libration frequencies appear on page 126 of Cosmo and Lorenzini with a reference (on C&L p 169) to the following item,

2. Beletskii, V. V. and Levin, E. M., "Dynamics of Space Tether Systems", Advances in the Astronautical Sciences , Vol. 83. (Univelt, Inc. 1993)

Reader Exercise: Is the in-plane libration solution stable against small perturbations?

F.6 Force analysis of the satellite in Frame S (Spherical Coordinates)

Having obtained the dumbbell satellite θ,φ equations of motion in (F.5.7) using the fictitious *torques* method. we now set out to rederive these same equations using Newton's linear second law with fictitious *forces*. This time we also obtain an expression for the stick tension T which enables us to comment on the tidal forces for a static satellite positioned at θ = 0.

Obtain the three equations of motion

The only true forces on a dumbbell mass are gravity and stick tension T. From (F.3.2) we then write

$$\mathbf{F'}_1 = -(GM_E m_1/r'^3_1)\,\mathbf{r'}_1 - T\,\hat{\mathbf{r}}_1 = -(GM_E m_1/r'^3_1)(\mathbf{b}+\mathbf{r}_1) - T\,\hat{\mathbf{r}}_1$$
$$\mathbf{F'}_2 = -(GM_E m_2/r'^3_2)\,\mathbf{r'}_2 - T\,\hat{\mathbf{r}}_2 = -(GM_E m_2/r'^3_2)(\mathbf{b}+\mathbf{r}_2) - T\,\hat{\mathbf{r}}_2\ . \qquad \text{(F.6.1)}$$

As noted earlier, the center of gravity is not quite at the center of mass in Fig (F.1.1), but the above equations are exact despite this fact. The forces are primed because they are forces in the inertial Frame S'. The reader is reminded that we are using the "swap notation" where prime↔noprime relative to the non-swap notation.

In order to use Newton's Law in rotating Frame S, we must include the fictitious forces. We translate the result of (8.1.8) to swap notation to obtain

$$\mathbf{F}_{fict,1} \approx -m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1) - 2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1$$
$$\mathbf{F}_{fict,2} \approx -m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_2) - 2m_2\,\boldsymbol{\omega}\times\mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}}\times\mathbf{r}_2\ . \qquad \text{(F.6.2)}$$
frame centrifugal Coriolis Euler

In these equations, the **b** acceleration is given by the swap version of (7.13) which then states

$$\ddot{\mathbf{b}}_{S'} = \dot{\boldsymbol{\omega}}\times\mathbf{b} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{b}). \qquad \text{// Special Case \#1} \qquad (7.13)_s \qquad \text{(F.6.3)}$$

and the vectors **b** and **ω** are given as in Fig (F.1.1) by

$$\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$$
$$\mathbf{b} = b\hat{\mathbf{z}}\ . \qquad \text{(F.6.4)}$$

Finally we may state Newton's Law for each mass,

$$\mathbf{F}_{\mathbf{eff},1} = m_1 \mathbf{a}_1 \tag{F.6.5}$$
$$\approx -(GM_E m_1/r'^3_1)\mathbf{r}'_1 - T\hat{\mathbf{r}}_1 - m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) - 2m_1\boldsymbol{\omega} \times \mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1$$

$$\mathbf{F}_{\mathbf{eff},2} = m_2 \mathbf{a}_2 \tag{F.6.6}$$
$$\approx -(GM_E m_2/r'^3_2)\mathbf{r}'_2 - T\hat{\mathbf{r}}_2 - m_2\ddot{\mathbf{b}}_{S'} - m_2\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_2) - 2m_2\boldsymbol{\omega} \times \mathbf{v}_2 - m_2\dot{\boldsymbol{\omega}} \times \mathbf{r}_2$$

where we have now a set of six scalar equations. Using (F.1.9) through (F.1.11), (F.6.6) can be rewritten,

$$\mathbf{F}_{\mathbf{eff},2} = -m_1 \mathbf{a}_1 \tag{F.6.7}$$
$$\approx -(GM_E m_2/r'^3_2)\mathbf{r}'_2 + T\hat{\mathbf{r}}_1 - m_2\ddot{\mathbf{b}}_{S'} + m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) + 2m_1\boldsymbol{\omega} \times \mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1 .$$

Adding (F.6.5) and (F.6.7) gives

$$0 = -(GM_E m_1/r'^3_1)\mathbf{r}'_1 - (GM_E m_2/r'^3_2)\mathbf{r}'_2 - (m_1+m_2)\ddot{\mathbf{b}}_{S'}$$

or

$$(m_1+m_2)\ddot{\mathbf{b}}_{S'} = -(GM_E m_1/r'^3_1)\mathbf{r}'_1 - (GM_E m_2/r'^3_2)\mathbf{r}'_2 \tag{F.6.8}$$

This equation is just $\mathbf{F} = m\mathbf{a}$ in inertial Frame S' for the total satellite where $\mathbf{b}$ is the center of mass. Ignoring the small offset between center of mass and center of gravity, the three equations (F.6.8) describe the circular orbit of the satellite around the Earth. We may then regard the equation (F.6.5) as a set of three scalar equations for the three unknowns θ,φ and T where recall $\mathbf{r}_1 = (r_1,\theta,\varphi)$ in the spherical coordinates of Fig (F.1.1).

Our next task is to write vector equation (F.6.5) in spherical coordinates to obtain the three equations of motion. After expanding the left side, we then consider the right side of (F.6.5) one term at a time:

$$\mathbf{F}_{\mathbf{eff},1} = m_1 \mathbf{a}_1 \qquad \textit{(F.6.5)}$$
$$\approx -(GM_E m_1/r'^3_1)\mathbf{r}'_1 - T\hat{\mathbf{r}}_1 - m_1\ddot{\mathbf{b}}_{S'} - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) - 2m_1\boldsymbol{\omega} \times \mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1$$
$$\qquad 1 \qquad\quad 2 \qquad 3 \qquad\quad 4 \qquad\quad 5 \qquad 6$$

Left side of (F.6.5): $m_1 \mathbf{a}_1 = m_1(a_r \hat{\mathbf{r}} + a_\theta \hat{\boldsymbol{\theta}} + a_\varphi \hat{\boldsymbol{\varphi}})$ *(E.3.6)*

$$= m_1 r_1[(-\dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta)\hat{\mathbf{r}}_1 + (\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta)\hat{\boldsymbol{\theta}} + (2\dot{\theta}\dot{\varphi}\cos\theta + \ddot{\varphi}\sin\theta)\hat{\boldsymbol{\varphi}}] \tag{F.6.9}$$

Term 1: $-(GM_E m_1/r'^3_1)\mathbf{r}'_1 = -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r}_1)$ // (F.1.8)

$$= -(GM_E m_1/r'^3_1)(b\hat{\mathbf{z}} + r_1\hat{\mathbf{r}_1}) \qquad \text{// (F.6.4)}$$

$$= -(GM_E m_1/r'^3_1)(b\cos\theta\,\hat{\mathbf{r}}_1 - b\sin\theta\,\hat{\boldsymbol{\theta}} + r_1\hat{\mathbf{r}_1}) \qquad \text{// (E.2.7)}$$

$$= -(GM_E m_1/r'^3_1)[(b\cos\theta + r_1)\hat{\mathbf{r}}_1 - b\sin\theta\,\hat{\boldsymbol{\theta}}] \tag{F.6.10}$$

Term 2: $-T\,\hat{\mathbf{r}}_1$ as is (F.6.11)

Term 3: $-m_1\ddot{\mathbf{b}}_{s'} = -m_1\dot{\boldsymbol{\omega}} \times \mathbf{b} - m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$ // (F.6.3)

$= -\,m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b})$ // satellite in circular orbit, $\dot{\boldsymbol{\omega}} = 0$

$= -\,m_1(\boldsymbol{\omega}\bullet\mathbf{b})\boldsymbol{\omega} + m_1\omega^2\mathbf{b}$ //$-\,\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = -(\mathbf{A}\bullet\mathbf{C})\mathbf{A} + A^2\mathbf{C}$

$= m_1\omega^2\mathbf{b} = m_1\omega^2 b\hat{\mathbf{z}}$ // (F.6.4)

$= m_1\omega^2 b\,[\cos\theta\,\hat{\mathbf{r}}_1 - \sin\theta\,\hat{\boldsymbol{\theta}}]$ // (E.2.7) (F.6.12)

Term 4: $-m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) = -m_1(\boldsymbol{\omega}\bullet\mathbf{r}_1)\boldsymbol{\omega} + m_1\omega^2\mathbf{r}_1$ // identity shown above

$= -m_1\omega^2 r_1(\hat{\mathbf{x}}\bullet\hat{\mathbf{r}}_1)\hat{\mathbf{x}} + m_1\omega^2 r_1\,\hat{\mathbf{r}}_1$ // (F.6.4)

$= -m_1\omega^2 r_1 \sin\theta\cos\varphi\,\hat{\mathbf{x}} + m_1\omega^2 r_1\,\hat{\mathbf{r}}_1$ // (E.2.4))

$= -m_1\omega^2 r_1 \sin\theta\cos\varphi\,[\sin\theta\cos\varphi\,\hat{\mathbf{r}}_1 + \cos\theta\cos\varphi\,\hat{\boldsymbol{\theta}} - \sin\varphi\,\hat{\boldsymbol{\varphi}}] + m_1\omega^2 r_1\,\hat{\mathbf{r}}_1$ // (E.2.7)

$= -\,m_1\omega^2 r_1\,[\,(\sin^2\theta\cos^2\varphi - 1)\,\hat{\mathbf{r}}_1 + (\sin\theta\cos\theta\cos^2\varphi)\,\hat{\boldsymbol{\theta}} + (-\sin\theta\cos\varphi\sin\varphi)\,\hat{\boldsymbol{\varphi}}$ (F.6.13)

Term 5: $-2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1 = -2m_1\,[\omega\hat{\mathbf{x}}] \times (v_\theta\hat{\boldsymbol{\theta}} + v_\varphi\hat{\boldsymbol{\varphi}})$ // (E.3.5)

$= -2m_1\omega\,[v_\theta\,\hat{\mathbf{x}} \times \hat{\boldsymbol{\theta}} + v_\varphi\,\hat{\mathbf{x}} \times \hat{\boldsymbol{\varphi}}\,]$

$= -2m_1\omega\,[v_\theta\,(\sin\theta\cos\varphi\,\hat{\boldsymbol{\varphi}} + \sin\varphi\,\hat{\mathbf{r}}_1) + v_\varphi(-\sin\theta\cos\varphi\,\hat{\boldsymbol{\theta}} + \cos\theta\cos\varphi\,\hat{\mathbf{r}}_1)]$ // (E.2.13)

$= -2m_1\omega\,[\,(v_\theta\sin\varphi + v_\varphi\cos\theta\cos\varphi)\hat{\mathbf{r}}_1 + (-v_\varphi\sin\theta\cos\varphi)\,\hat{\boldsymbol{\theta}} + (v_\theta\sin\theta\cos\varphi)\,\hat{\boldsymbol{\varphi}}\,]$

$= -2m_1\omega r_1\,[\,(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi)\hat{\mathbf{r}}_1 + (-\dot{\varphi}\sin^2\theta\cos\varphi)\,\hat{\boldsymbol{\theta}} + (\dot{\theta}\sin\theta\cos\varphi)\,\hat{\boldsymbol{\varphi}}\,]$ // (E.3.2) (F.6.14)

Term 6: $-m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1 = 0$ because we assume $\dot{\boldsymbol{\omega}} = 0$ (F.6.15)

Having all the bits and pieces, we now assemble the three component equations of (F.6.5).

$$\mathbf{F}_{\mathbf{eff},1} = m_1 \mathbf{a}_1 \qquad (F.6.5)$$

$$\approx -(GM_E m_1/r'^3_1)\mathbf{r}'_1 - T\,\hat{\mathbf{r}}_1 - m_1\ddot{\mathbf{b}}_S - m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) - 2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1 - m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1$$
$$\qquad\qquad 1 \qquad\quad 2 \qquad 3 \qquad\quad 4 \qquad\quad 5 \qquad\quad 6$$

$\hat{\mathbf{r}}_1$: $m_1 r_1(-\dot\theta^2 - \dot\varphi^2 \sin^2\theta) = -(GM_E m_1/r'^3_1)(b\cos\theta + r_1) - T + m_1\omega^2 b\cos\theta - m_1\omega^2 r_1(\sin^2\theta\cos^2\varphi - 1)$
$\qquad\qquad\qquad

$$r_1 (\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta) = + (\omega^2 b^3/r'^3_1)\, b \sin\theta - \omega^2 b \sin\theta \; - \omega^2 r_1 \sin\theta\cos\theta\cos^2\varphi \; + 2\omega r_1 \dot{\varphi} \sin^2\theta\cos\varphi$$

or

$$r_1 (\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta) = +\omega^2 b \sin\theta\, [(b/r'_1)^3 - 1] \; - \omega^2 r_1 \sin\theta\cos\theta\cos^2\varphi \; + 2\omega r_1 \dot{\varphi} \sin^2\theta\cos\varphi$$

or

$$\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta = +\omega^2 b \sin\theta\, [(b/r'_1)^3 - 1]/r_1 \; - \omega^2 \sin\theta\cos\theta\cos^2\varphi \; + 2\omega\dot{\varphi} \sin^2\theta\cos\varphi \; . \qquad \text{(F.6.23)}$$

Now assume the far approximation where $r'_1, b >> r$. Recall that

$$r'^2_1 = b^2 + r_1^2 + 2 b r_1 \cos\theta$$

$$(r'_1/b)^2 \; = 1 + (r_1/b)^2 + 2(r_1/b)\cos\theta$$

$$(r'_1/b)^3 = [\; 1 + (r_1/b)^2 + 2(r_1/b)\cos\theta \;]^{3/2}$$

$$(b/r'_1)^3 \; = [\; 1 + (r_1/b)^2 + 2(r_1/b)\cos\theta \;]^{-3/2} \; \approx \; 1 + (-3/2)\, [(r_1/b)^2 + 2(r_1/b)\cos\theta \;]$$

$$\approx \; 1 + (-3/2)\, 2(r_1/b)\cos\theta \; = \; 1 - 3(r_1/b)\cos\theta$$

so

$$[(b/r'_1)^3 - 1] \; \approx - 3(r_1/b)\cos\theta \; . \qquad \text{(F.6.24)}$$

Then (F.6.23) becomes

$$\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta = +\omega^2 b \sin\theta\, [- 3(r_1/b)\cos\theta]/r_1 \; - \omega^2 \sin\theta\cos\theta\cos^2\varphi \; + 2\omega\dot{\varphi} \sin^2\theta\cos\varphi$$

or

$$\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta = -3\omega^2 \sin\theta\cos\theta \; - \omega^2 \sin\theta\cos\theta\cos^2\varphi \; + 2\omega\dot{\varphi} \sin^2\theta\cos\varphi$$

or

$$\ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta + 3\omega^2 \sin\theta\cos\theta + \omega^2 \sin\theta\cos\theta\cos^2\varphi - 2\omega\dot{\varphi} \sin^2\theta\cos\varphi \; = 0$$

or

$$\ddot{\theta} + 3\omega^2 \sin\theta\cos\theta \; - \dot{\phi}^2 \sin\theta \cos\theta \; + \omega \sin\theta\cos\varphi(\omega\cos\theta\cos\varphi - 2\dot{\varphi}\sin\theta) \; = 0 \qquad \text{(F.6.25)}$$

which **matches** the $\hat{\boldsymbol{\varphi}}$ torque equation (F.5.7),

$$\ddot{\theta} + 3\omega^2 \sin\theta\cos\theta - \dot{\phi}^2 \sin\theta \cos\theta \; + \omega \sin\theta\cos\varphi \; (\omega \cos\theta \cos\varphi - 2\sin\theta \, \dot{\varphi}) \; = 0 \quad // \; \hat{\boldsymbol{\varphi}} \qquad \textit{(F.5.7)}$$

At this point we have derived the same angular equations of motion (F.5.7) in two different ways.

Obtaining the tension in the stick (or tether)

Finally we come to the $\hat{\mathbf{r}}_1$ equation (F.6.19),

$$\hat{\mathbf{r}}_1\text{:}\quad r_1(- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) = - (\omega^2 b^3/r'^3_1)(b\cos\theta + r_1) - T/m_1 + \omega^2 b \cos\theta - \omega^2 r_1(\sin^2\theta\cos^2\varphi - 1)$$
$$- 2\omega r_1 (\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta \cos\theta\cos\varphi) . \qquad (F.6.19)$$

Inserting the far approximation (F.6.24) that $(b/r'_1)^3 \approx 1 - 3(r_1/b)\cos\theta$ into the above gives

$$r_1(- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) = -\omega^2[1 - 3(r_1/b)\cos\theta] (b\cos\theta + r_1) - T/m_1 + \omega^2 b \cos\theta - \omega^2 r_1(\sin^2\theta\cos^2\varphi - 1)$$
$$- 2\omega r_1 (\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta \cos\theta\cos\varphi)$$

$$r_1(- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) = -\omega^2(b\cos\theta + r_1) + 3\omega^2(r_1/b)\cos\theta(b\cos\theta + r_1) - T/m_1 + \omega^2 b \cos\theta$$
$$- \omega^2 r_1(\sin^2\theta\cos^2\varphi - 1) - 2\omega r_1 (\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta \cos\theta\cos\varphi)$$

$$r_1(- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) = -\omega^2 b\cos\theta - \omega^2 r_1 + 3\omega^2 r_1\cos\theta(\cos\theta + r_1/b) - T/m_1 + \omega^2 b \cos\theta$$
$$- \omega^2 r_1\sin^2\theta\cos^2\varphi + \omega^2 r_1 - 2\omega r_1 (\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta \cos\theta\cos\varphi)$$

$$r_1(- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) = 3\omega^2 r_1\cos\theta(\cos\theta + r_1/b) - T/m_1 - \omega^2 r_1\sin^2\theta\cos^2\varphi - 2\omega r_1(\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta \cos\theta\cos\varphi)$$

$$(- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) \approx 3\omega^2\cos^2\theta - T/(m_1 r_1) - \omega^2\sin^2\theta\cos^2\varphi - 2\omega(\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta \cos\theta\cos\varphi)$$

$$(- \dot{\theta}^2 - \dot{\varphi}^2 \sin^2\theta) \approx \omega^2(3\cos^2\theta - \sin^2\theta\cos^2\varphi) - T/(m_1 r_1) - 2\omega(\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta\cos\theta\cos\varphi) .$$

The stick tension is then given by (this will later be verified in Section F.9),

$$T = m_1 r_1[\omega^2(3\cos^2\theta - \sin^2\theta\cos^2\varphi) + \dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta - 2\omega(\dot{\theta} \sin\varphi + \dot{\varphi} \sin\theta\cos\theta\cos\varphi)] . \qquad (F.6.26)$$

If the angular velocities are very small such that $|\dot{\theta}| << \omega$ and $|\dot{\varphi}| << \omega$, the result becomes

$$T \approx m_1\omega^2 r_1(3\cos^2\theta - \sin^2\theta\cos^2\varphi) . \qquad \text{// small velocities} \qquad (F.6.27)$$

In Cartesian coordinates this becomes,

$$T \approx (m_1\omega^2/r_1)(3r_1^2\cos^2\theta - r_1^2\sin^2\theta\cos^2\varphi)$$
$$= (m_1\omega^2/r_1)(3z^2 - x^2) . \qquad \text{// small velocities} \qquad (F.6.28)$$

In this small-velocity limit, the tension in the stick is positive as long as

$$|x| < \sqrt{3}\,|z| \quad \Rightarrow \quad T > 0 \qquad \text{// small velocity limit} \tag{F.6.29}$$

so for small x displacements T is always positive.

As noted, in general one must solve (F.6.20) and (F.6.21) for θ(t) and φ(t), then (F.6.26) gives T(t).

Tidal Force

If the dumbbell is static at θ = 0, we see from (F.6.26) or (F.6.27) that

$$T = 3m_1\omega^2 r_1 \tag{F.6.30}$$

which we associated with a "tidal force". The factor of 3 arises from (F.6.24) which in turn arises from the power 3 in the gravitational force factor in (F.6.1),

$$(GM_E m_1/r'_1{}^3) = (\omega^2 b^3\, m_1/r'_1{}^3) = \omega^2 m_1\,(b/r'_1)^3 .$$

It happens that in Frame S' the gradient of the radial gravitational field at mass m_1 is

$$\partial_{r_1'}(-GM_E m_1/r'_1{}^2) = \partial_{r_1'}(-\omega^2 b^3 m_1/r'_1{}^2) = -\omega^2 b^3 m_1\ \partial_{r_1'}(r'_1{}^{-2}) = 2\omega^2 b^3 m_1/r'_1{}^3$$

$$= 2\omega^2 m_1 (b/r'_1)^3 \tag{F.6.31}$$

so one can *associate* the factor of 3 in (F.6.30) with this gradient. However, the *result* (F.6.30) really comes from a sum of several terms in (F.6.5) for a static dumbbell, as we now review (the $\hat{\mathbf{r}}_1$ equation):

$$\mathbf{F}_{\mathbf{eff},1} = m_1\,\mathbf{a}_1 \tag{F.6.5}$$

$$\approx \underset{1}{-(GM_E m_1/r'_1{}^3)\mathbf{r}'_1} \underset{2}{-T\,\hat{\mathbf{r}}_1} \underset{3}{-m_1\ddot{\mathbf{b}}_{S'}} \underset{4}{-m_1\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1)} \underset{5}{-2m_1\,\boldsymbol{\omega}\times\mathbf{v}_1} \underset{6}{-m_1\dot{\boldsymbol{\omega}}\times\mathbf{r}_1}$$

$$\hat{\mathbf{r}}_1\text{:}\quad m_1 r_1(-\dot{\theta}^2 - \dot{\varphi}^2\sin^2\theta) = \underset{1}{-(GM_E m_1/r'_1{}^3)(b\cos\theta + r_1)} \underset{2}{-T} + \underset{3}{m_1\omega^2 b\cos\theta} \underset{4}{-m_1\omega^2 r_1(\sin^2\theta\cos^2\varphi - 1)}$$

$$\underset{5}{-2m_1\omega r_1(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi)} \tag{F.6.16}$$

$$m_1 r_1(0 - 0) = \underset{1}{-(GM_E m_1/r'_1{}^3)(b + r_1)} \underset{2}{-T} + \underset{3}{m_1\omega^2 b} \underset{4}{-m_1\omega^2 r_1(0 - 1)} \underset{5}{-2m_1\omega r_1(0 + 0)}$$

$$0 = \underset{1}{-m_1(\omega^2 b^3/r'_1{}^3)(b + r_1)} \underset{2}{-T} + \underset{3}{m_1\omega^2 b} + \underset{4}{m_1\omega^2 r_1}$$

$$0 = \underset{1}{\{-m_1\omega^2[1 - 3(r_1/b)](b + r_1)\}} \underset{2}{-T} + \underset{3}{m_1\omega^2 b} + \underset{4}{m_1\omega^2 r_1}$$

$$0 = \{ -m_1\omega^2 \quad -m_1\omega^2 b + 3m_1\omega^2 r_1 \} - T + m_1\omega^2 b + m_1\omega^2 r_1$$
$$\qquad\qquad\quad 1 \qquad\qquad\qquad\quad 2 \qquad 3 \qquad\quad 4$$

$$0 = 3m_1\omega^2 r_1 - T \qquad\qquad (F.6.32)$$
$$\quad 1,3,4 \quad 2$$

Thus in rotating Frame S the expression (F.6.30) for the tidal force T has contributions from the gravitational gradient (term 1) and from the "frame" term $-m_1\ddot{\mathbf{b}}_S$ (term 3) which is the centrifugal contribution due to the acceleration of Frame S toward the earth, and finally from the "local centrifugal term" $-m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)$ (term 4).

F.7 Numerical solutions of the equations of motion (Spherical Coordinates)

The angular equations of motion for mass m_1 of the dumbbell (or tether) satellite are stated in (F.5.7) which we replicate here,

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0$$
$$\ddot{\varphi} + 2\dot{\theta}\,\dot{\varphi}\cot\theta - \omega\cos\varphi\,(\omega\sin\varphi - 2\dot{\theta}) = 0 \ . \qquad (F.5.7) \qquad (F.7.1)$$

The first step is to enter these two equations into Maple:

```
restart; with(plots):
td := diff(theta(t),t): tdd := diff(theta(t),t,t):
pd := diff(phi(t),t):   pdd := diff(phi(t),t,t):
ct := cos(theta(t)):  st := sin(theta(t)):
cp := cos(phi(t)):    sp := sin(phi(t)):   w := omega:
eq1 := tdd+st*ct*(3*w^2-pd^2)+w*st*cp*(w*ct*cp-2*st*pd)=0;
```

$$eq1 := \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) + \sin(\theta(t))\cos(\theta(t))\left(3\,\omega^2 - \left(\frac{\partial}{\partial t}\phi(t)\right)^2\right) + \omega\sin(\theta(t))\cos(\phi(t))\left(\omega\cos(\theta(t))\cos(\phi(t)) - 2\sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)\right) = 0$$

```
eq2 := pdd+2*td*pd*cot(theta(t))-w*cp*(w*sp-2*td)=0;
```

$$eq2 := \left(\frac{\partial^2}{\partial t^2}\phi(t)\right) + 2\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right)\cot(\theta(t)) - \omega\cos(\phi(t))\left(\omega\sin(\phi(t)) - 2\left(\frac{\partial}{\partial t}\theta(t)\right)\right) = 0$$

```
eqs := {eq1,eq2}:  omega := 1:
funcs := {theta(t),phi(t)}:
```

(F.7.2)

For illustration purposes we have set the satellite orbit frequency to $\omega = 1$ sec^{-1} so

$$\omega = 1 \qquad \Rightarrow \qquad T = 2\pi = 6.28 \text{ sec}$$

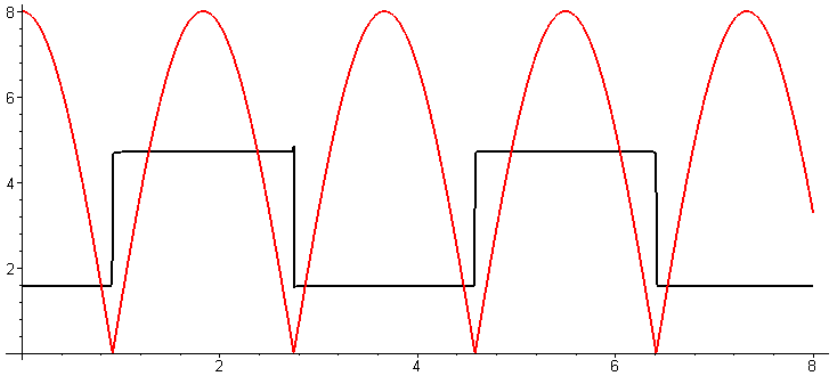
Verification of in-plane libration

The initial conditions are taken to be (see Fig (F.1.1) to see that $\varphi = \pi/2$ is the in-plane situation)

$$\theta = 0.2 \qquad \varphi = \pi/2 \qquad \dot{\theta} = 0 \qquad \dot{\varphi} = 0.01 \ . \qquad (F.7.3)$$

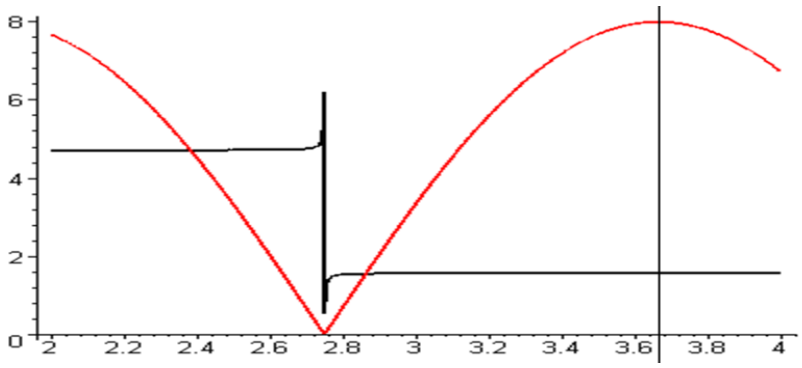
Here θ = 0.2 = 11.5° is a fairly small angle. We now call Maple's ODE solver routine dsolve and plot 40 times θ(t) in red and Mod2Pi(φ) in black. The peaks of the red curve are thus at 40*.2 = 8. Without this mod routine, the black φ curve just winds up without limit.

```
inits := {theta(0)=0.2,phi(0)=Pi/2,D(theta)(0)=0,D(phi)(0)=0.001}:
f := dsolve(eqs union inits,funcs,type=numeric,method=gear,output=listprocedure):
s1 := odeplot(f,[t,40*theta(t)],0..8,numpoints=1500,color=red,thickness=2):
s2 := odeplot(f,[t,Mod2pi(phi(t))],0..8,numpoints=1500,color=black,thickness=2):
display(s1,s2);
```



red = 40*θ black = Mod2p(φ) (F.7.4a)

Rather than plot a cosine wave, dsolve keeps θ positive all the time and has φ jump by π each time the solution passes through θ = 0. In spherical coordinates the figure shows the expected θ curve! In order to avoid exactly hitting θ = 0 which is a singular point in spherical coordinates (φ is undefined there), we have added a small $\dot{\varphi}$ = .001 to cause the dumbbell to slightly miss the z axis. One can see from the second equation in (F.7.1) that the numerical integrator is faced with $\ddot{\varphi} = 2\dot{\theta}\,\dot{\varphi}\cot\theta$ + stuff, and cotθ blows up at θ = 0 (and generates an error message in odeplot). We can expand the region t = (2,4) :



(F.7.4b)

From (F.5.10) for in-plane libration one predicts,

$$T_{ocs1} = T/\sqrt{3} = 6.28/1.73 = 3.63 \text{ sec} \qquad \text{(F.7.5)}$$

and this value is verified by the vertical line in the above figure (each tick is .04)

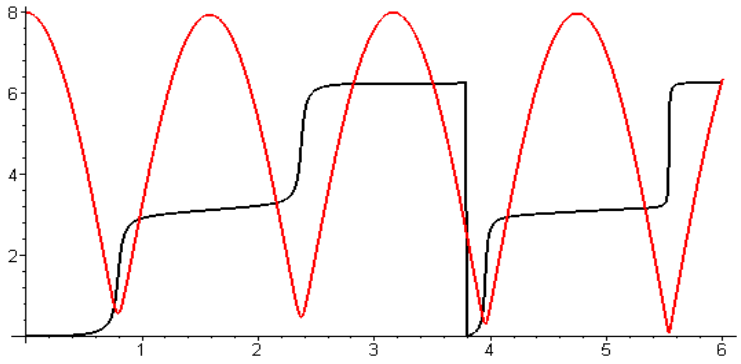
Verification of out-of-plane libration

The initial conditions are now taken to be

$$\theta = 0.2 \qquad \varphi = 0 \qquad \dot{\theta} = 0 \qquad \dot{\varphi} = 0 \tag{F.7.6}$$

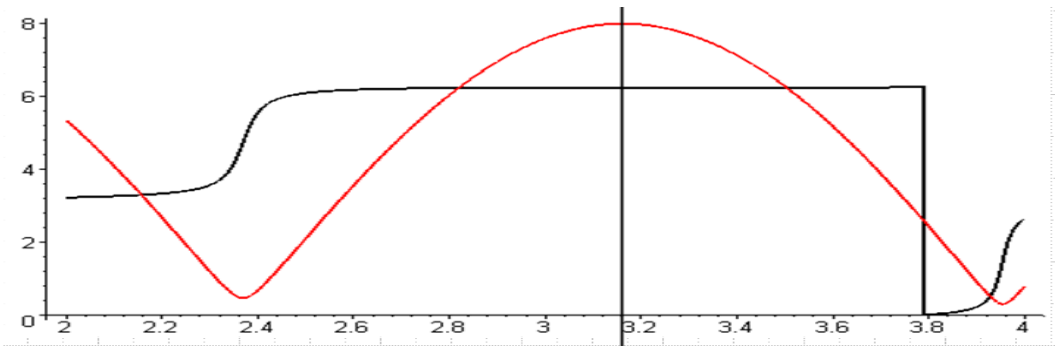
We then rerun the above code with a different set of "inits" :

```
inits := {theta(0)=0.2,phi(0)=0,D(theta)(0)=0,D(phi)(0)= 0}:
```



(F.7.7a)

Now the swing misses $\theta = 0$ of its own accord. One can see that the black φ curve starts moving away from $\varphi = 0$ at about $t = 0.5$ and in general the black φ curve has smooth rises near small θ. We again expand the region $t = (2,4)$:



(F.7.7b)

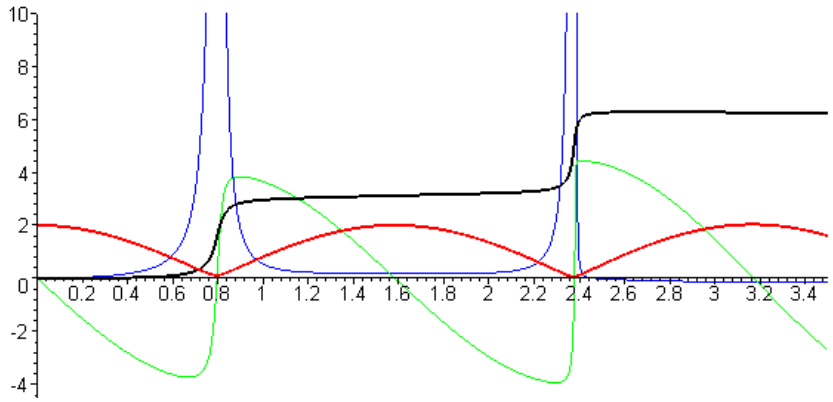
From (F.5.13) for out-of-plane libration one predicts,

$$T_{ocs2} = T/2 = 6.28/2 = 3.14 \tag{F.7.8}$$

while the figure shows about 3.16, close enough.

Maple can also plot $\dot{\theta}(t)$ and $\dot{\varphi}(t)$ and here is a plot showing all four curves :

```
s3 := odeplot(f,[t,10*diff(theta(t),t)],0..3.5,numpoints=1000,color=green):
s4 := odeplot(f,[t,diff(phi(t),t)],0..3.5,numpoints=1000,color=blue):
display(s1,s2,s3,s4,view = [0..3.5,-5..10]);
```



(F.7.9)

red = 10*θ black = φ green = 10*$\dot{\theta}$ blue = $\dot{\varphi}$

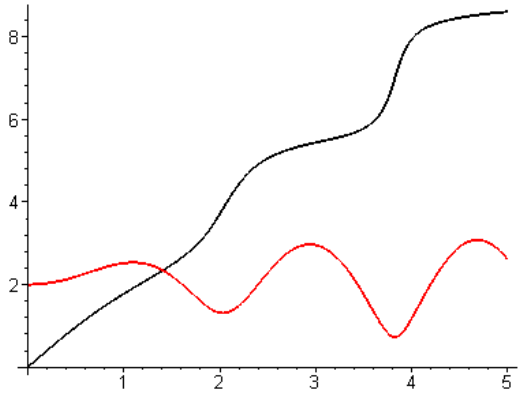
When red θ nears θ = 0, blue $\dot{\varphi}$ has major action since φ is quickly changing by π. And the green $\dot{\theta}$ of spherical coordinates has to make a radical change since it is in effect suddenly reversing course.

A more general solution

Here in addition to starting with θ(0) = 0.2 and φ(0) = 0, we provide a push in the azimuthal direction so that mass m_1 of the dumbbell satellite then swings around in azimuth while it oscillates in θ :

θ = 0.2 φ = 0 $\dot{\theta}$ = 0 $\dot{\varphi}$ = 2 . (F.7.10)

```
inits := {theta(0)=0.2,phi(0)=0,D(theta)(0)=0,D(phi)(0)=2}:
```

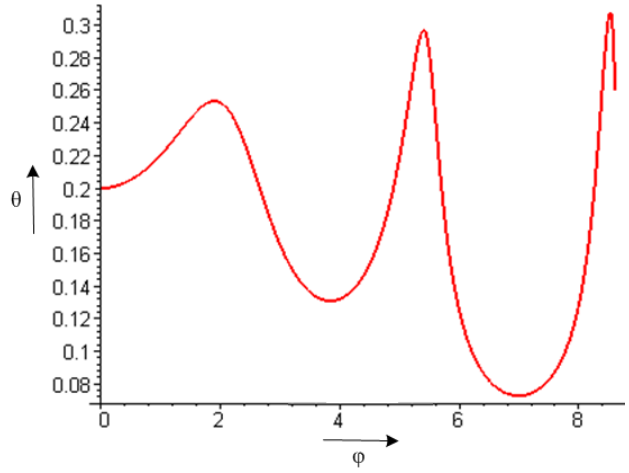


(F.7.11)

Notice that the azimuthal velocity slows down near the peaks of red θ(t). Energy is transferred back and forth between the θ and φ degrees of freedom in this system.

One can use the same odeplot routine used above to make "orbital" plots in angle space,

```
odeplot(f,[phi(t),theta(t)],0..5,numpoints=1500,color=red,thickness=2);
```



(F.7.12)

Rather than produce more plots here, we shall defer to Section F.9 where we shall plot x(t) and y(t) instead of θ(t) and φ(t).

F.8 Force analysis of the satellite in Frame S (Cartesian Coordinates)

Section F.6 used the force analysis to develop equations of motion for the dumbbell satellite in spherical coordinates. Here we repeat that development but in Cartesian coordinates where things are in many ways simpler. We follow Section F.6 down Newton's Law (F.6.5) ,

$$\mathbf{F}_{\mathbf{eff},1} = m_1 \mathbf{a}_1 \qquad \textit{(F.6.5)} \qquad \text{(F.8.1)}$$

$$\approx \underbrace{-(GM_E m_1/r'^3_1)\mathbf{r}'_1}_{1} \underbrace{- T\,\hat{\mathbf{r}}_1}_{2} \underbrace{- m_1 \ddot{\mathbf{b}}_{S'}}_{3} \underbrace{- m_1 \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)}_{4} \underbrace{- 2m_1 \boldsymbol{\omega} \times \mathbf{v}_1}_{5} \underbrace{- m_1 \dot{\boldsymbol{\omega}} \times \mathbf{r}_1}_{6} .$$

As a reminder, this is Newton's Law ($\mathbf{F}_{\mathbf{eff}} = m\mathbf{a}$) for mass m_1 of the satellite in Frame S where fictitious forces are included. Although this mass has coordinate $\mathbf{r}_1$ in Frame S, we shall refer to its components without 1 subscripts, so $\mathbf{r}_1 = (x,y,z)$. This is similar to how we used $\mathbf{r}_1 = (r_1,\theta,\varphi)$ in spherical coordinates where θ and φ had implied "1" subscripts. We also write $\mathbf{v}_1 = \mathbf{v}$ and $\mathbf{a}_1 = \mathbf{a}$ to reduce clutter. The quantity T is the tension in the stick (or tether).

We now evaluate the six terms of (F.8.1) in Cartesian coordinates, mimicking (F.6.9) through (F.6.14):

Left side of (F.8.1): $\quad m_1\mathbf{a}_1 = m_1\mathbf{a} = m_1(a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}})$ (F.8.2)

Term 1: $\quad -(GM_E m_1/r'^3_1)\mathbf{r}'_1 = -(GM_E m_1/r'^3_1)(\mathbf{b} + \mathbf{r}_1)$

$$= -(GM_E m_1/r'^3_1)\,[\, b\hat{\mathbf{z}} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\,]$$

$$= -(GM_E m_1/r'_1{}^3)\,[\, x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (b+z)\hat{\mathbf{z}}\,] \qquad \text{(F.8.3)}$$

Term 2: $$-T\,\hat{\mathbf{r}}_1 = -(T/r_1)\mathbf{r}_1 = -(T/r_1)\,[\, x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\,] \qquad \text{(F.8.4)}$$

Term 3: $$-m_1\ddot{\mathbf{b}}_{S'} = -m_1\dot{\boldsymbol{\omega}} \times \mathbf{b} - m_1\,\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad \text{// (F.6.3)}$$

$$= -\,m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \qquad \text{// satellite in circular orbit, } \dot{\boldsymbol{\omega}} = 0$$

$$= -\,m_1(\boldsymbol{\omega}\bullet\mathbf{b})\boldsymbol{\omega} + m_1\omega^2\mathbf{b} \qquad \text{//- } \mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = -(\mathbf{A}\bullet\mathbf{C})\mathbf{A} + A^2\mathbf{C}$$

$$= m_1\omega^2\mathbf{b} = m_1\omega^2 b\hat{\mathbf{z}} \qquad \text{(F.8.5)}$$

Term 4: $$-m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1) = -m_1(\boldsymbol{\omega}\bullet\mathbf{r}_1)\boldsymbol{\omega} + m_1\omega^2\mathbf{r}_1 \qquad \text{// identity shown above}$$

$$= -m_1\omega^2(\hat{\mathbf{x}}\bullet\mathbf{r}_1)\hat{\mathbf{x}} + m_1\omega^2\mathbf{r}_1 = -m_1\omega^2(x)\hat{\mathbf{x}} + m_1\omega^2\,[\, x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\,]$$

$$= m_1\omega^2\,[\, y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\,] \qquad \text{(F.8.6)}$$

Term 5: $$-2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1 = -2m_1\,[\omega\hat{\mathbf{x}}] \times (\, v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}})$$

$$= -2m_1\omega\,[\, v_y\hat{\mathbf{z}} - v_z\hat{\mathbf{y}}\,] \qquad \text{(F.8.7)}$$

Term 6: $$-m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1 = 0 \quad \text{because we assume } \dot{\boldsymbol{\omega}} = 0 \qquad \text{(F.8.8)}$$

Having all the bits and pieces, we now assemble the three component equations of (F.8.1). The numbers show the Term above associated with each piece:

$$\mathbf{F}_{\mathbf{eff},1} = m_1\,\mathbf{a} \qquad \textit{(F.8.1)}$$

$$\approx \underset{1}{-(GM_E m_1/r'_1{}^3)\mathbf{r}'_1} \;\underset{2}{-\,T\,\hat{\mathbf{r}}_1} \;\underset{3}{-\,m_1\ddot{\mathbf{b}}_{S'}} \;\underset{4}{-\,m_1\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_1)} \;\underset{5}{-\,2m_1\,\boldsymbol{\omega} \times \mathbf{v}_1} \;\underset{6}{-\,m_1\dot{\boldsymbol{\omega}} \times \mathbf{r}_1}$$

$\hat{\mathbf{x}}$: $$m_1 a_x = \underset{1}{-(GM_E m_1/r'_1{}^3)\,x} \;\underset{2}{-(T/r_1)x}$$

$\hat{\mathbf{y}}$: $$m_1 a_y = \underset{1}{-(GM_E m_1/r'_1{}^3)\,y} \;\underset{2}{-(T/r_1)y} + \underset{4}{m_1\omega^2 y} + \underset{5}{2m_1\omega v_z}$$

$\hat{\mathbf{z}}$: $$m_1 a_z = \underset{1}{-(GM_E m_1/r'_1{}^3)\,(b+z)} \;\underset{2}{-(T/r_1)z} + \underset{3}{m_1\omega^2 b} + \underset{4}{m_1\omega^2 z} - \underset{5}{2m_1\omega\, v_y} \qquad \text{(F.8.9)}$$

We now rewrite the three equations dividing by m_1 and using (F.5.1) that $GM_E = \omega^2 b^3$. At the same time we replace velocity and acceleration components with dot notation components like $\ddot{x}$ and $\dot{y}$:

$$\ddot{x} = -[(\omega^2 b^3/r'_1{}^3) + (T/m_1 r_1)]x$$

$$\ddot{y} = -[(\omega^2 b^3/r'_1{}^3) + (T/m_1 r_1) - \omega^2]y + 2\omega\dot{z}$$

$$\ddot{z} = -[(\omega^2 b^3/r'_1{}^3) - \omega^2](b+z) - (T/m_1 r_1)z - 2\omega\dot{y}$$

$$x^2+y^2+z^2 = r_1^2 \qquad \text{(F.8.10)}$$

where

$$r'_1{}^2 = (\mathbf{r}_1+\mathbf{b})^2 = r_1^2 + b^2 + 2\,\mathbf{r}_1 \bullet \mathbf{b} = r_1^2 + b^2 + 2\,\mathbf{r}_1 \bullet [b\hat{\mathbf{z}}] = r_1^2 + b^2 + 2bz\,. \qquad \text{(F.8.11)}$$

Eq. (F.8.10) is a system of 4 equations in 4 unknowns x,y,z,T. To simplify manipulations below, define

$A \equiv (\omega^2 b^3/r'_1{}^3)$
$B \equiv (T/m_1 r_1)$ // rescaled tension

so the system of equations becomes

1 $\ddot{x} = -(A + B)x$

2 $\ddot{y} = -(A + B - \omega^2)y + 2\omega\dot{z}$

3 $\ddot{z} = -(A - \omega^2)(b+z) - Bz - 2\omega\dot{y}$

4 $x^2+y^2+z^2 = r_1^2$ (F.8.12)

We now wish to eliminate the rescaled tension B from the equation set. We first eliminate B between equations 1 and 2 :

1*y $y\ddot{x} = -(A + B)xy$

2*x $x\ddot{y} = -(A + B - \omega^2)xy + 2\omega x\dot{z}$.

Subtract so that the -(A + B)xy terms cancel,

$$y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega x\dot{z}\,. \qquad \text{(F.8.13)}$$

Next, we eliminate B between equations 1 and 3:

1*z $z\ddot{x} = -(A + B)xz$

3*x $x\ddot{z} = -(A - \omega^2)(b+z)x - Bxz - 2\omega x\dot{y}$

$= -Abx - Azx + \omega^2(b+z)x - Bxz - 2\omega x\dot{y}$

$= -(A + B)xz - Abx + \omega^2(b+z)x - 2\omega x\dot{y}$.

Subtract so that the -(A + B)xy terms cancel,

$$z\ddot{x} - x\ddot{z} = Abx - \omega^2(b+z)x + 2\omega x\dot{y} \quad . \qquad \text{(F.8.14)}$$

We now have a system of 3 equations in three unknowns x,y,z, where we now restore $A = (\omega^2 b^3/r'_1{}^3)$

1 $y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega x\dot{z}$

2 $z\ddot{x} - x\ddot{z} = (\omega^2 b^3/r'_1{}^3)bx - \omega^2(b+z)x + 2\omega x\dot{y}$ where $r'_1{}^2 = r_1{}^2 + b^2 + 2bz$

3 $x^2+y^2+z^2 = r_1{}^2$. (F.8.15)

For convenience, we now reorder and rename equations 1 and 2 of this set as follows:

eq3 $z\ddot{x} - x\ddot{z} = (\omega^2 b^3/r'_1{}^3)bx - \omega^2(b+z)x + 2\omega x\dot{y}$

eq2 $y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega x\dot{z}$. (F.8.16)

Now apply the far approximation in equation eq3. From (F.6.24) we know that

$$[(b/r'_1)^3 - 1] \approx -3(r_1/b)\cos\theta \quad . \qquad \textit{(F.6.24)}$$

so

$(b/r'_1)^3 \approx 1 - 3(z/b)$ // $z = r_1\cos\theta$

and

$$A \equiv (\omega^2 b^3/r'_1{}^3) \approx \omega^2[1 - 3(z/b)] \quad . \qquad \text{(F.8.17)}$$

Equation eq3 above then becomes,

$z\ddot{x} - x\ddot{z} = (\omega^2 b^3/r'_1{}^3)bx - \omega^2(b+z)x + 2\omega x\dot{y}$

$= \omega^2[1 - 3(z/b)]bx - \omega^2(b+z)x + 2\omega x\dot{y} = \omega^2 bx - 3(z/b)\omega^2 bx - \omega^2 bx - \omega^2 zx + 2\omega x\dot{y}$

$$= -3(z/b)\omega^2 bx - \omega^2 zx + 2\omega x\dot{y} = -3\omega^2 xz - \omega^2 zx + 2\omega x\dot{y}$$

$$= -4\omega^2 xz + 2\omega x\dot{y} \ . \qquad \text{(F.8.18)}$$

This in the far approximation the equations of motion of mass m_1 of the dumbbell satellite are

eq3 $\quad z\ddot{x} - x\ddot{z} = -4\omega^2 xz + 2\omega x\dot{y}$

eq2 $\quad y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega x\dot{z}$

$$x^2+y^2+z^2 = r_1^2 \ . \qquad \text{(F.8.19)}$$

Once these three equations are solved for x(t), y(t) and z(t), we can find the tension T(t) from (say) the first equation of (F.8.12) :

$$\ddot{x}/x + A + B = 0 \quad \Rightarrow \quad B = -\ddot{x}/x - A \quad \Rightarrow \quad (T/m_1 r_1) = -\ddot{x}/x - (\omega^2 b^3/r'^3_1)$$

so

$$T = -m_1 r_1 [\ \ddot{x}/x + (\omega^2 b^3/r'^3_1)\ ]$$

so

$$T = -m_1 r_1 [\ \ddot{x}/x + \omega^2\ ] \ . \qquad \text{// far approximation (F.8.17)} \qquad \text{(F.8.20)}$$

F.9 Verification of the Cartesian equations of motion and stick tension

In Section C.7 we showed that the x,y,z Foucault pendulum equations of motion were the same as the θ,φ ones. Here we repeat that task for the dumbbell satellite equations of motion. We do these verifications to strengthen our confidence in all the equations since there are not many external sources for verification. A trusting reader can just skip this section.

First, here are the satellite angular equations of motion from (F.5.7),

eq1 $\quad \ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\ (\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) = 0$

eq2 $\quad \ddot{\varphi} + 2\cot\theta\,\dot{\theta}\,\dot{\varphi} - \omega\cos\varphi\ (\omega\sin\varphi - 2\dot{\theta}) = 0 \ .$ *(F.5.7)* $\qquad$ (F.9.1)

These were derived in (F.5.7) using the effective torque method and were then verified using the effective force method in (F.6.22) and (F.6.25).

Meanwhile, here are the Cartesian equations of motion from Section F.8,

eq3 $\quad z\ddot{x} - x\ddot{x} = -4\omega^2 xz + 2\omega x\dot{y}$

eq2 $\quad y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega x\dot{z} \ .$ *(F.8.19)* $\qquad$ (F.9.2)

Below we shall show that

Task (a): eq2 of (F.9.2) $\Rightarrow$ eq2 of (F.9.1)

Task (b): [(sinθ)*eq3 - (cosθsinφ)*eq2] of (F.9.2) $\Rightarrow$ eq1 of (F.9.1)

That is to say, angular eq1 of (F.9.1) is a certain linear combination of eq3 and eq2 of (F.9.2). If we can show Task (a) and Task (b) above, then we have shown that (F.9.2) $\Leftrightarrow$ (F.9.1), and this then serves as verification of (F.9.2).

Maple must replace x,y,z and derivatives with r_1,θ,φ and derivatives. For coordinates and first derivatives,

```
x := r1*sin(theta(t))*cos(phi(t));
```

$$x := r1\,\sin(\theta(t))\,\cos(\phi(t))$$

```
y := r1*sin(theta(t))*sin(phi(t));
```

$$y := r1\,\sin(\theta(t))\,\sin(\phi(t))$$

```
z := r1*cos(theta(t));
```

$$z := r1\,\cos(\theta(t))$$

```
xd := diff(x,t);
```

$$xd := r1\,\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\cos(\phi(t)) - r1\,\sin(\theta(t))\,\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)$$

```
yd := diff(y,t);
```

$$yd := r1\,\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t)) + r1\,\sin(\theta(t))\,\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)$$

```
zd := diff(z,t);
```

$$zd := -r1\,\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)$$

(F.9.3)

The second derivatives are messier, but Maple is happy to do the calculations,

```
> xdd := diff(x,t,t);
```

$$xdd := -r1\,\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2\cos(\phi(t)) + r1\,\cos(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\cos(\phi(t))$$

$$- 2\,r1\,\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right) - r1\,\sin(\theta(t))\,\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 - r1\,\sin(\theta(t))\,\sin(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$

```
> ydd := diff(y,t,t);
```

$$ydd := -r1\,\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2\sin(\phi(t)) + r1\,\cos(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\sin(\phi(t))$$

$$+ 2\,r1\,\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right) - r1\,\sin(\theta(t))\,\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 + r1\,\sin(\theta(t))\,\cos(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$

```
> zdd := diff(z,t,t);
```

$$zdd := -r1\,\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2 - r1\,\sin(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)$$

(F.9.4)

Task (a): Show that eq2 of (F.9.2) ⇒ eq2 of (F.9.1)

We enter eq2 of (F.9.2) and do some manipulations, suppressing the output except for the last step :

```
> eq2 := y*xdd-x*ydd +w^2*x*y + 2*w*x*zd =0:    eq2 := simplify(%):
> eq2a :=lhs(eq2)/r1^2:    eq2a :=simplify(%):
> eq2b := subs(cos(theta(t))^2=1-sin(theta(t))^2,eq2a):
> eq2c :=-expand(%);
```

$$eq2c := 2\sin(\theta(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right)+\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)\sin(\theta(t))^2 - w^2\cos(\phi(t))\sin(\phi(t))\sin(\theta(t))^2$$
$$+2\,w\cos(\phi(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\theta(t))^2 \qquad \text{(F.9.5)}$$

On the first red code line we enter eq2 of (F.9.2) and then divide the result by r_1^2. We then replace occurrences of $\cos^2\theta$ by $1-\sin^2\theta$. We use lhs = "left hand side" so we end up only with the left side of an equation which says stuff = 0. Symbol % refers to the last computed quantity. The blue result can be manually transcribed as

$$2\sin\theta\cos\theta\,\dot\theta\,\dot\varphi + \sin^2\theta\,\ddot\varphi - \omega^2\sin^2\theta\,\cos\varphi\sin\varphi + 2\omega\sin^2\theta\,\cos\varphi\,\dot\theta = 0 \ .$$

Now divide by $\sin^2\theta$ and reorder the four terms to get

$$\ddot\varphi + 2\cot(\theta)\,\dot\theta\,\dot\varphi - \omega^2\cos\varphi\sin\varphi + 2\omega\cos\varphi\,\dot\theta = 0$$

or

$$\ddot\varphi + 2\cot(\theta)\,\dot\theta\,\dot\varphi - \omega\cos\varphi(\omega\sin\varphi - 2\dot\theta) = 0 \qquad \text{(F.9.6)}$$

This is a **match** for eq2 of (F.9.1) so we have accomplished Task (a).

Task (b): $(\sin\theta)$eq3 $-(\cos\theta\sin\varphi)$eq2 of (F.9.2) ⇒ eq1 of (F.9.1)

The code continues from that shown above. Equation eq2 is already entered, so we now enter eq3, form the linear combination for eq1, then process the results with a series of typical tortuous Maple steps,

```
> eq3 := z*xdd-x*zdd + 4*w^2*x*z - 2*w*x*yd =0:    eq3 := simplify(%):
> eq1:= (sin(theta(t))*lhs(eq3)-cos(theta(t))*sin(phi(t))*lhs(eq2))/r1^2:
> eq1a := expand(eq1):
> eq1b := subs(cos(theta(t))^3=cos(theta(t))*(1-sin(theta(t))^2),eq1a):
> eq1c := factor(eq1b):
> eq1d := eq1c/(sin(theta(t))*cos(phi(t))):
> eq1e := subs(cos(theta(t))^2=1-sin(theta(t))^2,eq1d):
> expand(%);
```

$$-\sin(\theta(t))\cos(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 - 2\left(\frac{\partial}{\partial t}\phi(t)\right)\cos(\phi(t))\,w\,\sin(\theta(t))^2 + 4\,w^2\cos(\theta(t))\sin(\theta(t))$$
$$-\sin(\theta(t))\cos(\theta(t))\sin(\phi(t))^2 w^2 + \left(\frac{\partial^2}{\partial t^2}\theta(t)\right) \qquad \text{(F.9.7)}$$

We again manually transcribe the result

$$-\sin\theta\cos\theta\,\dot{\varphi}^2 - 2\omega\sin^2\theta\cos\varphi\,\dot{\varphi} + 4\omega^2\sin\theta\cos\theta - \omega^2\sin\theta\cos\theta\sin^2\varphi + \ddot{\theta}$$

or

$$-\sin\theta\cos\theta\,\dot{\varphi}^2 - 2\omega\sin^2\theta\cos\varphi\,\dot{\varphi} + 3\omega^2\sin\theta\cos\theta + \omega^2\sin\theta\cos\theta - \omega^2\sin\theta\cos\theta\sin^2\varphi + \ddot{\theta}$$

or

$$\ddot{\theta} + \sin\theta\cos\theta\,3\omega^2 - \sin\theta\cos\theta\,\dot{\varphi}^2 + \omega^2\sin\theta\cos\theta - \omega^2\sin\theta\cos\theta\sin^2\varphi - 2\omega\sin^2\theta\cos\varphi\,\dot{\varphi}$$

or

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega^2\sin\theta\cos\theta(1-\sin^2\varphi) - 2\omega\sin^2\theta\cos\varphi\,\dot{\varphi}$$

or

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega^2\sin\theta\cos\theta\cos^2\varphi - 2\omega\sin^2\theta\cos\varphi\,\dot{\varphi}$$

or

$$\ddot{\theta} + \sin\theta\cos\theta(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta\cos\varphi\,(\omega\cos\theta\cos\varphi - 2\sin\theta\,\dot{\varphi}) \qquad \text{(F.9.8)}$$

and after "pulling teeth" we do end up with a **match** for eq1 of (F.9.1), so Task (b) is accomplished.

Tension equation verification

Using the angular equations of motion (F.9.1), we now show that the following two tension expressions are the same (the first is angular (F.6.26) while the second is Cartesian (F.8.20)) ,

$$T/(m_1 r_1) = \omega^2(3\cos^2\theta - \sin^2\theta\cos^2\varphi) + \dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta - 2\omega(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi) \quad \textit{(F.6.26)}$$
$$T/(m_1 r_1) = -\left[\ \ddot{x}/x + \omega^2\ \right]. \qquad \textit{(F.8.20)} \qquad \text{(F.9.9)}$$

Our task of showing (F.9.9) is the same as showing that

$$x\,[\omega^2(3\cos^2\theta - \sin^2\theta\cos^2\varphi) + \dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta - 2\omega(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi)\,] = -\ddot{x} - \omega^2 x$$

or

$$x\,[\omega^2(3\cos^2\theta - \sin^2\theta\cos^2\varphi + 1) + \dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta - 2\omega(\dot{\theta}\sin\varphi + \dot{\varphi}\sin\theta\cos\theta\cos\varphi)\,] = -\ddot{x}$$

or

$$\text{LHS} = \text{RHS}\,. \qquad \text{(F.9.10)}$$

We first get the complicated left hand side LHS entered:

```
> restart;
> ct := cos(theta(t)):     st := sin(theta(t)):
> cp := cos(phi(t)):       sp := sin(phi(t)):
> td := diff(theta(t),t): pd := diff(phi(t),t):
> LHS := x*(w^2*(3*ct^2-st^2*cp^2+1)+td^2+pd^2*st^2-2*w*(td*sp+pd*st*ct*cp));
```

$$LHS := x\left(w^2\left(3\cos(\theta(t))^2 - \sin(\theta(t))^2\cos(\phi(t))^2 + 1\right) + \left(\frac{\partial}{\partial t}\theta(t)\right)^2 + \left(\frac{\partial}{\partial t}\phi(t)\right)^2\sin(\theta(t))^2 - 2w\left(\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t)) + \left(\frac{\partial}{\partial t}\phi(t)\right)\sin(\theta(t))\cos(\theta(t))\cos(\phi(t))\right)\right)$$

(F.9.11)

We then compute RHS = $-\ddot{x}$ as done earlier in this section,

```
> x := r1*sin(theta(t))*cos(phi(t)):
> xd := diff(x,t):          xdd := diff(x,t,t):
> RHS := -xdd;
```

$$RHS := r1\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2\cos(\phi(t)) - r1\cos(\theta(t))\left(\frac{\partial^2}{\partial t^2}\theta(t)\right)\cos(\phi(t)) + 2\,r1\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right) + r1\sin(\theta(t))\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2 + r1\sin(\theta(t))\sin(\phi(t))\left(\frac{\partial^2}{\partial t^2}\phi(t)\right)$$

(F.9.12)

Notice that RHS contains second derivatives $\ddot{\theta}$ and $\ddot{\varphi}$. We shall eliminate these derivatives by manually solving the angular equations of motion (F.9.1) for Tdd = $\ddot{\theta}$ and Pdd = $\ddot{\varphi}$:

```
Tdd := -st*ct*(3*w^2-pd^2)-w*st*cp*(w*ct*cp-2*st*pd);
```

$$Tdd := -\sin(\theta(t))\cos(\theta(t))\left(3w^2 - \left(\frac{\partial}{\partial t}\phi(t)\right)^2\right) - w\sin(\theta(t))\cos(\phi(t))\left(w\cos(\theta(t))\cos(\phi(t)) - 2\sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)\right)$$

```
Pdd := -2*td*pd*cot(theta(t))+w*cp*(w*sp-2*td);
```

$$Pdd := -2\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right)\cot(\theta(t)) + w\cos(\phi(t))\left(w\sin(\phi(t)) - 2\left(\frac{\partial}{\partial t}\theta(t)\right)\right)$$

To show that LHS = RHS, we define d = LHS-RHS and show that d = 0:

(F.9.13)

```
> d := LHS-RHS:
> d1 := subs(diff(theta(t),t,t)=Tdd,diff(phi(t),t,t)=Pdd,d);
```

$$d1 := rl\sin(\theta(t))\cos(\phi(t))\left(w^2(3\cos(\theta(t))^2-\sin(\theta(t))^2\cos(\phi(t))^2+1)+\left(\frac{\partial}{\partial t}\theta(t)\right)^2+\left(\frac{\partial}{\partial t}\phi(t)\right)^2\sin(\theta(t))^2\right.$$
$$\left.-2w\left(\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t))+\left(\frac{\partial}{\partial t}\phi(t)\right)\sin(\theta(t))\cos(\theta(t))\cos(\phi(t))\right)\right)-rl\sin(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)^2\cos(\phi(t))+rl\cos(\theta(t))$$
$$\left(-\sin(\theta(t))\cos(\theta(t))\left(3w^2-\left(\frac{\partial}{\partial t}\phi(t)\right)^2\right)-w\sin(\theta(t))\cos(\phi(t))\left(w\cos(\theta(t))\cos(\phi(t))-2\sin(\theta(t))\left(\frac{\partial}{\partial t}\phi(t)\right)\right)\right)\cos(\phi(t))$$
$$-2rl\cos(\theta(t))\left(\frac{\partial}{\partial t}\theta(t)\right)\sin(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)-rl\sin(\theta(t))\cos(\phi(t))\left(\frac{\partial}{\partial t}\phi(t)\right)^2$$
$$-rl\sin(\theta(t))\sin(\phi(t))\left(-2\left(\frac{\partial}{\partial t}\theta(t)\right)\left(\frac{\partial}{\partial t}\phi(t)\right)\cot(\theta(t))+w\cos(\phi(t))\left(w\sin(\phi(t))-2\left(\frac{\partial}{\partial t}\theta(t)\right)\right)\right)$$

```
> d2 := simplify(d1);
```

$$d2 := 0$$

(F.9.14)

Thus d = 0 and LHS = RHS and the two expressions for T in (F.9.9) are the same.

F.10 Numerical solutions of the equations of motion (Cartesian Coordinates)

We have done a lot of "work" in this Appendix F, and now it is time to "play", making use of our hard-won Cartesian equations of motion which don't have the singularity problems had by the angular equations at θ = 0.

Our task is to solve the set of equations (F.8.19) (eq1 now has a new meaning) :

eq1 $x^2+y^2+z^2 = r_1^2$

eq2 $y\ddot{x} - x\ddot{y} = -\omega^2 xy - 2\omega x\dot{z}$

eq3 $z\ddot{x} - x\ddot{z} = -4\omega^2 xz + 2\omega x\dot{y}$ *(F.8.19)* (F.10.1)

These equations describe the motion of mass m_1 of the dumbbell satellite in rotating Frame S as depicted in Fig (F.1.1). The position of mass m_1 is (x,y,z) where $x^2+y^2+z^2 = r_1^2$. The motion of mass m_2 is then determined by (D.2.8) $m_1\mathbf{r}_1 = -m_2\mathbf{r}_2$ so $(x_2,y_2,z_2) = -(m_1/m_2)(x,y,z)$. The length of the stick of the dumbbell satellite is $s = r_1 + r_2 = r_1 + (m_1/m_2)r_1 = [1 + (m_1/m_2)]\, r_1$.

We enter eq2 and eq3 writing derivatives for example as $\ddot{x}$ = xdd (w = ω) ,

```
xdd := diff(x(t),t,t):  ydd := diff(y(t),t,t): yd := diff(y(t),t):
eq2 := y(t)*xdd - x(t)*ydd = -w^2*x(t)*y(t)-2*w*x(t)*zd;
```

$$eq2 := y(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right)-x(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right) = -w^2 x(t)\,y(t) - 2w\,x(t)\,zd$$

```
eq3 := z(t)* xdd - x(t)*zdd = -4*w^2*x(t)*z(t) + 2*w*x(t)*yd;
```

$$eq3 := z(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right)-x(t)\,zdd = -4w^2 x(t)\,z(t) + 2w\,x(t)\left(\frac{\partial}{\partial t}y(t)\right)$$

(F.10.2)

At this point zd = $\dot{z}$ and zdd = $\ddot{z}$ are unspecified. We use eq1 to compute $\dot{z}$ and $\ddot{z}$ in terms of x and y, using eq1 above:

```
z := (t) -> sqrt(r1^2-x(t)^2-y(t)^2);
```

$$z := t \to \sqrt{r1^2 - x(t)^2 - y(t)^2}$$

```
zd := diff(z(t),t);
```

$$zd := \frac{1}{2}\frac{-2\,x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right)}{\sqrt{r1^2 - x(t)^2 - y(t)^2}}$$

```
zdd := diff(z(t),t,t);
```

$$zdd := -\frac{1}{4}\frac{\left(-2\,x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right)\right)^2}{\left(r1^2 - x(t)^2 - y(t)^2\right)^{\left(\frac{3}{2}\right)}} + \frac{1}{2}\frac{-2\left(\frac{\partial}{\partial t}x(t)\right)^2 - 2\,x(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - 2\left(\frac{\partial}{\partial t}y(t)\right)^2 - 2\,y(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right)}{\sqrt{r1^2 - x(t)^2 - y(t)^2}}$$

(F.10.3)

When these expressions are installed, eq2 and eq3 becomes these formidable-looking equations which contain two unknown functions x(t) and y(t) and constants r_1 and ω :

```
> eq2;
```

$$y(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - x(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right) = -w^2\,x(t)\,y(t) - \frac{w\,x(t)\left(-2\,x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right)\right)}{\sqrt{r1^2 - x(t)^2 - y(t)^2}}$$

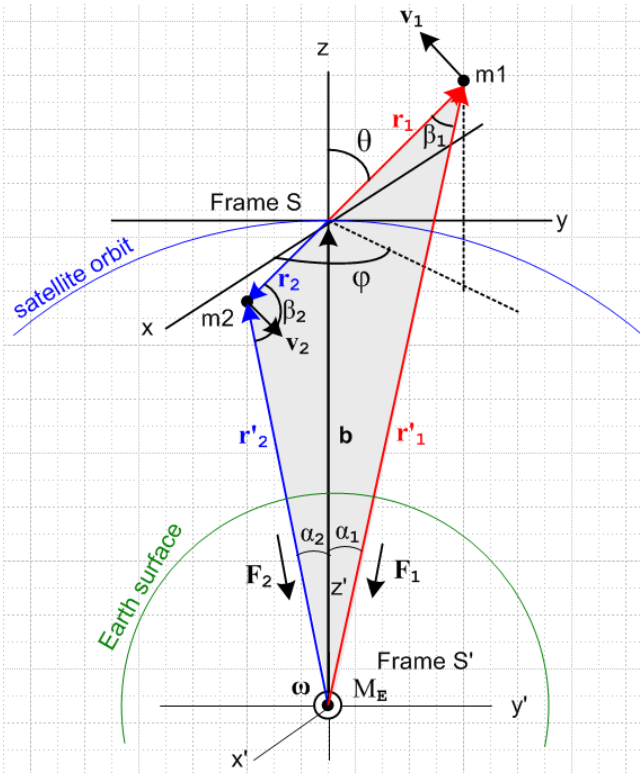
```
> eq3;
```

$$\sqrt{r1^2 - x(t)^2 - y(t)^2}\left(\frac{\partial^2}{\partial t^2}x(t)\right) - x(t)\left(-\frac{1}{4}\frac{\left(-2\,x(t)\left(\frac{\partial}{\partial t}x(t)\right) - 2\,y(t)\left(\frac{\partial}{\partial t}y(t)\right)\right)^2}{\left(r1^2 - x(t)^2 - y(t)^2\right)^{\left(\frac{3}{2}\right)}} + \frac{1}{2}\frac{-2\left(\frac{\partial}{\partial t}x(t)\right)^2 - 2\,x(t)\left(\frac{\partial^2}{\partial t^2}x(t)\right) - 2\left(\frac{\partial}{\partial t}y(t)\right)^2 - 2\,y(t)\left(\frac{\partial^2}{\partial t^2}y(t)\right)}{\sqrt{r1^2 - x(t)^2 - y(t)^2}}\right) =$$

$$-4\,w^2\,x(t)\sqrt{r1^2 - x(t)^2 - y(t)^2} + 2\,w\,x(t)\left(\frac{\partial}{\partial t}y(t)\right)$$

(F.10.4)

The reader is reminded of the geometry of Fig (F.1.1) where z points up, away from Earth center, y points to the right and is in the plane of the satellite orbit, while x is perpendicular to the plane of the satellite.



(F.10.5)

Our plots below in the (x,y) plane are what a viewer would see looking at mass m_1 "from above", that is, from a point at perhaps z = b+2s on the z axis in the above figure.

In-Plane Libration

As our first test, we shall look for the "in-plane libration" behavior. We examined this behavior earlier below (F.7.3) in angular coordinates, and we now look in Cartesian coordinates. The initial conditions are:

$$x(0) = 0 \qquad y(0) = 1 \qquad \dot{x}(0) = 0 \qquad \dot{y}(0) = 0 \qquad \text{(F.10.6)}$$

With ω = 1, we expect to get a simple swinging back in the x=0 plane with period T = 3.63 sec as shown in (F.7.5). A half period is then 1.82 seconds.

The Maple code to invoke a solution is as follows (for a numerical integration from t = 0 to t = 1.82 sec):

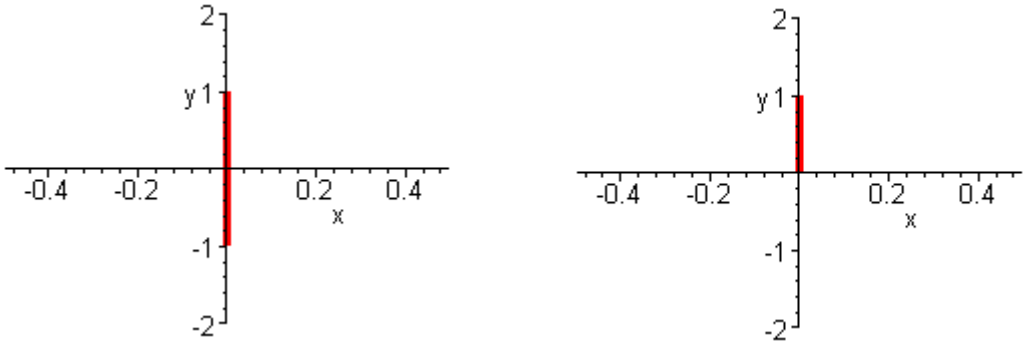
```
eqs := {eq2,eq3}:
funcs := {x(t),y(t)}:
inits := {x(0)=0,y(0)=1, D(x)(0)=0,D(y)(0)=0}:
f := dsolve(eqs union inits,funcs,type=numeric,method=rkf45,output=listprocedure);
```

$$f := \left[t = (\text{proc}(t) \ \ldots \ \text{end}),\ y(t) = (\text{proc}(t) \ \ldots \ \text{end}),\ \frac{\partial}{\partial t} y(t) = (\text{proc}(t) \ \ldots \ \text{end}),\ x(t) = (\text{proc}(t) \ \ldots \ \text{end}),\ \frac{\partial}{\partial t} x(t) = (\text{proc}(t) \ \ldots \ \text{end}) \right]$$

```
odeplot(f,[x(t),y(t)],0..1.82,numpoints=1000,axes = boxed,labels = [x,y], view =
[-1..1,-2..2],thickness=4,axes=normal );
```

(F.10.7)

We show the result below on the left, and then from t = 0 to t = 0.91 (quarter period) on the right :



(F.10.8)

Thus both the "orbit" and the period for in-plane libration are visually confirmed. If we run from t = 0 to t = 10, the graph is as on the left above since mass m_1 just swings back an forth in the same orbit, never leaving the y-axis.

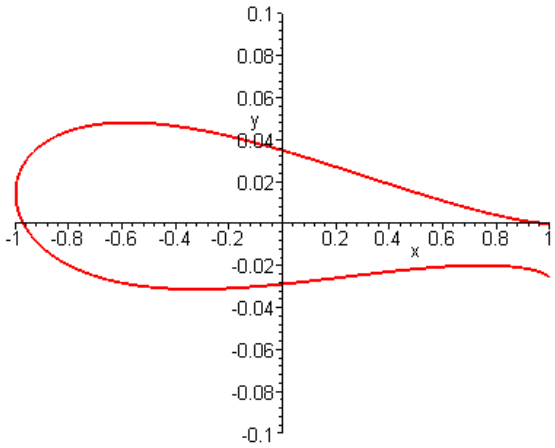
Out-of-Plane Libration

We examined this behavior earlier below (F.7.6) in angular coordinates, and we now look in Cartesian coordinates. The initial conditions are now,

$$x(0) = 1 \qquad y(0) = 0 \qquad \dot{x}(0) = 0 \qquad \dot{y}(0) = 0 \qquad \text{(F.10.9)}$$

With ω = 1, we expect to get a swinging back in the y=0 with period T = 3.14 sec as shown in (F.7.8). Here is what Maple has to say:

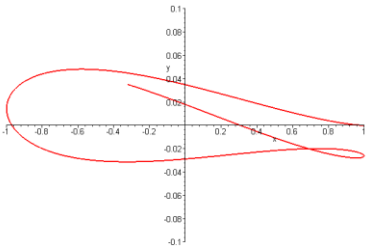
```
eqs := {eq2,eq3}:
funcs := {x(t),y(t)}:
inits := {x(0)=1,y(0)=0, D(x)(0)=0,D(y)(0)=0}:
f := dsolve(eqs union inits,funcs,type=numeric,method=rkf45,output=listprocedure):
odeplot(f,[x(t),y(t)],0..3.14,numpoints=1000,axes = boxed,labels = [x,y], view =
[-1..1,-0.1..0.1],thickness=2,axes=normal );
```



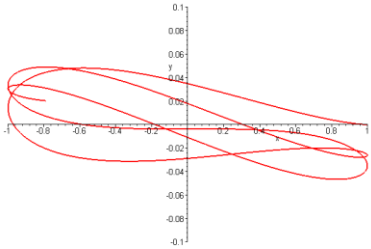
(F.10.10)

The period looks right since mass m_1 swings back close to its initial position after 3.14 seconds, but one sees that the motion is not quite in the y=0 plane, so out-of-plane libration is an approximate concept as we noted earlier below (F.5.13). Note in the figure the fine scale of the vertical axis relative to horizontal.

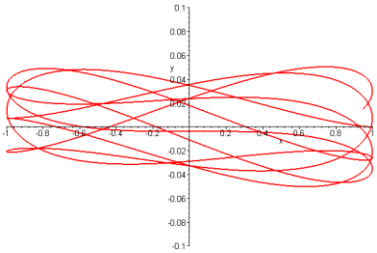
Here are orbits for a selection of final integration times:



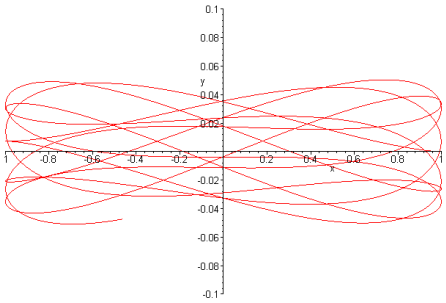
t = 4.1



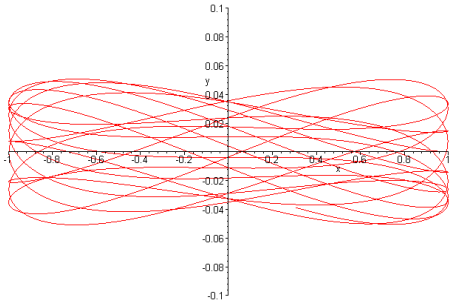
t = 8.2



t = 15.9



t = 21



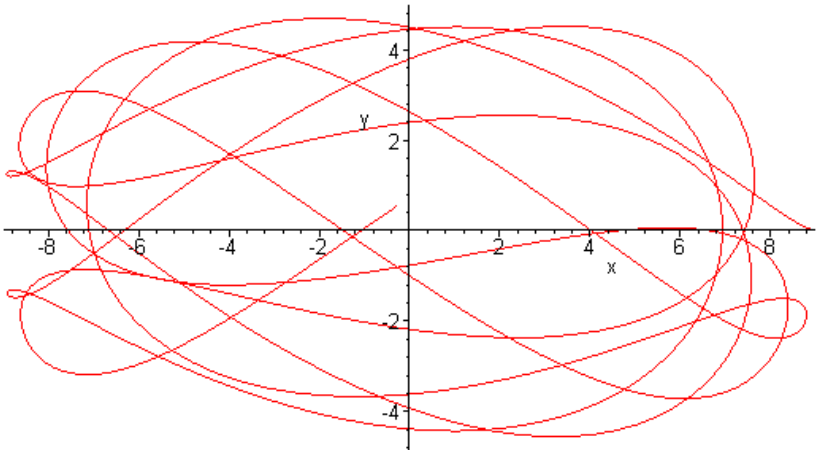
t = 41.6

(F.10.11)

It does seem that the out-of-plane libration stays within a certain small band of deviation in the y direction which we shall leave to the reader to theoretically calculate. In each plot the time was selected to make the tail of the trace clearly visible. The author is reminded of a lecture given by Shelly Glashow on the question: What can be said about orbits on an arbitrarily-shaped-but-convex billiard table? Do they all eventually close on themselves, or might some never close? (Exercise for the reader). As regards the above plots, the fact that the ratio of the two libration frequencies is an irrational number $\sqrt{3}/2$ might have some bearing on the closure of the orbits.

In the examples above $r_1 = 10$ and we have used x(0) = 1 or y(0) = 1 to obtain "small oscillation". If in the in-plane libration case we use y(0) = 9, there is no change in the orbit, but the oscillation period is slightly altered. If in the out-of-plane libration case we use x(0) = 9, the orbit no longer maintains the narrow band as in the above examples. For example, going again to t = 64 seconds with x(0) = 9,

```
inits := {x(0)=9,y(0)=0, D(x)(0)=0,D(y)(0)=0}:  Digits := 14:
f := dsolve(eqs union inits,funcs,type=numeric,method=rkf45,output=listprocedure):
odeplot(f,[x(t),y(t)],0..80,numpoints=2000,axes = boxed,labels = [x,y],view =
[-9..9,-5..5],thickness=1,axes=normal,scaling=constrained );
```

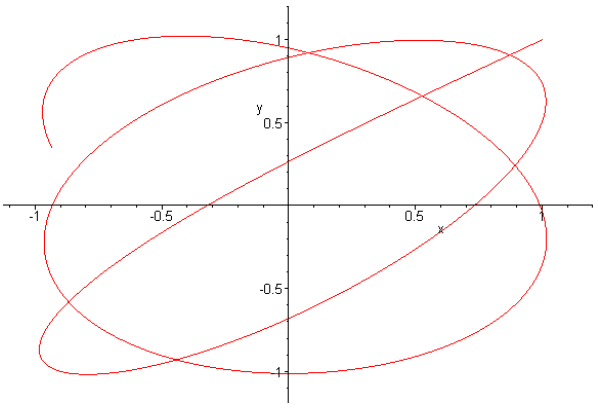


(F.10.12)

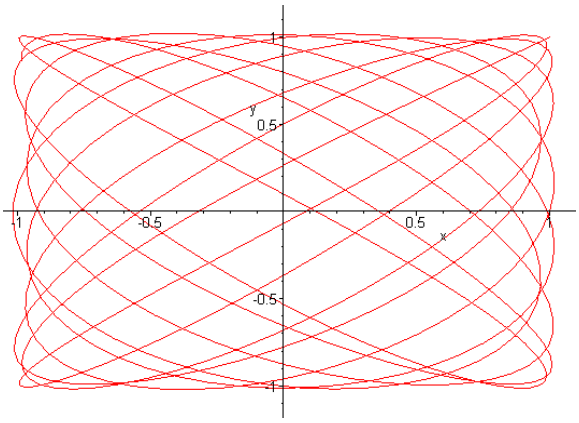
The Digits := 14 command tells Maple to compute the numerical integration with 14 decimal places of accuracy instead of the default 10 digits.

<u>Starting with a diagonal initial position</u>

x(0) = 1 y(0) = 1 $\dot{x}(0) = 0$ $\dot{y}(0) = 0$



t = 8



t = 64

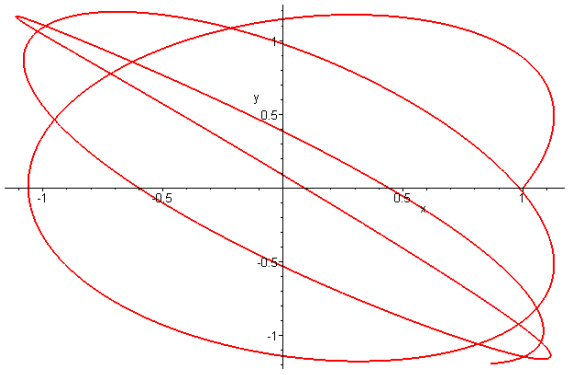
(F.10.13)

The t=64 result is reminiscent of a Lissajous pattern on an oscilloscope screen when the x and y axes are driven by different frequency sine ways. (See sine plots below.) In some sense these are the two libration frequencies.

Attempting a Circular Orbit (Conical Solution)

We have made many attempts to get a circular-like orbit by giving the mass m_1 an initial velocity kick in some useful direction, but this system does not want to cooperate. Here is an example :

```
inits := {x(0)=1,y(0)=0, D(x)(0)=1,D(y)(0)=2}:  Digits := 14:
f := dsolve(eqs union inits,funcs,type=numeric,method=rkf45,output=listprocedure):
odeplot(f,[x(t),y(t)],0..10,numpoints=1000,axes = boxed,labels =
[x,y],thickness=1,axes=normal );
```



(F.10.14)

What starts as a rough circle is soon distorted into a narrow orbit. In the case of the spherical pendulum we had a Conical Solution in (C.5.20) where $\theta = \theta_0$ and $\dot{\varphi}$ = constant. Assuming $\theta = \theta_0$ in the satellite angular equations (F.5.7) gives

$$\sin\theta_0\cos\theta_0(3\omega^2 - \dot{\varphi}^2) + \omega\sin\theta_0\cos\varphi\,(\omega\cos\theta_0\cos\varphi - 2\sin\theta_0\,\dot{\varphi}) = 0 \qquad // \hat{\boldsymbol{\varphi}}$$

$$\ddot{\varphi} - \omega\cos\varphi\,(\omega\sin\varphi) = 0\ . \qquad // \hat{\boldsymbol{\theta}}$$

Since this is two ODE's for the one function $\varphi(t)$, it seems unlikely there is any general non-static solution. If $\dot{\varphi} = 0$ the equations become

$$\sin\theta_0\cos\theta_0(3\omega^2) + \omega\sin\theta_0\cos\varphi\,(\omega\cos\theta_0\cos\varphi) = 0$$

$$-\ \omega\cos\varphi\,(\omega\sin\varphi) = 0\ .$$

If $\varphi = 0$ the first equation requires that $\theta_0 = 0$ or $\pi/2$ which are static vertical and horizontal positions. The same is true for $\varphi = \pi/2$.

Three-dimensional plots

To make such plots, one must first extract the solution functions from the dsolve environment. For details on how this works and other information on dsolve (including a debugger's guide), see the author's Maple User Guide. Here we extract the functions calling them X,Y and Z ,

```
X := subs(f,x(t));
```

$$X := \mathrm{proc}(t) \ldots \mathrm{end}$$

```
Y := subs(f,y(t));
```

$$Y := \mathrm{proc}(t) \ldots \mathrm{end}$$

```
Z := (t) ->sqrt(r1^2-X(t)^2-Y(t)^2);
```

$$Z := t \rightarrow \sqrt{r1^2 - X(t)^2 - Y(t)^2}$$

(F.10.15)

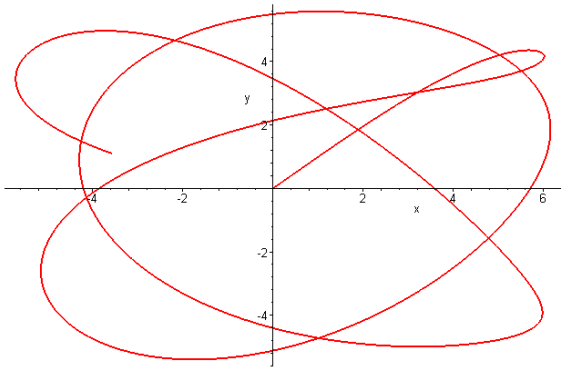
The following code then creates a 3D orbit and superposes it on a contour sphere of radius r_1 ,

```
s1 := spacecurve(['X(t)','Y(t)','Z(t)'],t=0..10,numpoints=100,thickness=2,color=red):
s2 :=implicitplot3d(x^2+y^2+z^2 =
r1^2,x=-r1..r1,y=-r1..r1,z=-r1..r1,scaling=constrained,style=contour):
display(s1,s2);
```

(F.10.16)

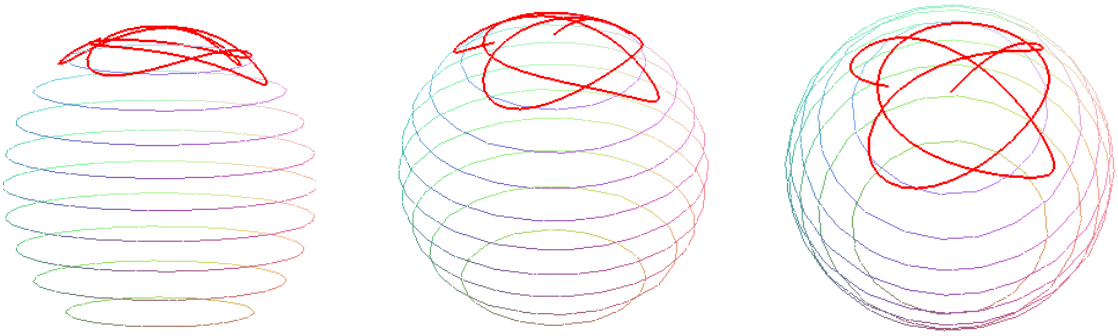
For a sample application, we start mass m_1 on the z axis and give it a good kick in the x and y directions with $\dot{x}(0) = 10$ and $\dot{y}(0) = 10$ to get an x,y plot :

```
inits := {x(0)=0,y(0)=0, D(x)(0)=10,D(y)(0)=10}:  Digits := 14:
f := dsolve(eqs union inits,funcs,type=numeric,method=rkf45,output=listprocedure):
odeplot(f,[x(t),y(t)],0..10,numpoints=1000,axes = boxed,labels =
[x,y],thickness=2,axes=normal );
```



(F.10.17)

Here then is the corresponding 3D plot



(F.10.18)

where the sphere of radius r_1 is gradually tipped down toward the viewer.

Conventional plots

In order to plot x(t), y(t) and so on, we first extract all functions from the dsolve system and then crudely add missing pieces like Xdd:

```
dt := 1e-8:

X := subs(f,x(t)):
Xd := subs(f,diff(x(t),t)):
Xdd := (t) -> (Xd(t+dt)- Xd(t))/dt:

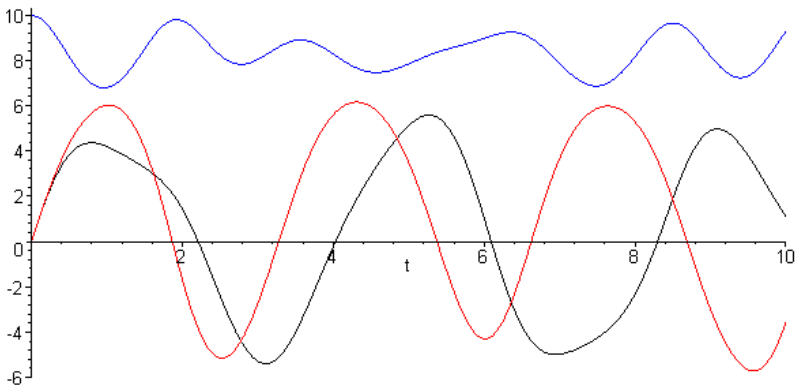
Y := subs(f,y(t)):
Yd := subs(f,diff(y(t),t)):
Ydd := (t) -> (Yd(t+dt)- Yd(t))/dt:

Z := (t) -> sqrt(r1^2-X(t)^2-Y(t)^2):
Zd := (t) -> (Z(t+dt)- Z(t))/dt:
Zdd := (t) -> (Zd(t+dt)- Zd(t))/dt:
```

(F.10.19)

Here then is a plot of x(t),y(t),z(t) = red,black,blue for the above example:

```
plot(['X(t)','Y(t)','Z(t)'],t=0..10, color = [red,black,blue],numpoints=100);
```



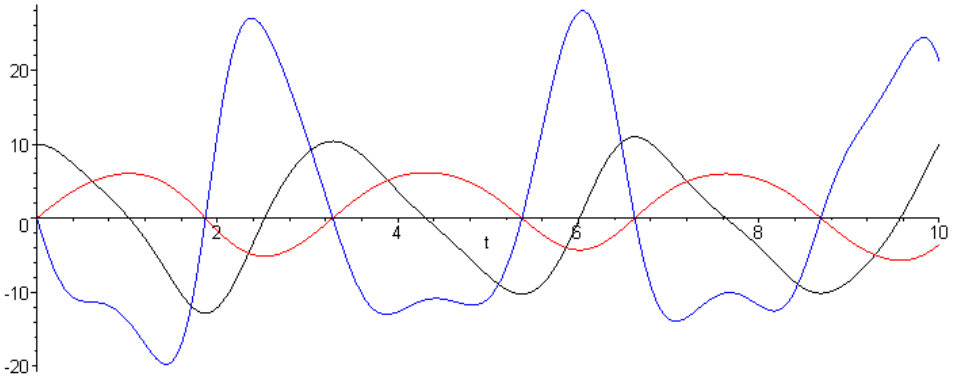
red = x(t) black = y(t) blue = z(t)

(F.10.20)

Here one sees red x and black y executing roughly sinusoidal motions. In a ballpark sense x and y are sinusoidal at their respective libration periods (3.14 and 3.63), and at least we see that black y has a longer period than red x (this black y period seems more like ~4). This is what creates the Lissajous pattern in our earlier figures. Meanwhile, blue z(t) is not coming down much from its maximum value of z = 10.

Here is a plot of $x,\dot{x},\ddot{x}$ = red.black,blue :

```
plot(['X(t)','Xd(t)','Xdd(t)'],t=0..10, color = [red,black,blue],numpoints=100);
```



red = x(t) black = $\dot{x}$(t) blue = $\ddot{x}$(t) (F.10.21)

Notice that red x and blue $\ddot{x}$ always cross the axis at the same time, which allows $\ddot{x}/x$ to be finite at all values of t (see below).

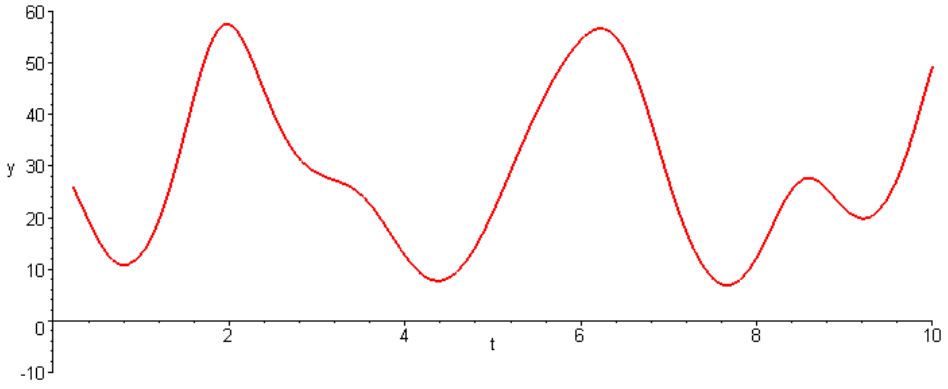
Tension in the the stick or tether

The tension in the stick (tether) is stated in (F.8.20),

$$T = - m_1 r_1 [\ \ddot{x}/x + \omega^2\ ], \qquad (F.8.20)$$

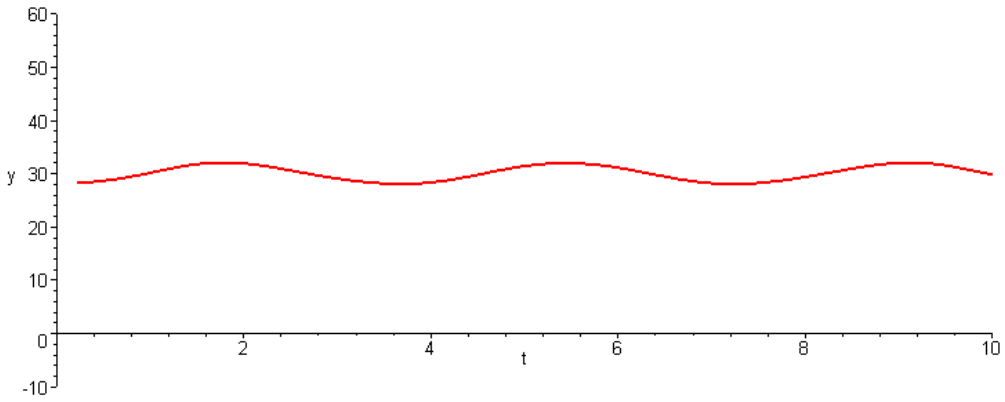
which we then plot with $m_1 = 1$ for the above example :

```
m1 := 1: T := (t) -> -m1*r1*('Xdd'(t)/'X(t)' + w^2):
plot(T(t),t=0..10,y=-10..60,thickness=2);
```



(F.10.22)

If we lower $\dot{x}(0) = \dot{y}(0) = 10$ to $\dot{x}(0) = \dot{y}(0) = 1$ to get a milder motion, the tension is less variable,



(F.10.23)

Here T is roughly equal to the "DC" value for the static dumbbell with $\theta = 0$. From (F.6.30) that value is

$$T \approx 3m_1\omega^2 r_1 = 3 * 1 * 1^2 * 10 = 30 \text{ N} . \qquad \text{// tether tension} \qquad \text{(F.10.24)}$$

We have been using $\omega = 1$, but for a low-Earth orbit satellite one has $T_{orbit} \approx 88*60$ seconds so

$$\omega = 2\pi/T_{orbit} \approx .0012 . \qquad \text{(F.10.25)}$$

Then mass $m_1 = 1$ kg on a 20 meter tether with equal masses ($r_1 = 10$ m) would feel a tidal force of

$$T \approx 3m_1\omega^2 r_1 = 3 * 1 * (.0012)^2 * 10 = .4320\text{e-}4 \text{ N} = 43 \ \mu\text{N} \qquad \text{(F.10.26)}$$

which is a very small tension.

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