An Essay on frames doc Section 1_1 PhL 2.17.17

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I have several confusions that are very annoying, they just refused to be nailed down. You nail them down, and they just flop up again the next day or the next week.

So what exactly is going on in Section 1.1 as it currently exists today.

I could start it a little differently as follows.

[Here I am rewriting Section 1.1 and most of these in fact got installed there]

Let Frame S have orthonormal basis vectors $\mathbf{e}_n$ which form a complete set.
Let Frame S; have orthonormal basis vectors $\mathbf{e'}_n$ which form a complete set. Thus

$$\mathbf{e}_n \bullet \mathbf{e}_m = \delta_{nm} \qquad \mathbf{e'}_n \bullet \mathbf{e'}_m = \delta_{nm} \qquad (1)$$

Assume for the moment that the two sets are related in this manner:

$$\mathbf{e'}_n = R_{nm}\, \mathbf{e}_m \qquad n = 1,2,3 \qquad (2)$$

This says that each basis vector of Frame S' is a certain linear combination of Frame S basis vectors. Let us assume also that the matrix R of coefficients is real orthogonal so that $R^{-1}{}_{ij} = R^{T}{}_{ji}$ or $RR^{T} = 1$.

We now define "components" of all the basis vectors this way,

$$\mathbf{e'}_n = (\mathbf{e}_i \bullet \mathbf{e'}_n)\, \mathbf{e}_i = (\mathbf{e'}_n)_i\, \mathbf{e}_i \quad \text{expansion} \qquad (\mathbf{e'}_n)_i = (\mathbf{e}_i \bullet \mathbf{e'}_n) \quad \text{projection} \qquad (3)$$

$$\mathbf{e'}_n = (\mathbf{e'}_i \bullet \mathbf{e'}_n)\, \mathbf{e'}_i = (\mathbf{e'}_n)'_i\, \mathbf{e'}_i \quad \text{expansion} \qquad (\mathbf{e'}_n)'_i = (\mathbf{e'}_i \bullet \mathbf{e'}_n) \quad \text{projection}$$

$$\mathbf{e}_n = (\mathbf{e}_i \bullet \mathbf{e}_n)\, \mathbf{e}_i = (\mathbf{e}_n)_i\, \mathbf{e}_i \quad \text{expansion} \qquad (\mathbf{e}_n)_i = (\mathbf{e}_i \bullet \mathbf{e}_n) \quad \text{projection}$$

$$\mathbf{e}_n = (\mathbf{e'}_i \bullet \mathbf{e}_n)\, \mathbf{e'}_i = (\mathbf{e}_n)'_i\, \mathbf{e'}_i \quad \text{expansion} \qquad (\mathbf{e}_n)'_i = (\mathbf{e'}_i \bullet \mathbf{e}_n) \quad \text{projection} \qquad (4)$$

Lines 2 and 3 are not very interesting because they just say

$$\mathbf{e'}_n = \delta_{in}\, \mathbf{e'}_i = (\mathbf{e'}_n)'_i\, \mathbf{e'}_i \quad \text{expansion} \qquad (\mathbf{e'}_n)'_i = \delta_{in} \quad \text{projection}$$

$$\mathbf{e}_n = \delta_{in}\, \mathbf{e}_i = (\mathbf{e}_n)_i\, \mathbf{e}_i \quad \text{expansion} \qquad (\mathbf{e}_n)_i = \delta_{in} \quad \text{projection}$$

These two lines are just identities.

Now let's examine equation **(**2**)** in both frame's components

$(\mathbf{e}'_n)_i = R_{nm}\,(\mathbf{e}_m)_i$ Frame S got this by applying $\mathbf{e}_i\bullet$

$(\mathbf{e}'_n)'_i = R_{nm}\,(\mathbf{e}_m)'_i$ Frame S' got this by applying $\mathbf{e}'_i\bullet$ (5)

We can rewrite these as

$(\mathbf{e}'_n)_i = R_{nm}\,\delta_{mi} = R_{ni}$

$\delta_{ni} = R_{nm}\,(\mathbf{e}_m)'_i$

In this second equation multiply both sides by R^T_{kn} and sum on n to get

$R^T_{kn}\delta_{ni} = R^T_{kn}R_{nm}\,(\mathbf{e}_m)'_i$

or

$R^T_{ki} = \delta_{km}\,(\mathbf{e}_m)'_i = (\mathbf{e}_k)'_i = R_{ik}$

We have thus shown that

$(\mathbf{e}'_n)_i = R_{ni} = (\mathbf{e}_i \bullet \mathbf{e}'_n)$

$(\mathbf{e}_n)'_i = R_{in} = (\mathbf{e}'_i \bullet \mathbf{e}_n) = (\mathbf{e}_n \bullet \mathbf{e}'_i)$

So now we know all about the components of the basis vectors in each frame

$(\mathbf{e}'_n)_i = R_{ni}$ $(\mathbf{e}_n)_i = \delta_{ni}$ Frame S components

$(\mathbf{e}_n)'_i = R_{in}$ $(\mathbf{e}'_n)'_i = \delta_{ni}$ Frame S' components

Theorem: We claim that [early proof of what now is the Basis Theorem]

$\mathbf{e}_n = R\mathbf{e}'_n$

Once we prove this, it then follows that $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$.

Proof: Show that the equation is true in terms of Frame S components:

$(\mathbf{e}_n)_i = (R\mathbf{e}'_n)_i$?

$\delta_{ni} = R_{ik}\,(\mathbf{e}'_n)_k$?

$\delta_{ni} = R_{ik} R_{nk}$?

$\delta_{ni} = R_{ik} R^{T}_{kn}$ yes QED

We can also verify the equation in terms of Frame S' components

$(\mathbf{e}_n)'_i = (R\mathbf{e}'_n)'_i$?

$R_{in} = \mathbf{e}'_i \bullet [R\mathbf{e}'_n]$?

$R_{in} = (\mathbf{e}'_i)_k [R\mathbf{e}'_n]_k$?

$R_{in} = (\mathbf{e}'_i)_k R_{kj} (\mathbf{e}'_n)_j$?

$R_{in} = R_{ik} R_{kj} R_{nj}$?

$R_{in} = R_{ik} R_{kj} R^{T}_{jn}$?

$R_{in} = R_{ik} \delta_{kn}$ yes QED

Another way to do this same verification is this

$(\mathbf{e}_n)'_i = (R\mathbf{e}'_n)'_i$?

$(\mathbf{e}_n)'_i = R_{ij}(\mathbf{e}'_n)'_j$?

$R_{in} = R_{ij}\delta_{nj}$ yes

So we have shown that, whether we take Frame S or Frame S' components, the equations

$\mathbf{e}_n = R\mathbf{e}'_n$ and $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$

are true. The left equation says that each $\mathbf{e}_n$ is just a rotation of the corresponding $\mathbf{e}'_n$ by rotation matrix R. The same R works for all n.

Fact: If you choose to say

$(R\mathbf{e}'_n)'_i = R'_{ij} (\mathbf{e}'_n)_j$

with a different matrix called R', that is fine, but it turns out that R' = R so both matrices are the same.

Here is a more explicit statement of the above [Basis Theorem]

$\mathbf{e}'_n = R^{(S)}_{nm} \mathbf{e}_m$ $\mathbf{e}_n = R\mathbf{e}'_n$ $(\mathbf{e}_n)_i = (R\mathbf{e}'_n)_i = R^{(S)}_{ij} (\mathbf{e}'_n)_j$

where the superscript (S) means that the matrix is in the Frame S basis.

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NEW TOPIC! I am quoting my Goldstein ω derivation from some other doc.

Now let's return to our Goldstein **ω** problem. We start off there with

$$\mathbf{e}_n = R\,\mathbf{e'}_n \qquad (1.1.1) \qquad (G.4.1)$$

$$R = R_z(\psi)\,R_x(\theta)\,R_z(\varphi) = R(\mathbf{\Phi}) = \exp(-i\,\mathbf{\Phi}\bullet\mathbf{J}) = \exp(-i\,\Phi\,\mathbf{n}\bullet J) \qquad (G.4.2)$$

which then determines MY meaning of the Euler angles. If I wanted to do this from the Start in Goldstein notation I would write

$$R = R_z(-\psi)\,R_x(-\theta)\,R_z(-\varphi) = R(-\mathbf{\Phi}) = \exp(+i\,\mathbf{\Phi}\bullet\mathbf{J}) = \exp(+i\,\Phi\,\mathbf{n}\bullet J) \qquad (G.4.2)$$

R = B C D // Goldstein page 109

For now I will stick with my original notation. I then quote an earlier equation which contains **ω**,

$$(d\mathbf{e}_n/dt)_{S'} = -\,\boldsymbol{\omega}\times\mathbf{e}_n\,. \qquad (1.7.4) \qquad (G.4.5)$$

There is only one matrix R, the matrices I formerly called R^{ss} and $R^{s's'}$ are exactly the same. And also the matrix is that product of rotation matrices! Things then go on :

$$\mathbf{e}_n = R(\mathbf{\Phi})\,\mathbf{e'}_n \qquad (G.4.4)$$

$$(d\mathbf{e}_n)_{S'} = R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e'}_n - R(\mathbf{\Phi})\mathbf{e'}_n \equiv dQ\,\mathbf{e'}_n \qquad (G.4.8)$$

$$dQ \equiv [\,R_z(\psi+d\psi)\,R_x(\theta+d\theta)\,R_z(\varphi+d\varphi) - R_z(\psi)\,R_x(\theta)\,R_z(\varphi)\,]\,. \qquad (G.4.9)$$

We have some new objects here. But there is only one matrix R(**Φ**+d**Φ**) and only one R(**Φ**) and thus there is only one dQ. Everything is well-defined. I continue,

$$(d\mathbf{e}_n)_{S'} = dQ\,\mathbf{e'}_n = dQ[R^{-1}(\mathbf{\Phi})\mathbf{e}_n] = [dQ\,R^{-1}(\mathbf{\Phi})]\,\mathbf{e}_n \qquad (G.4.10)$$

I then divide by dt to get

$$(d\mathbf{e}_n/dt)_{S'} = [(dQ/dt)_{S'}\,R^{-1}(\mathbf{\Phi})]\,\mathbf{e}_n = A\,\mathbf{e}_n$$

I need the S' label on the left, and am not sure whether I need it on the $(dQ/dt)_{S'}$. There is only one matrix A and it works the same way that R worked in terms of indices. I show that in fact A is this matrix:

$$A = -[(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)]$$

Now claim that you can evaluate this thing in either components

$$[(d\mathbf{e}_n/dt)_{S'}]_i = [A\,\mathbf{e}_n]_i = A_{ij}(\mathbf{e}_n)_j = A_{in}$$

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = [A\,\mathbf{e}_n]'_i = A_{ij}(\mathbf{e}_n)'_j = A_{ij}R_{jn} = (AR)_{in}$$ [error here, need A'_{ij}]

As expected, the components are different.

Meanwhile we have this equation

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n$$

which similarly can be evaluated in either frame

$$[(d\mathbf{e}_n/dt)_{S'}]_i = -[\boldsymbol{\omega} \times \mathbf{e}_n]_i = -\varepsilon_{ijk}\omega_j(\mathbf{e}_n)_k = -\varepsilon_{ijk}\omega_j\delta_{nk} = -\varepsilon_{ijn}\omega_j$$

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = -[\boldsymbol{\omega} \times \mathbf{e}_n]'_i = -\varepsilon_{ijk}(\omega)'_j(\mathbf{e}_n)'_k = -\varepsilon_{ijk}(\omega)'_jR_{kn}$$

Again the results are different. Now comparing the results we get

Frame S comp: $A_{in} = -\varepsilon_{ijn}\omega_j$

Frame S' comp: $A_{ik}R_{kn} = -\varepsilon_{ijk}(\omega)'_jR_{kn}$ or $A_{in} = -\varepsilon_{ijn}(\omega)'_j$

This is our Big Problem. The two answers are exactly the same but one gives $(\omega)_j$ and the other $(\omega)'_j$. If this is all correct, we end up with the components of $\boldsymbol{\omega}$ being exactly the same in both Frames. But any simple example shows this is generally not true.

So the paradox has survived another 14 hour day of effort to resolve it.

[I think if you don't omit that prime, you have matrix A' in the Frame S' evaluation and in fact this is the A matrix you have been dealing with all the time and should have had a prime on it, for example

$$A' = -[(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)]$$

Then you get the right answer in Frame S' and I think with $A' = RAR^{-1}$ you then also get the right answer in a Frame S evaluation.]

[I now go on to try and resolve my Paradox, not knowing about A']

Once again with gusto:

$$A\,\mathbf{e}_n = -\boldsymbol{\omega} \times \mathbf{e}_n$$

How exactly do you take components ? Well,

$$\mathbf{e}_i \bullet (A\,\mathbf{e}_n) = -\mathbf{e}_i \bullet (\boldsymbol{\omega} \times \mathbf{e}_n) \qquad \text{Frame S components}$$

$$\mathbf{e'}_i \bullet (A\,\mathbf{e}_n) = -\mathbf{e'}_i \bullet (\boldsymbol{\omega} \times \mathbf{e}_n) \qquad \text{Frame S' components}$$

Now do these dot products

$$\mathbf{e'}_i \bullet (A\,\mathbf{e}_n) = -\mathbf{e'}_i \bullet (\boldsymbol{\omega} \times \mathbf{e}_n) \qquad \text{Frame S' components}$$

$$(\mathbf{e'}_i)_k\,(A\,\mathbf{e}_n)_k = -(\mathbf{e'}_i)_k\,(\boldsymbol{\omega} \times \mathbf{e}_n)_k$$

$$R_{ik}\,A_{kj}\,(\mathbf{e}_n)_j = -R_{ik}\,\varepsilon_{kab}(\omega)_a(\mathbf{e}_n)_b$$

$$A_{kj}\,(\mathbf{e}_n)_j = -\varepsilon_{kab}(\omega)_a(\mathbf{e}_n)_b$$

$$A_{kj}\,(\mathbf{e}_n)_j = -\varepsilon_{kaj}(\omega)_a(\mathbf{e}_n)_j$$

$$A_{kj} = -\varepsilon_{kaj}(\omega)_a$$

So doing it this way, even though I took Frame S' components, the result involves $(\omega)_a$ because I use the Frame S dot products.

Let's just look at the dot product all by itself

$$-\mathbf{e}_i \bullet (\boldsymbol{\omega} \times \mathbf{e}_n) = -(\mathbf{e}_i)_k\,(\boldsymbol{\omega} \times \mathbf{e}_n)_k = -(\mathbf{e}_i)_k\,\varepsilon_{kab}(\omega)_a(\mathbf{e}_n)_b = -\varepsilon_{ian}(\omega)_a$$

$$-\mathbf{e}_i \bullet (\boldsymbol{\omega} \times \mathbf{e}_n) = -(\mathbf{e}_i)'_k\,(\boldsymbol{\omega} \times \mathbf{e}_n)'_k = -(\mathbf{e}_i)'_k\,\varepsilon_{kab}(\omega)'_a(\mathbf{e}_n)'_b = -R_{ki}\,\varepsilon_{kab}(\omega)'_a\,R_{bn}$$

This seems to suggest that

$$\varepsilon_{ian}(\omega)_a = R_{ki}\,\varepsilon_{kab}(\omega)'_a\,R_{bn}$$

Well how about this:

$$\mathbf{e}_i \bullet \boldsymbol{\omega} = (\mathbf{e}_i)_k(\omega)_k = (\omega)_i$$

$$\mathbf{e}_i \bullet \boldsymbol{\omega} = (\mathbf{e}_i)'_k(\omega)'_k = R_{ki}\,(\omega)'_k$$

so then

$$(\boldsymbol{\omega})_i = R_{ki}\,(\boldsymbol{\omega})'_k$$

which seems reasonable more or less.

Continuing 12.18.17

I have an idea about why there might be distinct A and A' objects, whereas the R and R' objects are the same. To wit, we start over once again on the Goldstein stuff. But first, just to make totally sure:

$$R_{ij} = \langle \mathbf{e}_i | R | \mathbf{e}_j \rangle$$

$$R'_{ij} = \langle \mathbf{e}'_i | R | \mathbf{e}'_j \rangle = \langle \mathbf{e}'_i | \mathbf{e}_m \rangle \langle \mathbf{e}_m | R | \mathbf{e}_n \rangle \langle \mathbf{e}_n | \mathbf{e}'_j \rangle$$

$$= (\mathbf{e}_m)'_i \, R_{mn} \, (\mathbf{e}'_j)_n = R_{im} \, R_{mn} \, R_{jn} = R_{im} \, R_{mn} \, R^T_{nj}$$

$$= R_{im}(RR^T)_{mj} = R_{im}\delta_{mj} = R_{ij}$$

The general rule is this

$$G'_{ij} = \langle \mathbf{e}'_i | G | \mathbf{e}'_j \rangle = \langle \mathbf{e}'_i | \mathbf{e}_m \rangle \langle \mathbf{e}_m | G | \mathbf{e}_n \rangle \langle \mathbf{e}_n | \mathbf{e}'_j \rangle$$

$$= (\mathbf{e}_m)'_i \, G_{mn} \, (\mathbf{e}'_j)_n = R_{im} \, G_{mn} \, R_{jn} = R_{im} \, G_{mn} \, R^T_{nj} = (RGR^T)_{ij}$$

or

$$G' = RGR^T = RGR^{-1}$$ // agrees with comments in new App G.4

Good. Now back to Gold:

$$\mathbf{e}_n = R \, \mathbf{e}'_n \qquad (1.1.1) \qquad (G.4.1)$$

$$R = R_z(\psi) \, R_x(\theta) \, R_z(\varphi) = R(\mathbf{\Phi}) = \exp(-i \, \mathbf{\Phi} \bullet \mathbf{J}) = \exp(-i \, \Phi \, \mathbf{n} \bullet J) \quad \text{ok} \quad (G.4.2)$$

For this R, we have R' = R just as above. Next,

$$\mathbf{e}_n = R(\mathbf{\Phi}) \, \mathbf{e}'_n \quad \text{ok} \quad (G.4.4)$$

$$(d\mathbf{e}_n)_{S'} = [R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}'_n] - [R(\mathbf{\Phi})\mathbf{e}'_n] \qquad (G.4.8)$$

There are no matrices in this equation yet! $[R(\mathbf{\Phi})\mathbf{e}'_n]$ is some vector. Now while working in Frame S', I write out the Frame S' components of the above equation

$$[(d\mathbf{e}_n)_S]'_i = [R(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}'_n]'_i - [R(\mathbf{\Phi})\mathbf{e}'_n]'_i$$

Now I show some matrices for the first time

$$= [R'(\mathbf{\Phi}+d\mathbf{\Phi})]_{ij} \, (\mathbf{e}'_n)'_j - [R'(\mathbf{\Phi})]_{ij} \, (\mathbf{e}'_n)'_j$$

$$= \{ [R'(\mathbf{\Phi}+d\mathbf{\Phi})]_{ij} - [R'(\mathbf{\Phi})]_{ij} \} \, (\mathbf{e}'_n)'_j$$

The primes on these matrices mean they are matrices as observed in Frame S'.

We now define the Euler angles by writing :

$$[R'(\mathbf{\Phi})]_{ij} = [R_z(\psi)\, R_x(\theta)\, R_z(\varphi)]_{ij} \qquad // = [R(\mathbf{\Phi})]_{ij}$$

$$[R'(\mathbf{\Phi}+d\mathbf{\Phi})]_{ij} = R_z(\psi+d\psi)\, R_x(\theta+d\theta)\, R_z(\varphi+d\varphi)$$

Comment: Worker in the body frame would perhaps like to say

$$R'(\mathbf{\Phi}) = R_{z'}(\psi)\, R_{x'}(\theta)\, R_{z'}(\varphi)$$

because he sees axes like z' and x', not x and z. But for him these would be the same matrices.

From the first equation, we could in theory obtain an expression for $\mathbf{\Phi}$ and from the second we could obtain an expression for $d\mathbf{\Phi}$. These expressions would be very ugly as Section ** shows. In general the differential vector $d\mathbf{\Phi}$ is not in the same direction as $\mathbf{\Phi}$.

We now write

$$[(d\mathbf{e}_n)_{S'}]'_i = \{ [R'(\mathbf{\Phi}+d\mathbf{\Phi})]_{ij} - [R'(\mathbf{\Phi})]_{ij} \}\ (\mathbf{e}'_n)'_j$$

$$= [dR']_{ij}\ (\mathbf{e}'_n)'_j$$

Then

$$[(d\mathbf{e}_n/dt)_{S'}]'_i = [dR'/dt]_{ij}\ (\mathbf{e}'_n)'_j = [dR'/dt]_{ij}\ [R'^{-1}(\mathbf{\Phi})]_{jk}\ (\mathbf{e}_n)'_k$$

$$= A'_{ik}\ (\mathbf{e}_n)'_k \qquad A'_{ik} = [dR'/dt]_{ij}\ [R'^{-1}(\mathbf{\Phi})]_{jk}$$

In matrix notation we have

$$A' = [dR'/dt]\ R'^{-1}$$

[ok, now I have the correct notation A']

If we continue on in this manner, we obtain the correct result for the Frame S' coordinates of $\boldsymbol{\omega}$ from this equation

$$[(d\mathbf{e}_n/dt)_S]'_i = -[\boldsymbol{\omega} \times \mathbf{e}_n]'_i = -\varepsilon_{ijk}(\omega)'_i\ (\mathbf{e}_n)'_j$$

and that solution is this

$$A'_{ij} = \varepsilon_{ijk}\,(\omega)'_k \quad \Rightarrow \quad (\omega)'_1 = A'_{23}\,,\ (\omega)'_2 = A'_{31},\ (\omega)'_3 = A'_{12}$$

Question: How then do we get results in Frame S components instead of Frame S' ? We would have a matrix A given by

$$A = R^{-1}A'R$$

In Frame S components the two equations are

$$[(d\mathbf{e}_n/dt)_{S'}]_i = A_{ik}\,(\mathbf{e}_n)_k$$

$$[(d\mathbf{e}_n/dt)_S]_i = -[\boldsymbol{\omega} \times \mathbf{e}_n]_i = -\varepsilon_{ijk}(\omega)_i\,(\mathbf{e}_n)_j$$

The solution is then

$$A_{ij} = \varepsilon_{ijk}\,(\omega)_k \quad \Rightarrow \quad (\omega)_1 = A_{23}\,,\ (\omega)_2 = A_{31},\ (\omega)_3 = A_{12}$$

Now do a little more:

$$A_{ij} = \varepsilon_{ijk}\,(\omega)_k$$

$$[R^{-1}A'R]_{ij} = \varepsilon_{ijk}\,(\omega)_k$$

$$R^{-1}{}_{ik}\,A'_{kn}\,R_{nj} = \varepsilon_{ijk}\,(\omega)_k$$

$$R^{-1}{}_{ik}\,[\varepsilon_{kna}\,(\omega)'_a]\,R_{nj} = \varepsilon_{ijk}\,(\omega)_k \qquad A'_{kn} = \varepsilon_{kna}\,(\omega)'_a$$

$$\varepsilon_{kna}\,R_{nj}R_{ki}\,(\omega)'_a = \varepsilon_{ijk}\,(\omega)_k$$

$$[\varepsilon_{akn}\,R_{ki}R_{nj}]\,(\omega)'_a = \varepsilon_{ijk}\,(\omega)_k$$

$$[\,R_{ak}\varepsilon_{kij}]\,(\omega)'_a = \varepsilon_{ijk}\,(\omega)_k \qquad \text{//(A.19)}$$

$$\varepsilon_{kij}[\,R^{-1}{}_{ka}]\,(\omega)'_a = \varepsilon_{ijk}\,(\omega)_k$$

$$\varepsilon_{ijk}[\,R^{-1}{}_{ka}]\,(\omega)'_a = \varepsilon_{ijk}\,(\omega)_k$$

$$\varepsilon_{ijk}[\,R^{-1}{}_{ka}]\,(\omega)'_a = \varepsilon_{ijk}\,(\omega)_k$$

$$[\,R^{-1}{}_{ka}]\,(\omega)'_a = (\omega)_k$$

$$[\,R^{-1}{}_{ka}]\,(\omega')_a = (\omega)_k$$

$$R^{-1}\boldsymbol{\omega}' = \boldsymbol{\omega}$$

$\boldsymbol{\omega}' = R\boldsymbol{\omega}$

$\boldsymbol{\omega} = R^{-1}\boldsymbol{\omega}'$

Wow! It works. So you just want to do this operation.

$(\omega)_k = [\ R^{-1}_{ka}]\ (\omega')_a$

Let's compute this!

In My notation we ended up with

$(\omega)'_1 = A'_{23} = -(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi) = -\dot{\varphi}\sin\theta\sin\psi - \dot{\theta}\cos\psi$

$(\omega)'_2 = A'_{31} = -(\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi) = \dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$

$(\omega)'_3 = A'_{12} = -(\dot{\psi} + \dot{\varphi}\cos\theta) = -\dot{\varphi}\cos\theta - \dot{\psi}$. // our result (G.4.22)

I am going to have Maple compute this thing and see what it is!! This is a first!

$R = R_z(\psi)\ R_x(\theta)\ R_z(\varphi)]$

$Rinv := R_z(-\varphi)R_x(-\theta)\ R_z(-\psi)$

Here we go!

```
Rinv := evalm(Rz(-phi) &* Rx(-theta) &* Rz(-psi));
```

$$Rinv := \begin{bmatrix} \cos(\psi)\cos(\phi) - \sin(\psi)\cos(\theta)\sin(\phi) & \sin(\psi)\cos(\phi) + \cos(\psi)\cos(\theta)\sin(\phi) & \sin(\theta)\sin(\phi) \\ -\cos(\psi)\sin(\phi) - \sin(\psi)\cos(\theta)\cos(\phi) & -\sin(\psi)\sin(\phi) + \cos(\psi)\cos(\theta)\cos(\phi) & \sin(\theta)\cos(\phi) \\ \sin(\psi)\sin(\theta) & -\cos(\psi)\sin(\theta) & \cos(\theta) \end{bmatrix}$$

```
wp1 := -phid*sin(theta)*sin(psi) -thetad*cos(psi):
wp2 := phid*sin(theta)*cos(psi) -thetad*sin(psi):
wp3 := -phid * cos(theta) - psid:
wp := matrix(3,1,[wp1,wp2,wp3]);
```

$$wp := \begin{bmatrix} -phid\sin(\theta)\sin(\psi) - thetad\cos(\psi) \\ phid\sin(\theta)\cos(\psi) - thetad\sin(\psi) \\ -phid\cos(\theta) - psid \end{bmatrix}$$

```
evalm(Rinv &* wp):simplify(%);
```

$$\begin{bmatrix} -\cos(\phi)\,thetad - \sin(\theta)\sin(\phi)\,psid \\ \sin(\phi)\,thetad - \sin(\theta)\cos(\phi)\,psid \\ -phid - \cos(\theta)\,psid \end{bmatrix}$$

So I have computed $\boldsymbol{\omega} = R^{-1}\boldsymbol{\omega}'$ and the answer is this

$\boldsymbol{\omega}$ = (- $\dot{\theta}$ cosφ - $\dot{\varphi}$ sinθsinφ, $\dot{\theta}$ sinφ - $\dot{\varphi}$ sinθcosφ, - $\dot{\varphi}$ - $\dot{\psi}$cosθ)

So lets compare

$(\omega)'_1$ = - $\dot{\varphi}$sinθsinψ - $\dot{\theta}$cosψ

$(\omega)'_2$ = $\dot{\varphi}$sinθcosψ - $\dot{\theta}$sinψ

$(\omega)'_3$ = - $\dot{\varphi}$ cosθ - $\dot{\psi}$. // our result (G.4.22)

$(\omega)_1$ = - $\dot{\varphi}$ sinθsinφ - $\dot{\theta}$ cosφ

$(\omega)_2$ = - $\dot{\varphi}$ sinθcosφ + $\dot{\theta}$ sinφ

$(\omega)_3$ = - $\dot{\psi}$cosθ- $\dot{\varphi}$ // what my result would be.

The Frame S form is as simple, but it is different. [have some error above, did not track down]

Question: Is this whole thing much easier if I take a different starting G Rule formula like

$(d\mathbf{e'}_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e'}_n$

Now we are "working in Frame S". Then

$\mathbf{e'}_n = R^{-1}(\boldsymbol{\Phi})\, \mathbf{e}_n$ ok *(G.4.4)*

$(d\mathbf{e'}_n)_S = [R^{-1}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})\mathbf{e}_n] - [R^{-1}(\boldsymbol{\Phi})\mathbf{e}_n]$

Then suppose I set

$R(\boldsymbol{\Phi}) = [R_z(\varphi)\, R_x(\theta)\, R_z(\psi)]$

$R(\boldsymbol{\Phi} + d\boldsymbol{\Phi}) = [R_z(\varphi+d\varphi)\, R_x(\theta+d\theta)\, R_z(\psi+d\psi)]$

Then I will have

$R^{-1}(\boldsymbol{\Phi}) = [R_z(-\psi)\, R_x(-\theta)\, R_z(-\varphi)] = B\ C\ D$ // Goldstein page 109

Then how would it go?

$[(d\mathbf{e'}_n)_S]_i = \{ [R^{-1}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})]_{ij} - [R^{-1}(\boldsymbol{\Phi})]_{ij} \}\ (\mathbf{e}_n)_j$

$= [d(R^{-1})]_{ij}\ (\mathbf{e}_n)_j$

$[(d\mathbf{e'}_n/dt)_S]_i = [d(R^{-1})/dt]_{ij}\ (\mathbf{e}_n)_j = [d(R^{-1})/dt]_{ij}\, [R(\boldsymbol{\Phi})]_{jk}\ (\mathbf{e'}_n)_k$

$= A_{ik}\ (\mathbf{e'}_n)_k \qquad A_{ik} = [d(R^{-1})/dt]_{ij}\, [R(\boldsymbol{\Phi})]_{jk}$

I would then have to compute a few things.

$R^{-1}(\boldsymbol{\Phi}) = [R_z(-\psi)\, R_x(-\theta)\, R_z(-\varphi)]$

$R^{-1}(\boldsymbol{\Phi}+d\boldsymbol{\Phi}) = [R_z(-\psi-d\psi)\, R_x(-\theta-d\theta)\, R_z(-\varphi-d\varphi)]$

Then I would show that

$[d(R^{-1})]_{ij} = [R^{-1}(\boldsymbol{\Phi}+d\boldsymbol{\Phi})]_{ij} - [R^{-1}(\boldsymbol{\Phi})]_{ij}$

$[d(R^{-1})] = [R_z(-\psi-d\psi)\, R_x(-\theta-d\theta)\, R_z(-\varphi-d\varphi)] - [R_z(-\psi)\, R_x(-\theta)\, R_z(-\varphi)]$

$A = [d(R^{-1})/dt]\, [R(\boldsymbol{\Phi})]$

$= \{ [R_z(-\psi-d\psi)\, R_x(-\theta-d\theta)\, R_z(-\varphi-d\varphi)] - [R_z(-\psi)\, R_x(-\theta)\, R_z(-\varphi)] \}\, [R_z(\varphi)\, R_x(\theta)\, R_z(\psi)]$

$$= \; - [(-\dot{\theta}\cos\psi - \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (-\dot{\psi} - \dot{\varphi}\cos\theta)(iJ_3)]$$

$$= \; [(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (-\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)]$$

$$= \; [(\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)(iJ_1) + (\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)(iJ_2) + (\dot{\varphi}\cos\theta + \dot{\psi})(iJ_3)]$$

My solution is just sitting there! At this point I have

$$(d\mathbf{e'}_n)_S = [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi})\mathbf{e}_n] - [R^{-1}(\mathbf{\Phi})\mathbf{e}_n]$$

$$= [R^{-1}(\mathbf{\Phi}+d\mathbf{\Phi}) - R^{-1}(\mathbf{\Phi})]\mathbf{e}_n$$

$$= [d(R^{-1})]\,\mathbf{e}_n$$

$$= [d(R^{-1})]\,R(\mathbf{\Phi})\,\mathbf{e'}_n$$

Then

$$(d\mathbf{e'}_n/dt)_S = [d(R^{-1})/dt]\,R(\mathbf{\Phi})\,\mathbf{e'}_n = A\,\mathbf{e'}_n$$

with A as shown above. Meanwhile we have the G rule

$$(d\mathbf{e'}_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e'}_n \quad . \qquad (1.7.1)$$

So we have the equation

$$A\,\mathbf{e'}_n = \boldsymbol{\omega} \times \mathbf{e'}_n$$

$$A = (\dot{\varphi}\sin\theta\sin\psi + \dot{\theta}\cos\psi)(iJ_1) + (\dot{\varphi}\sin\theta\cos\psi - \dot{\theta}\sin\psi)(iJ_2) + (\dot{\varphi}\cos\theta + \dot{\psi})(iJ_3)$$

$$\equiv A_1(iJ_1) + A_2(iJ_2) + A_3(iJ_3) = A_k(iJ_k)$$

$$A_{ij} = A_k(iJ_k)_{ij} = A_k\,\varepsilon_{kij} = \varepsilon_{ijk}A_k$$

and then

$$[A\,\mathbf{e'}_n]_i = A_{ij}\,(\mathbf{e'}_n)_j = \varepsilon_{ijk}A_k\,(\mathbf{e'}_n)_j$$

MEANWHILE, we write

$$[\boldsymbol{\omega} \times \mathbf{e'}_n]_i = \varepsilon_{ikj}\,\omega_k(\mathbf{e'}_n)_j =$$

So we then have

$$\varepsilon_{ijk} A_k \, (\mathbf{e'}_n)_j \;=\; -\,\varepsilon_{ijk}\, \omega_k (\mathbf{e'}_n)_j$$

This seems to suggest that

$$\omega_k \;= A_k$$

and of course this is the wrong result!!!

OK, I think I am getting the message slowly here. Why not use swap notation so the body frame is Frame S. This has to give the same as my first result today. This does not match his pictures very well for the Euler angles. It was just an idea.

[Have I ever verified that $A' = RAR^{-1}$ gives the right result in Frame S? Meaning $A = R^{-1}A' R$. No I have not done this in detail, but I think it implies $\boldsymbol{\omega}' = R\boldsymbol{\omega}$ so I then know it works.]