Here I reviewed Sections 1.1 and 1.2 and 1.3 adding red comments. This review was in light of some horrible confusions about $\mathbf{V}' = R\mathbf{V}$ and $\hat{\mathbf{x}}' = R\hat{\mathbf{x}}$ versus $\hat{\mathbf{x}}' = R^{-1}\hat{\mathbf{x}}$ and why they are not both true causing a Paradox. In line here I ended up rewriting Sections 1.2 and 1.3 completely, and adding a small comment to Section 1.1 where $\mathbf{V}' = R\mathbf{V}$ appears. These two sections have been installed. Do not edit here!!

Rotating Frames of Reference

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Maple code is available upon request. Comments and errata are welcome.
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The graphics look ratty in Windows Adobe PDF viewers when not scaled up, but look just fine in this excellent freeware viewer: https://www.tracker-software.com/product/pdf-xchange-editor .

The table of contents has live links, and use of a wide Bookmarks pane is recommended.

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Overview

This monograph presents an extended discussion of doing physics in non-inertial frames of reference. The first chapters present the general theory, while the latter chapters and appendices contain examples concerning ant paths on turntables, tides, pendulums, fluid flows and tethered/dumbbell satellites. The presentation is entirely self-contained with all support material provided. Where possible, connections are made to external sources on this subject. Both linear (force) and rotational (torque) viewpoints are considered. Maple is used extensively to plot particle trajectories, to obtain and plot numerical solutions of differential equations (dsolve), and to verify complicated equalities.

Our general context is an Apparatus containing a Particle observed from two frames of reference called S and S'. Frame S' is rotating and translating in some arbitrary manner with respect to Frame S as indicated in this drawing,

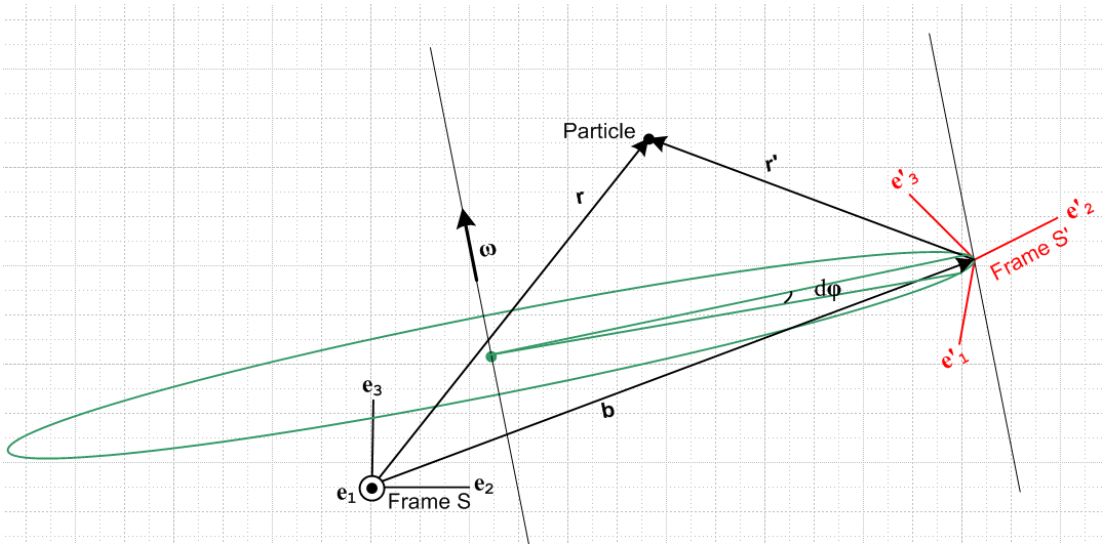


Fig 1

The Particle is located at position **r** relative to the Frame S origin, and at position **r'** relative to the Frame S' origin. These vectors are related by **r** = **r'** + **b** where **b** is a dynamic vector connecting the frame origins. Rotation is about some possibly moving axis with some angular velocity **ω** which might be changing in both direction and magnitude.

We describe the relationship between the properties of the Particle as measured in these two frames of reference. The entire discussion takes place in a non-relativistic framework where time is the same in the two frames. Even in this limited context, things are fairly complicated.

An important subtopic of the rotating frames discussion might be called "Newtonian mechanics in non-inertial frames" where one considers the fate of **F** = m**a** in a non-inertial frame. This is where the famous fictitious forces and less-famous fictitious torques appear.

Most mechanics textbooks which treat rotating frames, having a multitude of other topics to address, spend 10-20 pages on the subject with the following itinerary: state the G Rule (see below), use it to derive an inter-frame velocity and/or acceleration relation, discuss fictitious forces in a rotating frame with emphasis on the Coriolis force, do a few basic problems, and end up treating the Foucault pendulum.

A notable exception is the book of Taylor which devotes 40 pages to the subject including a nice discussion of the tides. (In his book, our frames S and S' are called S_0 and S.)

In this document, having the luxury of no space limitations, we try to probe more deeply into the technical nuts and bolts of rotating frames analysis. Almost all calculations are done in line for the reader to see.

The "papal we" mode of presentation is often used below, as if this paper had multiple authors who seem to own the equations, drawings and experiments as their personal possessions. The approved mode of course is to use passive or impersonal sentence constructions as if the author did not exist. Interestingly, Taylor uses the "I mode" which is perhaps more honest and is certainly refreshing.

Summary

Section 1 sets up the notation used throughout the document. As shown in Fig 1, a decision was made that Frame S' is the rotating frame, even though this conflicts with the choice made by many textbook authors. We refer to this notation as our **non-swap notation** and all our development work is done in this notation. One can imagine another version of Fig 1 which has S ↔ S', which is then our **swap notation**. Often it is more convenient to have Frame S be the rotating frame to avoid an avalanche of primes in the equations of interest, and in that case the "swap" notation is more useful. All key results are summarized in Sections 12 and 13 in both "non-swap" and "swap" notations.

Unit basis vectors are $\mathbf{e}_i$ and $\mathbf{e}'_i$ for Frame S and Frame S'. A generic vector **a** can be expanded on either basis. The presence of two frames of reference often associates with a vector **a** another vector **a**' which leads to the need for a compact notation which distinguishes the components $(a')_i$ and $(a)'_i$ which are often different. **The prime symbol '** plays a central role in the notation and is never used to indicate time differentiation (overdots are used for that purpose).

After considering the meaning of equality for two vectors, we develop the notion of "conical rotational motion" according to $\dot{\mathbf{a}} = \boldsymbol{\omega} \times \mathbf{a}$ and then apply that notion to the basis vectors. The need for putting a frame label on the time derivative of a vector (but not a scalar or vector component) is demonstrated. It is then shown that the Particle in Fig 1 has four distinct velocities and eight distinct accelerations, reinforcing the need for a precise notation. Finally, it is noted that the operations of computing a time derivative and taking a component generally do not commute.

Section 2 derives and discusses what we call "**the G Rule**", namely, $(d\mathbf{a}/dt)_S = (d\mathbf{a}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{a}$ where Frame S' rotates at rate $\boldsymbol{\omega}$ relative to Frame S and **a** is an arbitrary vector.

Section 3 describes an **Observer** and an **Apparatus** containing a Particle, all of which are in the rotating Frame S'.

Section 4 explains the relationship between Frames S and S' for a general placement of the instantaneous rotation axis about which Frame S' rotates at vector angular frequency $\boldsymbol{\omega}$. Two **special cases** are identified for the location of this axis, and then three applications are roughly outlined.

Section 5 (which is a very brief) states **the goal** of subsequent sections which is basically to determine how, given Particle properties in Frame S', one may determine these properties in Frame S. The results are eventually summarized in Section 12.

Section 6 derives the relationships among the four **velocities** mentioned above.

Section 7 derives the relationship among three of the eight **accelerations**. This and the preceding section are as dry as dust (mud's thirsty sister), but the results are of key importance, so everything is done step by step.

Section 8 addresses the traditional subject of **fictitious forces** and interprets them. First the centrifugal and Euler forces are interpreted with the aid of some drawings, then comes the Coriolis force. An arm-waving interpretation of this force is provided for a simple four-projectile problem (on a turntable), and an exact solution to this problem later appears in Section 15. Three **applications** involving fictitious forces are then considered, but not fully analyzed: problems with moving objects near the surface of the Earth, **tethered satellites**, and **ocean tides**.

Section 9 relates our notation to that used by **Marion** (1970) and **Thornton & Marion** [T&M] (2003). It is found that the Marion texts are very close to our "swap" notation.

Section 10 then relates our notation to that used by **Goldstein** (1950) and **Goldstein, Poole and Safko** [GPS] (2001). These texts assume that both reference frames have the same origin which simplifies their presentations.

Section 11 comments on the **angular momentum** vector **L** and its time derivative, and establishes how these vectors are related in the two frames. The notion of **fictitious torques** is introduced. It is demonstrated how both fictitious forces and torques are applied in **fluid dynamics**. Brief comments are made concerning fluid material and **control volumes** and the Reynolds Transport Theorem.

Section 12 summarizes the set of equations which fulfill the goal stated in Section 5: given properties in Frame S', what are they in Frame S? The results are given in both "non-swap" and "swap" notation.

Section 13 then considers the **Inverse Problem**: given properties in Frame S, what are they in Frame S'? The inverse equations are first obtained by laboriously inverting those presented in Section 12.1, and are then reobtained by a simple symmetry operation. The Inverse Problem equations are then summarized in both "non-swap" and "swap" notation.

Section 14 briefly adds the complication of having a different **orthogonal curvilinear coordinate** system in each of the Frames S and S'. Up to this point, only Cartesian coordinates have been used.

Section 15 treats three **"ant on turntable" problems** in some detail. In the first two problems, the ant crawls in a certain manner on the turntable as it rotates (Frame S') and the ant's position, velocity and acceleration are computed in inertial Frame S. In the third problem, the ant flies in a straight line in Frame S just over the turntable surface, and the ant's position, velocity and acceleration are computed in Frame S'. Many (hopefully entertaining) Maple trajectory plots are presented, along with the very simple code for these plots. In the final section, the projectile problem of Section 8.3 is solved using the third problem results, and it is noted that there are hard ways to solve rotation problems that can be avoided.

Appendix A computes a certain matrix R(ξ) which relates spherical unit vectors to Cartesian ones.

Appendix B derives the **G Rule** for a general tensor of rank-n.

Appendix C contains a detailed discussion of the plane (**simple**) **pendulum**, the **spherical pendulum**, and the **Foucault** mode of the spherical pendulum, including details of the **Airy precession**. A compact summary may be found at the start of Appendix C.

Appendix D defines the notion of a **center of gravity** (as distinct from a center of mass) and computes the location of the center of gravity for various orientations of a dumbbell satellite. Certain intuitive notions regarding the location of the center of gravity are seen to be not fully accurate.

Appendix E summarizes useful **facts about spherical coordinates**. An emphasis is on relations involving the spherical unit vectors including their spatial and time derivatives. The position, velocity and acceleration of a point particle are expressed in terms of these spherical unit vectors. A small section provides similar information for polar coordinates.

Appendix F undertakes a detailed study of the motion of a **dumbbell or tethered satellite** in circular orbit around the Earth. The equations of motion are obtained both from a fictitious torque analysis and a fictitious force analysis. Numerical solutions are presented for satellite librations and more general motions. A compact summary may be found at the start of Appendix F.

References are then given for all works mentioned.

1. Notation, important role of the Prime Symbol, and other Preliminaries

I am reviewing Section 1 for stability in terms of Active / Passive Views.

1.1 Basis vectors $\mathbf{e}_n$, $\mathbf{e'}_n$, rotation R, Dirac notation, the Basis Theorem, and Concatenation

Unless otherwise specified, repeated indices have implicit sums. For example $(\mathbf{e'}_n)_i\mathbf{e}_i$ means $\Sigma_i(\mathbf{e'}_n)_i\mathbf{e}_i$. This is known as the Einstein convention. We write δ_{ij} in place of the usual $\delta_{i,j}$. Both these conventions are efforts to reduce symbol clutter.

We have in mind that we are operating in Euclidian space E^3, but most everything in this section is generally valid in E^N.

Let Frame S have orthonormal basis vectors $\mathbf{e}_n$.
Let Frame S' have orthonormal basis vectors $\mathbf{e'}_n$. Thus,

$$\mathbf{e}_n \bullet \mathbf{e}_m = \delta_{nm} \qquad \mathbf{e'}_n \bullet \mathbf{e'}_m = \delta_{nm} \quad . \tag{1.1.1}$$

Assume that the two basis vector sets are related in this manner:

$$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \qquad n = 1,2,3 \qquad RR^T = 1 \quad . \tag{1.1.2}$$

This says that each basis vector of Frame S' is a certain *linear combination* of Frame S basis vectors. We shall assume that the matrix R of coefficients is real orthogonal ($R^{-1} = R^T$ or $RR^T = R^TR = 1$). Since $RR^T = 1$, we know that det(R) = ± 1. Real orthogonal matrices with det(R) = - 1 are combinations of regular rotations with a reflection, whereas for regular rotations one has det(R) = +1.

Notice that the n on $\mathbf{e}_n$ is a label and not a component index.

Using a notation described more in the Section 1.2, we expand each basis vector onto both bases:

<u>expansions</u>	<u>projections</u>
$\mathbf{e'}_n = (\mathbf{e}_i \bullet \mathbf{e'}_n)\,\mathbf{e}_i = (\mathbf{e'}_n)_i\,\mathbf{e}_i$	$(\mathbf{e'}_n)_i = (\mathbf{e}_i \bullet \mathbf{e'}_n)$
$\mathbf{e'}_n = (\mathbf{e'}_i \bullet \mathbf{e'}_n)\,\mathbf{e'}_i = (\mathbf{e'}_n)'_i\,\mathbf{e'}_i$	$(\mathbf{e'}_n)'_i = (\mathbf{e'}_i \bullet \mathbf{e'}_n)$
$\mathbf{e}_n = (\mathbf{e}_i \bullet \mathbf{e}_n)\,\mathbf{e}_i = (\mathbf{e}_n)_i\,\mathbf{e}_i$	$(\mathbf{e}_n)_i = (\mathbf{e}_i \bullet \mathbf{e}_n)$
$\mathbf{e}_n = (\mathbf{e'}_i \bullet \mathbf{e}_n)\,\mathbf{e'}_i = (\mathbf{e}_n)'_i\,\mathbf{e'}_i$	$(\mathbf{e}_n)'_i = (\mathbf{e'}_i \bullet \mathbf{e}_n)$.

(1.1.3)

Lines 2 and 3 are not very interesting because they just say what we already know,

$$\mathbf{e'}_n = \delta_{in}\,\mathbf{e'}_i = (\mathbf{e'}_n)'_i\,\mathbf{e'}_i \qquad (\mathbf{e'}_n)'_i = \delta_{in} \quad \text{projection}$$

$$\mathbf{e}_n = \delta_{in}\,\mathbf{e}_i = (\mathbf{e}_n)_i\,\mathbf{e}_i \qquad\qquad (\mathbf{e}_n)_i = \delta_{in} \quad \text{projection} . \qquad (1.1.4)$$

Now dot (1.1.2) [$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$] first with $\mathbf{e}_i$ and then with $\mathbf{e'}_i$ and use the projections in (1.1.3) to get

$$(\mathbf{e'}_n)_i = R_{nm}\,(\mathbf{e}_m)_i \qquad \text{Frame S components}$$

$$(\mathbf{e'}_n)'_i = R_{nm}\,(\mathbf{e}_m)'_i \qquad \text{Frame S' components} . \qquad (1.1.5)$$

Using (1.1.1) write the first equation as

$$(\mathbf{e'}_n)_i = R_{nm}\,\delta_{mi} = R_{ni} . \qquad (1.1.6)$$

For the second equation, one has

$$\delta_{ni} = R_{nm}\,(\mathbf{e}_m)'_i \;\Rightarrow\; R^T_{kn}\delta_{ni} = R^T_{kn}R_{nm}\,(\mathbf{e}_m)'_i$$
$$\Rightarrow\; R^T_{ki} = (R^TR)_{km}\,(\mathbf{e}_m)'_i = \delta_{km}(\mathbf{e}_m)'_i = (\mathbf{e}_k)'_i$$

and therefore

$$(\mathbf{e}_k)'_i = R^T_{ki} = R_{ik} \qquad \Rightarrow \qquad (\mathbf{e}_n)'_i = R_{in} . \qquad (1.1.7)$$

So now we know all about the components of the basis vectors in each Frame :

$$(\mathbf{e'}_n)_i = R_{ni} \qquad (\mathbf{e}_n)_i = \delta_{ni} \qquad \text{Frame S components}$$

$$(\mathbf{e}_n)'_i = R_{in} \qquad (\mathbf{e'}_n)'_i = \delta_{ni} \qquad \text{Frame S' components} \qquad (1.1.8)$$

Equation (1.1.1) says that either set of basis vectors is **orthonormal**. It is also true that each set of basis vectors is **complete**. In Frame S this means that *any* vector **a** can be expanded as $\mathbf{a} = a_n\mathbf{e}_n$. As in (1.1.3) one can then write

$$\mathbf{a} = a_n\mathbf{e}_n \quad \text{where } a_n = (\mathbf{e}_n \bullet \mathbf{a}) \qquad \Rightarrow \qquad \mathbf{a} = (\mathbf{e}_n \bullet \mathbf{a})\mathbf{e}_n \quad .$$

There is no mention here of any vector called **a'**. I think so far all is OK.

Writing this last equation out in Frame S components, one gets

$$a_j = (\,(\mathbf{e}_n)_i a_i\,)(\mathbf{e}_n)_j \qquad \text{or} \qquad a_j = [(\mathbf{e}_n)_i(\mathbf{e}_n)_j]\,a_i \quad .$$

In order that this last equation be valid for *any* **a**, it must be true that (implicit sum on n) ,

$$(\mathbf{e}_n)_i(\mathbf{e}_n)_j = \delta_{ij} \qquad \text{// completeness of the } \mathbf{e}_n \qquad (1.1.9a)$$

This is the formal statement that the $\mathbf{e}_n$ are complete. The equation is obvious since with (1.1.1) it just says $\delta_{ni}\delta_{nj} = \delta_{ij}$. Starting instead with $\mathbf{a} = (\mathbf{e'}_n \bullet \mathbf{a})\mathbf{e'}_n$ one finds $a_i = ((\mathbf{e'}_n)_i a_i)(\mathbf{e'}_n)_j$ and concludes that,

$$(\mathbf{e'}_n)_i(\mathbf{e'}_n)_j = \delta_{ij} \qquad \text{// completeness of the } \mathbf{e'}_n \qquad (1.1.9b)$$

From (1.1.8) this says $R_{ni}R_{jn} = \delta_{ij}$ which we know is true since $RR^T = 1$. Thus the completeness statements are "nothing new". As we shall see in the Dirac world, completeness is very useful.

ok to here

The Notation Problem

We shall soon be pondering equations of the following form,

$$\mathbf{a} = T\mathbf{b} \ .$$

If we had only one basis $\mathbf{e}_n$ to worry about, we would simply state that T was a matrix and the meaning of the equation $\mathbf{a} = T\mathbf{b}$ is $a_i = T_{ij}b_j$ where a_i and b_i are Frame S components of **a** and **b**. However, when there are multiple bases involved (such as in dealing with "rotating frames of reference"), the meaning of the above equation is not so clear, especially when **a** and **b** are basis vectors in different bases. As we shall see, the existence of multiple bases implies the existence of tensors which are defined in terms of the transformation between those bases.

We have found (after a lifetime of pain regarding this subject) that the so-called Dirac notation described below always provides a clean, efficient and unambiguous meaning for expressions of the above type. In a sense, it is the Gold Standard, although we generally use simpler vector notations that have more ambiguity. Whenever an equation's meaning seems unclear, one should ask what that equation looks like in Dirac notation. For this reason, we ask the reader to absorb the following Dirac Notation digression.

The Dirac notation was invented for use in quantum mechanics by Paul Dirac (1947). It appears in most quantum mechanics texts (including Saxon, Schiff, Messiah and Shankar). Various Hilbert Spaces are associated with the notation in quantum mechanics applications (spin space, configuration space, momentum space, etc.) but we will only be concerned about the Hilbert Space E^3 whose operators can always be represented by 3x3 real matrices in any given basis.

The Dirac notation does not add any new math or physics, it just makes things clearer. For example, we will say things like the following,

$$T'_{ij} = \langle \mathbf{e'}_i | \mathcal{T} | \mathbf{e'}_j \rangle = (\mathbf{e'}_i)^T T (\mathbf{e'}_j) = (* \ * \ *)\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}\begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

$$= (\mathbf{e'}_i)^T{}_n T_{nm} (\mathbf{e'}_j)_m = R_{in}T_{nm}R_{jm} = (RTR^T)_{mn}$$

In $\langle \mathbf{e'}_i | \mathcal{T} | \mathbf{e'}_j \rangle$ we imagine the existence of an operator $\mathcal{T}$ whose matrix in the $\mathbf{e'}_n$ basis is T'_{ij} . As the vector notation on the right shows, this can all be done with normal vector/matrix notation and no "operator" is needed.

Various other notations have appeared in the literature from time to time to express the above idea. For example

$$\langle \mathbf{e'}_i | \mathcal{T} | \mathbf{e'}_j \rangle = (\mathbf{e'}_i)^T{}_n T_{nm} (\mathbf{e'}_j)_m = \mathbf{e'}_i \bullet (T\mathbf{e'}_j) = \mathbf{e'}_i \bullet T \bullet \mathbf{e'}_j = \mathbf{e'}_i \overleftrightarrow{T} \mathbf{e'}_j$$

Often the Dirac notation is made even more compact by writing

$$\langle \mathbf{e'}_i | \mathcal{T} | \mathbf{e'}_j \rangle = \langle i' | \mathcal{T} | j' \rangle \qquad \text{or} \qquad \mathbf{1} = |\mathbf{e'}_j\rangle\langle \mathbf{e'}_j| = |j'\rangle\langle j'| \quad \text{(completeness)}$$

where only the minimal necessary information is displayed. We shall not take things this far.

ok to here. Only primed vectors are $\mathbf{e'}_m$ which are just another basis relative to $\mathbf{e}_n$.

Dirac Notation

In this notation, one writes

$$\mathbf{a} = |\mathbf{a}\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \text{"vector"} \qquad \text{// known as a "ket"}$$

$$\mathbf{a}^T = \langle \mathbf{a}| = (a_1, a_2, a_3) = \text{"transpose vector"} \qquad \text{// known as a "bra"} \qquad (1.1.10)$$

In formal language $|\mathbf{a}\rangle$ is a vector in the space $\mathcal{H}$ while $\langle \mathbf{a}|$ is a corresponding vector in the "dual space" $\mathcal{H}^*$ (sometimes $\langle \mathbf{a}|$ called a covector). Notice how the dot product (scalar product, inner product) works in the following example,

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{a}^T\mathbf{b} = (a_1, a_2, a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \langle \mathbf{a}|\mathbf{b}\rangle \quad . \qquad (1.1.11)$$

On the other hand, one writes

$$|\mathbf{a}\rangle\langle\mathbf{b}| = \mathbf{a}\mathbf{b}^T = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1, b_2, b_3) = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} = \text{a 3x3 matrix} \qquad (1.1.12)$$

where the vector components are implicitly in the $\mathbf{e}_n$ basis.

Comments:
Somtimes $\mathbf{a}\mathbf{b}^T$ is written $\mathbf{a}\mathbf{b}$ and is called a "dyadic product" or a "dyad".
A Hilbert Space is basically a vector space with an inner product like $\mathbf{a} \bullet \mathbf{b} = \langle \mathbf{a}|\mathbf{b}\rangle$.

Since our space $\mathcal{H}$ is real (not complex), we know that

$$\langle \mathbf{a}|\mathbf{b}\rangle = \langle \mathbf{b}|\mathbf{a}\rangle \text{ or } \mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \; . \qquad (1.1.13)$$

We can of course let **a** and **b** be any of the basis vectors $\mathbf{e}_n$, $\mathbf{e'}_n$ or $\mathbf{e''}_n$. For example, using (1.1.8),

$$\delta_{nm} = (\mathbf{e}_m)_n = \mathbf{e}_n \bullet \mathbf{e}_m = \mathbf{e}_n{}^T\mathbf{e}_m = \langle \mathbf{e}_n|\mathbf{e}_m\rangle = \langle \mathbf{e}_m|\mathbf{e}_n\rangle$$

$$R_{mn} = (\mathbf{e'}_m)_n = \mathbf{e}_n \bullet \mathbf{e'}_m = \mathbf{e}_n{}^T\mathbf{e'}_m = \langle \mathbf{e}_n|\mathbf{e'}_m\rangle = \langle \mathbf{e'}_m|\mathbf{e}_n\rangle \ . \qquad (1.1.14)$$

We now imagine that $\mathcal{T}$ is some **operator** in $\mathcal{H}$, and $|\mathbf{a}\rangle$ is some vector in $\mathcal{H}$. We write,

$$\mathcal{T}\,|\mathbf{a}\rangle = |T\mathbf{a}\rangle \quad \text{where } |T\mathbf{a}\rangle \text{ is some new vector in } \mathcal{H}. \qquad (1.1.15)$$

In particular, we can write for the basis vectors $\mathbf{e}_n$

$$\mathcal{T}\,|\mathbf{e}_n\rangle = |T\mathbf{e}_n\rangle \ . \qquad (1.1.16)$$

The definition of $|T\mathbf{e}_n\rangle$ is that it is what you get by applying operator $\mathcal{T}$ to the vector $|\mathbf{e}_n\rangle$.

If we want to know the Frame S components of the vector $|T\mathbf{e}_n\rangle$, we calculate them :

$$\langle \mathbf{e}_m|\,\mathcal{T}\,|\mathbf{e}_n\rangle = \langle \mathbf{e}_m|\,T\mathbf{e}_n\rangle = [T\mathbf{e}_n]_m \ . \qquad (1.1.17)$$

We then define the *matrix* T to be

$$T_{mn} \equiv \langle \mathbf{e}_m|\,\mathcal{T}\,|\mathbf{e}_n\rangle \quad \text{so then} \quad [T\mathbf{e}_n]_m = T_{mn} \ . \qquad (1.1.18)$$

Repeating the above in Frame S' gives,

$$T'_{mn} \equiv \langle \mathbf{e'}_m|\,\mathcal{T}\,|\mathbf{e'}_n\rangle = \langle \mathbf{e'}_m|T\mathbf{e'}_n\rangle = [T\mathbf{e'}_n]'_m \qquad (1.1.19)$$

where we have now found the Frame S' components of $T\mathbf{e'}_n = \mathcal{T}\,|\mathbf{e'}_n\rangle$.

Notice the distinction between the matrices T and T', and the symbol T in the vectors $[T\mathbf{e}_n]$ and $[T\mathbf{e'}_n]$. It is the same symbol T because these vectors are $\mathcal{T}\,|\mathbf{e}_n\rangle$ and $\mathcal{T}\,|\mathbf{e'}_n\rangle$ with the same operator $\mathcal{T}$. The symbol T in $[T\mathbf{e'}_n]$ is not itself a matrix, it is part of the name of the vector $[T\mathbf{e'}_n]$.

In all of what follows, repeated indices are implicitly summed.

One operator in $\mathcal{H}$ of special interest is the unity operator $\mathcal{1}$ such that $\mathcal{1}\,|\mathbf{a}\rangle = |\,1\mathbf{a}\rangle = |\mathbf{a}\rangle$ for any vector $|\mathbf{a}\rangle$ in $\mathcal{H}$. In the Dirac notation one can write, for example in the $\mathbf{e'}_n$ basis,

$$\mathcal{1} = |\mathbf{e'}_n\rangle\langle \mathbf{e'}_n| \ . \qquad \text{// implied sum on n !!}$$

This is in fact the statement of completeness in the $\mathbf{e'}_n$ basis. "Closing" with $\langle \mathbf{e}_i|$ on the left and $|\mathbf{e}_j\rangle$ on the right, one gets

$$\delta_{ij} = \langle \mathbf{e}_i | \mathbf{e}_j \rangle = \langle \mathbf{e}_i | \boldsymbol{1} | \mathbf{e}_j \rangle = \langle \mathbf{e}_i | \mathbf{e}'_n \rangle \langle \mathbf{e}'_n | \mathbf{e}_j \rangle = (\mathbf{e}'_n)_i (\mathbf{e}'_n)_j$$

and this replicates the completeness statement (1.1.9b). Completeness is valid in any basis, so

$$\boldsymbol{1} = |\mathbf{e}_i\rangle\langle\mathbf{e}_i| = |\mathbf{e}'_i\rangle\langle\mathbf{e}'_i| = |\mathbf{e}''_i\rangle\langle\mathbf{e}''_i| \qquad \text{completeness in Frames S, S' and S''} \ . \qquad (1.1.20)$$

One might wonder how the matrices T and T' are related. Consider,

$$T'_{mn} \equiv \langle \mathbf{e}'_m | \mathcal{T} | \mathbf{e}'_n \rangle = \langle \mathbf{e}'_m | \boldsymbol{1} \mathcal{T} \boldsymbol{1} | \mathbf{e}'_n \rangle = \langle \mathbf{e}'_m | \mathbf{e}_i \rangle \langle \mathbf{e}_i | \mathcal{T} | \mathbf{e}_j \rangle \langle \mathbf{e}_j | \mathbf{e}'_n \rangle$$

$$= R_{mi}\, T_{ij}\, R_{nj} = R_{mi}\, T_{ij}\, (R^{-1})_{jn} = [R\, T\, R^{-1}]_{mn}$$

Thus the relationship is

$$T' = RTR^{-1} \ . \qquad (1.1.21)$$

This is in fact the transformation rule for the rank-2 tensor T, analogous to the transformation rule for a rank-1 tensor which is simply $\mathbf{V}' = R\mathbf{V}$. (****)

First mention of a primed vector! The meaning of $\mathbf{V}' = R\mathbf{V}$ is undefined, I have not yet discussed Apparatus Vectors vs Basis Vectors vs Active vs Passive. Right here, this is just a quick passing reference which probably does not need detailed explaining here. But it must be done somewhere. Let's just continue with this critical review and see if there are any other primed vectors mentioned.

The actual tensor is the operator $\mathcal{T}$ while T is the matrix which represents $\mathcal{T}$ in the $\mathbf{e}_n$ basis. This tensor $\mathcal{T}$ can be expanded on the various bases in this manner

$$\mathcal{T} = T_{ij} |\mathbf{e}_i\rangle\langle\mathbf{e}_j| = T'_{ij} |\mathbf{e}'_i\rangle\langle\mathbf{e}'_j| = T''_{ij} |\mathbf{e}''_i\rangle\langle\mathbf{e}''_j| \ . \quad \text{// implied sum on i and j} \qquad (1.1.22)$$

To verify that this is true, we can close for example with $\langle \mathbf{e}'_m |$ and $|\mathbf{e}'_n\rangle$ to get

$$\langle \mathbf{e}'_m | \mathcal{T} | \mathbf{e}'_n \rangle = T'_{ij} \langle \mathbf{e}'_m | \mathbf{e}'_i \rangle \langle \mathbf{e}'_j | \mathbf{e}'_n \rangle = T'_{ij}\, \delta_{mi} \delta_{jm} = T'_{mn}$$

Similarly,

$$\langle \mathbf{e}'_m | \mathcal{T} | \mathbf{e}'_n \rangle = T_{ij} \langle \mathbf{e}'_m | \mathbf{e}_i \rangle \langle \mathbf{e}_j | \mathbf{e}'_n \rangle = T_{ij}\, R_{mi} R_{nj} = R_{mi} T_{ij} (R^{-1})_{jn} = [RTR^{-1}]_{mn} = T'_{mn}$$

and

$$\langle \mathbf{e}_m | \mathcal{T} | \mathbf{e}_n \rangle = T'_{ij} \langle \mathbf{e}_m | \mathbf{e}'_i \rangle \langle \mathbf{e}'_j | \mathbf{e}_n \rangle = T'_{ij}\, R_{im} R_{jn} = (R^{-1})_{mi}\ T'_{ij}\, R_{jn} = [R^{-1} T' R]_{mn} = T_{mn}$$

If $\mathcal{T} = \boldsymbol{1}$, the above expansion replicates the statement of completeness,

$$\boldsymbol{1} = (1)_{ij} |\mathbf{e}_i\rangle\langle\mathbf{e}_j| = \delta_{ij} |\mathbf{e}_i\rangle\langle\mathbf{e}_j| = |\mathbf{e}_i\rangle\langle\mathbf{e}_i| \ .$$

Sometimes one writes $\mathcal{T} = |\mathbf{e}_i\rangle T_{ij} \langle\mathbf{e}_j|$ so then $\boldsymbol{1} = |\mathbf{e}_i\rangle \delta_{ij} \langle\mathbf{e}_j| = |\mathbf{e}_i\rangle\langle\mathbf{e}_i|$.

The Basis Theorem

Recall now our assumed linear combination sum (1.1.2) which states

$$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \qquad \text{or} \qquad |\mathbf{e'}_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \ . \tag{1.1.23}$$

Again the prime on $\mathbf{e'}_n$ just indicates a basis vector that is different from $\mathbf{e}_n$ and is the Frame S' basis.

One regards the matrix R_{nm} as the representation of an operator $\boldsymbol{\mathcal{R}}$ in the $\mathbf{e}_n$ basis, as in (1.1.18) for T, so

$$R_{nm} = \langle \mathbf{e}_n \,|\, \boldsymbol{\mathcal{R}}\, |\mathbf{e}_m\rangle \ . \tag{1.1.24}$$

Note that

$$\delta_{nm} = \langle \mathbf{e}_n | \mathbf{e}_m\rangle = \langle \mathbf{e}_n \,|\, \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}^{-1}\, |\mathbf{e}_m\rangle = \langle \mathbf{e}_n \,|\, \boldsymbol{\mathcal{R}}\, |\mathbf{e}_i\rangle\langle \mathbf{e}_i|\, \boldsymbol{\mathcal{R}}^{-1}\, |\mathbf{e}_m\rangle = R_{ni}\,\langle \mathbf{e}_i|\, \boldsymbol{\mathcal{R}}^{-1}\, |\mathbf{e}_m\rangle$$

so one must conclude that

$$\langle \mathbf{e}_i|\, \boldsymbol{\mathcal{R}}^{-1}\, |\mathbf{e}_m\rangle = (R^{-1})_{im} \ . \tag{1.1.25}$$

Now apply $\boldsymbol{\mathcal{R}}$ to (1.1.23) to get

$$\boldsymbol{\mathcal{R}}|\mathbf{e'}_n\rangle = R_{nm}\,\boldsymbol{\mathcal{R}}|\mathbf{e}_m\rangle \ .$$

From (1.1.16) the left side is $|R\mathbf{e'}_n\rangle$ while the right side is

$$R_{nm}\,\boldsymbol{\mathcal{R}}|\mathbf{e}_m\rangle = R_{nm}\,|\mathbf{e}_i\rangle\langle \mathbf{e}_i|\boldsymbol{\mathcal{R}}|\mathbf{e}_m\rangle = R_{nm}\,|\mathbf{e}_i\rangle\, R_{im} = R_{nm}R_{im}|\mathbf{e}_i\rangle = R_{nm}R^{T}_{mi}|\mathbf{e}_i\rangle$$

$$= (RR^{T})_{ni}\,|\mathbf{e}_i\rangle = \delta_{ni}\,|\mathbf{e}_i\rangle = |\mathbf{e}_n\rangle \ .$$

Thus we have shown that

$$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \qquad \Rightarrow \qquad |R\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle \ \text{ or } \ \boldsymbol{\mathcal{R}}|\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle \ \text{ or } \ R\mathbf{e'}_n = \mathbf{e}_n \tag{1.1.26}$$

where on the right we show three equivalent forms of the same equation.

[In the following sequence of steps, we show Dirac notation on the left and vector notation on the right.]

Conversely to the above, suppose we know that

$$\boldsymbol{\mathcal{R}}|\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle \qquad\qquad R\mathbf{e'}_n = \mathbf{e}_n \ .$$

Inverting we get

$$|\mathbf{e'}_n\rangle = \mathcal{R}^{-1}|\mathbf{e}_n\rangle \qquad\qquad \mathbf{e'}_n = R^{-1}\mathbf{e}_n \,.$$

Since the basis $\mathbf{e}_n$ is complete, we know we can write, for some unknown coefficients A_{nm} ,

$$|\mathbf{e'}_n\rangle = A_{nm}\,|\mathbf{e}_m\rangle \qquad\qquad \mathbf{e'}_n = A_{nm}\mathbf{e}_m \,. \tag{1.1.27}$$

Comparing the last two equations one has,

$$A_{nm}\,|\mathbf{e}_m\rangle = \mathcal{R}^{-1}|\mathbf{e}_n\rangle = |R^{-1}\mathbf{e}_n\rangle \qquad\qquad A_{nm}\mathbf{e}_m = R^{-1}\mathbf{e}_n \;.$$

Now close with $\langle\mathbf{e}_i|$ on the left to get

$$A_{nm}\,\langle\mathbf{e}_i\,|\mathbf{e}_m\rangle = \langle\mathbf{e}_i\,|\mathcal{R}^{-1}|\mathbf{e}_n\rangle \qquad\qquad A_{nm}\,(\mathbf{e}_m)_i \;=\; [R^{-1}\mathbf{e}_n]_i \;= (R^{-1})_{ik}(\mathbf{e}_n)_k$$

or

$$A_{nm}\,\delta_{im} = (R^{-1})_{in} \qquad\qquad A_{nm}\,\delta_{mi} = (R^{-1})_{ik}\,\delta_{nk} = (R^{-1})_{in}$$

or

$$A_{ni} = R_{ni} \qquad\qquad A_{ni} = R_{ni} \,.$$

Therefore (1.1.27) becomes,

$$|\mathbf{e'}_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \,.$$

Thus we have shown that

$$\mathcal{R}|\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle \quad\Rightarrow\quad |\mathbf{e'}_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \tag{1.1.28}$$

or

$$R\mathbf{e'}_n = \mathbf{e}_n \quad\Rightarrow\quad \mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \;.$$

We have now proved a simple theorem which seems to have no name, so we give it a name:

The Basis Theorem:

$$\begin{aligned} &\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \quad\Leftrightarrow\quad R\mathbf{e'}_n = \mathbf{e}_n && \text{// vector notation} \\ &|\mathbf{e'}_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \quad\Leftrightarrow\quad |R\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle && \text{// Dirac notation} \\ &\qquad\qquad\qquad\quad \mathcal{R}|\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle \end{aligned} \tag{1.1.29}$$

I think this is still OK, since no mention of any **V'** type vector.

The equation on the left concerns Frame S' basis vectors being linear combinations of Frame S basis vectors. The equation on the right says that the rotation operator $\mathcal{R}$ acting on $|\mathbf{e'}_n\rangle$ creates a rotated vector called $|R\mathbf{e'}_n\rangle$ (or $R\mathbf{e'}_n$) which is equal to $|\mathbf{e}_n\rangle$ (or $\mathbf{e}_n$).

We can invert both sides of this theorem to get

$$\mathbf{e}_n = (R^{-1})_{nm}\,\mathbf{e}'_m \qquad \Leftrightarrow \qquad \mathbf{e}'_n = R^{-1}\mathbf{e}_n \qquad \text{// vector notation}$$

$$|\mathbf{e}_n\rangle = (R^{-1})_{nm}\,|\mathbf{e}'_m\rangle \qquad \Leftrightarrow \qquad |\mathbf{e}'_n\rangle = |R^{-1}\mathbf{e}_n\rangle \qquad \text{// Dirac notation} \qquad (1.1.30)$$
$$|\mathbf{e}'_n\rangle = \boldsymbol{\mathcal{R}}^{-1}\,|\mathbf{e}_n\rangle$$

Alternate shorthand notations

$$\mathbf{e}_n = R\mathbf{e}'_n \qquad \Leftrightarrow \qquad (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = R\,(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) = \boldsymbol{\mathcal{R}}\,(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) \qquad (1.1.31)$$

$$\mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \qquad \Leftrightarrow \qquad \begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = R \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} . \qquad (1.1.32)$$

The first alternate notation implies for example that $\mathbf{e}_1 = R\mathbf{e}'_1$. This is *not* implied by the second alternate notation which is meant to say $\mathbf{e}'_1 = R_{11}\mathbf{e}_1 + R_{12}\mathbf{e}_2 + R_{13}\mathbf{e}_3$ = linear combination of vectors. These alternate notations are useful when the basis vectors have names like $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ or $\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}$.

The matrices R and R'

Consider now the relation $\mathbf{e}_n = R\mathbf{e}'_n$ so that R relates the Frame S and Frame S' bases, as above. In this case, taking components one gets,

$$(\mathbf{e}_n)_i = [R\mathbf{e}'_n]_i = R_{ij}\,(e'_n)_j \qquad \text{Frame S components}$$

$$(\mathbf{e}_n)'_i = [R\mathbf{e}'_n]'_i = R'_{ij}\,(e'_n)'_j \qquad \text{Frame S' components} \quad \text{// note prime on } R'_{ij} \qquad (1.1.33)$$

In Dirac notation the above two lines may be expressed as,

$$(\mathbf{e}_n)_i = \langle \mathbf{e}_i \mid \mathbf{e}_n \rangle = \langle \mathbf{e}_i \mid R\mathbf{e}'_n \rangle = \langle \mathbf{e}_i \mid \boldsymbol{\mathcal{R}} \mid \mathbf{e}'_n \rangle = \langle \mathbf{e}_i \mid \boldsymbol{\mathcal{R}} \mid \mathbf{e}_j \rangle\langle \mathbf{e}_j \mid \mathbf{e}'_n \rangle = R_{ij}(\mathbf{e}'_n)_j$$

$$(\mathbf{e}_n)'_i = \langle \mathbf{e}'_i \mid \mathbf{e}_n \rangle = \langle \mathbf{e}'_i \mid R\mathbf{e}'_n \rangle = \langle \mathbf{e}'_i \mid \boldsymbol{\mathcal{R}} \mid \mathbf{e}'_n \rangle = \langle \mathbf{e}'_i \mid \boldsymbol{\mathcal{R}} \mid \mathbf{e}'_j \rangle\langle \mathbf{e}'_j \mid \mathbf{e}'_n \rangle = R'_{ij}\,(\mathbf{e}'_n)'_j \ .$$

We encounter two matrices here,

$$R_{ij} = \langle \mathbf{e}_i \mid \boldsymbol{\mathcal{R}} \mid \mathbf{e}_j \rangle$$

$$R'_{ij} = \langle \mathbf{e}'_i \mid \boldsymbol{\mathcal{R}} \mid \mathbf{e}'_j \rangle \ . \qquad (1.1.34)$$

Because the operator $\boldsymbol{\mathcal{R}}$ is the same operator which relates the two bases, these two matrices are the same,

$$R'_{ij} = \langle \mathbf{e}'_i \mid \boldsymbol{\mathcal{R}} \mid \mathbf{e}'_j \rangle = \langle \mathbf{e}'_i \mid \mathbf{e}_n \rangle\langle \mathbf{e}_n \mid \boldsymbol{\mathcal{R}} \mid \mathbf{e}_m \rangle\langle \mathbf{e}_m \mid \mathbf{e}'_j \rangle = R_{in}\,R_{nm}\,R_{jm}$$

$$= R_{in}\,R_{nm}\,R^{T}_{mj} = R_{in}\,(RR^{T})_{nj} = R_{in}\,\delta_{nj} = R_{ij} \ . \qquad (1.1.35)$$

This fact is abundantly clear from (1.1.21) $T' = RTR^{-1}$ which in this case says $R' = RRR^{-1} = R$.

We have managed so far to avoid the following Dirac notation facts, but now is a good time to get them on the table. Here we show Dirac notation on the left, and vector notation on the right :

$$|\mathbf{a}\rangle = |X\mathbf{b}\rangle = \mathcal{X}\,|\mathbf{b}\rangle \qquad\qquad \mathbf{a} = X\mathbf{b}$$

$$\langle\mathbf{a}| = \langle X\mathbf{b}| = \langle\mathbf{b}|\,\mathcal{X}^{\mathbf{T}} \qquad\qquad \mathbf{a}^{\mathbf{T}} = (X\mathbf{b})^{\mathbf{T}} = \mathbf{b}^{\mathbf{T}}X^{\mathbf{T}}$$

$$\langle\mathbf{c}\,|\,\mathcal{X}\,|\,\mathbf{d}\rangle = \langle\mathbf{c}\,|\,X\mathbf{d}\rangle = \langle X\mathbf{d}\,|\mathbf{c}\rangle = \langle\mathbf{d}|\,\mathcal{X}^{\mathbf{T}}\,|\mathbf{c}\rangle \qquad\qquad \mathbf{c}^{\mathbf{T}}X\mathbf{d} = (\mathbf{c}^{\mathbf{T}}X\mathbf{d})^{\mathbf{T}} = \mathbf{d}^{\mathbf{T}}X^{\mathbf{T}}\mathbf{c}\ . \qquad (1.1.36)$$

Notice in $\langle\mathbf{a}| = \langle\mathbf{b}|\,\mathcal{X}^{\mathbf{T}}$ that the operator $\mathcal{X}^{\mathbf{T}}$ acts to the left, just as the matrix in $\mathbf{b}^{\mathbf{T}}X^{\mathbf{T}}$ acts to the left on the transpose vector $\mathbf{b}^{\mathbf{T}}$. Also, $\mathbf{c}^{\mathbf{T}}X\mathbf{d}$ is just a number, so $(\mathbf{c}^{\mathbf{T}}X\mathbf{d}) = (\mathbf{c}^{\mathbf{T}}X\mathbf{d})^{\mathbf{T}}$.

Suppose one knows that $XX^{\mathbf{T}} = 1$. That says $X_{ij}(X^{\mathbf{T}})_{jk} = \delta_{ik}$ or $X_{ij}X_{kj} = \delta_{ik}$ or

$$\langle\mathbf{e}_i|\,\mathcal{X}\,|\mathbf{e}_j\rangle\langle\mathbf{e}_k|\,\mathcal{X}\,|\mathbf{e}_j\rangle = \delta_{ik}$$

or

$$\langle\mathbf{e}_i|\,\mathcal{X}\,|\mathbf{e}_j\rangle\langle\mathbf{e}_j|\,\mathcal{X}^{\mathbf{T}}\,|\mathbf{e}_k\rangle = \delta_{ik} \qquad \text{// (1.1.36)}$$

or

$$\langle\mathbf{e}_i|\,\mathcal{X}\mathcal{X}^{\mathbf{T}}\,|\mathbf{e}_k\rangle = \delta_{ik}\ . \qquad \text{// (1.1.20)}$$

Thus it must be that

$$XX^{\mathbf{T}} = 1 \ \Leftrightarrow\ \mathcal{X}\mathcal{X}^{\mathbf{T}} = \mathit{1} \qquad (1.1.37)$$

as one would expect.

As an application of the above consider this fact, where R is our usual real orthogonal rotation matrix,

$$\mathbf{a}\bullet\mathbf{b} = [R\mathbf{a}]\bullet[R\mathbf{b}] \qquad \text{or} \qquad \langle\mathbf{a}|\mathbf{b}\rangle = \langle R\mathbf{a}|R\mathbf{b}\rangle\ . \qquad (1.1.38)$$

Proof: In vector notation one has, using $\mathbf{e}_n$ vector components,

$$[R\mathbf{a}]\bullet[R\mathbf{b}] = [R\mathbf{a}]_k[R\mathbf{b}]_k = R_{ki}a_iR_{kj}b_j \overset{(1.1.33)}{=} (R^{\mathbf{T}})_{jk}\,R_{ki}a_ib_j$$

$$= (R^{\mathbf{T}}R)_{ji}a_ib_j = \delta_{ji}a_ib_j = a_ib_i = \mathbf{a}\bullet\mathbf{b}\ .$$

In Dirac notation the proof reads, using (1.1.36) and (1.1.37),

$$\langle R\mathbf{a}|R\mathbf{b}\rangle = \langle\mathbf{a}|\mathcal{R}^{\mathbf{T}}\mathcal{R}|\mathbf{b}\rangle = \langle\mathbf{a}|\,\mathit{1}\,|\mathbf{b}\rangle = \langle\mathbf{a}|\mathbf{b}\rangle\ .$$

Notice no mention at this point of $\langle\mathbf{a}'|\mathbf{b}'\rangle$ which would bring into question the meaning of $\mathbf{a}'$.

Time dependence of the R_{ij}

If Frame S is fixed and Frame S' is rotating, then we really have $\mathbf{e}_n = R(t)\mathbf{e}'_n(t)$ where the R matrix is a function of time, so we have $R_{ij}(t)$. Similarly, if Frame S is moving and Frame S' is fixed, $\mathbf{e}_n(t) = R(t)\,\mathbf{e}'_n$ and again one has $R_{ij}(t)$. Only in the case where there is no rotation between the frames are the R_{ij} independent of time. This means that $\boldsymbol{\omega} = 0$ in Fig 1. Since our document is about "rotating frames of reference" we exclude this no-rotation case from consideration.

ok to here

Dealing with Concatenated Transformations

Up to this point, we have dealt with a single rotation transformation $\mathbf{e}_n = R\,\mathbf{e}'_n$ for which the following facts are true (the left side is the Basis Theorem),

$$\mathbf{e}_n = R\,\mathbf{e}'_n \overset{(1.1.29)}{\Leftrightarrow} \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \qquad \mathbf{e}'_i \bullet \mathbf{e}_j \overset{(1.1.8)}{=} R_{ij}\ ,\quad T' \overset{(1.1.21)}{=} RTR^{-1}\ ,\quad R \overset{(1.1.35)}{=} R'\ . \tag{1.1.39}$$

Suppose we have a second rotation transformation $\mathbf{e}'_n = S\,\mathbf{e}''_n$. Then the claim is that,

$$\mathbf{e}'_n = S\,\mathbf{e}''_n \Leftrightarrow \mathbf{e}''_n = S'_{nm}\,\mathbf{e}'_m \qquad \mathbf{e}''_i \bullet \mathbf{e}'_j = S'_{ij}\,,\quad T'' = S'T'S'^{-1}\ ,\quad S' = S''\ . \tag{1.1.40}$$

Again, the left side is just a statement of the Basis Theorem for this second transformation. To verify the items on the right, consider

$$\mathbf{e}''_i \bullet \mathbf{e}'_j = [S'_{im}\,\mathbf{e}'_m] \bullet \mathbf{e}'_j = S'_{im}\,[\,\mathbf{e}'_m \bullet \mathbf{e}'_j\,] = S'_{im}\,\delta_{mj} = S'_{ij}\ .$$

Then,

$$T''_{ij} = \langle \mathbf{e}''_i \,|\, \mathcal{T}\, |\mathbf{e}''_j\rangle = \langle \mathbf{e}''_i \,|\, e'_m\rangle\langle e'_m|\, \mathcal{T}\, |e'_n\rangle\langle e'_n\,|\,\mathbf{e}''_j\rangle = S'_{im}\,T'_{mn}\,S'_{jn}$$

$$= S'_{im}\,T'_{mn}\,S''^{T}_{jn} = S'_{im}\,T'_{mn}\,S'^{-1}_{jn} = (S'T'S'^{-1})_{jj}$$

which shows that

$$T'' = S'T'S'^{-1}\ .$$

Finally, applying this last equation to T'' = S'' one finds,

$$S'' = S'S'S'^{-1} = S'\ .$$

so all three items in the right of (1.1.40) are verified.

Notice now that

$$\mathbf{e}_n = R\,\mathbf{e}'_n = R\,(S\,\mathbf{e}''_n) = RS\,\mathbf{e}''_n\ . \qquad //\ |\mathbf{e}_n\rangle = \mathcal{RS}\,|\mathbf{e}''_n\rangle$$

Also,

$$\mathbf{e''}_n = S'_{nm}\,\mathbf{e'}_m = S'_{nm}\,(R_{mk}\,\mathbf{e}_k) = S'_{nm}\,R_{mk}\,\mathbf{e}_k = (S'R)_{nk}\mathbf{e}_k \ .$$

On the other hand, we can apply the Basis Theorem directly to Q ≡ SR to get

$$\mathbf{e}_n = RS\mathbf{e''}_n \qquad \Leftrightarrow \qquad \mathbf{e''}_n = (RS)_{nk}\,\mathbf{e}_k \ .$$

Comparing the right sides of the last two $\mathbf{e''}_n$ expressions one finds the seemingly contradictory result that

$$(S'R)_{nk}\mathbf{e}_k = (RS)_{nk}\,\mathbf{e}_k$$

where the matrices seem to have reverse order on the two sides. But there is no contradiction because we know that $S' = RSR^{-1}$ and therefore S'R = RS.

Since we are going to be dealing later with Euler angles which involve *three* basis transformations, we consider finally a third rotation transformation U whose facts the reader can easily verify,

$$\mathbf{e''}_n = U\,\mathbf{e'''}_n \ \Leftrightarrow \ \mathbf{e'''}_n = U''_{nm}\,\mathbf{e''}_m \qquad \mathbf{e'''}_i \bullet \mathbf{e''}_j = U''_{ij}\ ,\ T''' = U''T''U''^{-1}\ ,\ U'' = U'''\ . \quad (1.1.41)$$

Concatentating transformations in the ways shown above one gets,

$$\mathbf{e}_n = R\mathbf{e'}_n = RS\mathbf{e''}_n = RSU\mathbf{e'''}_n \qquad // = \mathcal{RSU}\,|\mathbf{e'''}_n\rangle \quad (1.1.42)$$

$$\mathbf{e'''}_n = U''_{nm}\,\mathbf{e''}_m = U''_{nm}S'_{mi}\,\mathbf{e'}_i = U''_{nm}S'_{mi}\,R_{ij}\,\mathbf{e}_j = (U''S'R)_{nj}\,\mathbf{e}_j \ ,$$

Now thinking RSU = Q, the Basis Theorem says,

$$\mathbf{e}_n = RSU\mathbf{e'''}_n \qquad \Leftrightarrow \qquad \mathbf{e'''}_n = (RSU)_{nk}\,\mathbf{e}_k \ . \quad (1.1.43)$$

Comparing the last two expressions for $\mathbf{e'''}_n$ we get (again with reverse ordered matrices),

$$(U''S'R)_{nj}\,\mathbf{e}_j = (RSU)_{nk}\,\mathbf{e}_k \ .$$

Can we show that in fact U''S'R = RSU ? Consider, using the tensor rules shown above,

$$U''S'R = (S'U'S'^{-1})S'R = S'U'R = (RSR^{-1})(RUR^{-1})R = RSU$$

so again there is no contradiction.

Since the matrices like R, S and U in basis $\mathbf{e}_n$ (Frame S) are likely to be known, whereas the others might have to be calculated, we prefer the RSU form shown in (1.1.43). In the alternate shorthand notation of (1.1.31) and (1.1.32) we shall write (1.1.42) and (1.1.43) in this manner

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathrm{RSU}(\mathbf{e}'''_1, \mathbf{e}'''_2, \mathbf{e}'''_3) \quad \text{or} \quad (|\mathbf{e}_1>, |\mathbf{e}_2>, |\mathbf{e}_3>) = \mathcal{RSU}(|\mathbf{e}'''_1>, |\mathbf{e}'''_2>, |\mathbf{e}'''_3>) \tag{1.1.44}$$

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = \mathrm{RSU} \begin{pmatrix} \mathbf{e}'''_1 \\ \mathbf{e}'''_2 \\ \mathbf{e}'''_3 \end{pmatrix} . \tag{1.1.45}$$

In (1.1.45), R S and U are always matrices in the $\mathbf{e}_n$ (Frame S) basis. In (1.1.44), recall that [RSU$\mathbf{e}'''_1$] is the name of the vector obtained by applying $\mathcal{RSU}$ to the vector $\mathbf{e}'''_1$. If we take $\mathbf{e}_n$ components of this equation, *then* we may regard R, S and U as Frame S matrices, since in that case,

$$(\mathbf{e}_n)_i = (\mathrm{RSU})_{ij} (\mathbf{e}'''_n)_j = R_{in} S_{nm} U_{mj} (\mathbf{e}'''_n)_j .$$

However, if we take Frame S' (or some other frame) components, we get different matrices, for example,

$$(\mathbf{e}_n)'_i = (\mathrm{RSU})'_{ij} (\mathbf{e}'''_n)'_j = R'_{in} S'_{nm} U'_{mj} (\mathbf{e}'''_n)'_j$$

We shall make use of (1.1.44) and (1.1.45) in our discussion of Euler Angle rotations in Appendix G.

OK, apart from the one inconsistency noted between T' = RTR^{-1} and unstated V' = RV, I think all of the above is OK. But I smell trouble coming below.

1.2 Expansions of a vector and use of primes and parentheses

Note: We write $(a)_i$ as a component of vector $\mathbf{a}$, but $(\mathbf{e}_n)_i$ as a component of $\mathbf{e}_n$. In the first case the a in $(a)_i$ is not bolded, but since $\mathbf{e}_n$ is decorated with a label n, it gets bolded. It is just our convention.

Any vector $\mathbf{a}$ can be expanded on either set of basis vectors $\mathbf{e}_i$ (Frame S) or $\mathbf{e}'_i$ (Frame S') so that, with implied summation on i,

$$\mathbf{a} = a_i \mathbf{e}_i = a'_i \mathbf{e}'_i \quad . \qquad a_i = \mathbf{a} \bullet \mathbf{e}_i \qquad a'_i = \mathbf{a} \bullet \mathbf{e}'_i \quad . \tag{1.2.1}$$

If some other vector named $\mathbf{a}'$ is lurking in the wings, one might want to be more careful labeling components. A safe method is this:

$$\begin{aligned} \mathbf{a} &= (a)_i \mathbf{e}_i = (a)'_i \mathbf{e}'_i & (a)_i &= \mathbf{a} \bullet \mathbf{e}_i & (a)'_i &= \mathbf{a} \bullet \mathbf{e}'_i \\ \mathbf{a}' &= (a')_i \mathbf{e}_i = (a')'_i \mathbf{e}'_i & (a')_i &= \mathbf{a}' \bullet \mathbf{e}_i & (a')'_i &= \mathbf{a}' \bullet \mathbf{e}'_i \quad . \end{aligned} \tag{1.2.2}$$

Here, a prime *inside* a parentheses is part of the vector name, whereas a prime *outside* a parentheses denotes a vector component in Frame S' (whereas no prime outside means a component in Frame S). Unless the relationship between vectors $\mathbf{a}$ and $\mathbf{a}'$ has a certain simple form, it is very likely that $(a')_i \neq (a)'_i$. In this case the notation a'_i would be ambiguous since one doesn't know whether it refers to $(a')_i$ or $(a)'_i$. It is true that the notation a_i could be unambiguously identified with $(a)_i$, but we shall maintain the parentheses just to be uniform.

Matrix Notation to show how components are related.

Let R be the transformation appearing in the Basis Theorem (1.1.29) such that $\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$ and $\mathbf{e}_n = R\mathbf{e'}_n$. Consider the following expansion of vector **a** on the basis vectors $\mathbf{e'}_j$,

$\mathbf{a} = (a)'_j\,\mathbf{e'}_j$ // (1.2.2)

$= (a)'_j\,\{ R_{ji}\,\mathbf{e}_i \}$ // $\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$, linear combination of vectors

$= (a)'_j\,\{ (R^{-1})_{ij}\,\mathbf{e}_i \}$ // $R = (R^{-1})^T$ real orthogonal rotation

$= \{ (R^{-1})_{ij}(a)'_j \)\mathbf{e}_i$. // reorder (1.2.3)

Comparing this to $\mathbf{a} = (a)_i\,\mathbf{e}_i$ of (1.2.2) we conclude that, since $\mathbf{e}_i$ is a complete basis,

$(a)_i = (R^{-1})_{ij}(a)'_j$ so $(a)'_i = R_{ij}(a)_i$ (1.2.4)

where R^{-1} is a 3x3 rotation matrix.

We can repeat the above discussion replacing **a** with **a'** with this result,

$(a')_i = (R^{-1})_{ij}(a')'_j$. so $(a')'_i = R_{ij}(a')_i$ (1.2.5)

These matrix equations are convenient for computing the components of a vector on the $\mathbf{e}_i$ basis if they are known in the $\mathbf{e'}_i$ basis (and vice versa) .

1.3 Active and Passive Views of rotation, and a review of dot products

In this discussion, Frame S and Frame S' have the same origin, which we shall later call **Special Case #4**. This is a more restricted situation than that illustrated in Fig 1 where **r** is a particle in an Apparatus but the Frames have different origins separated by vector **b**.

Basis Vectors, Kinematic Vectors, Active and Passive Views

We imagine an Apparatus in Frame S which "contains" (is described by, is associated with) various vectors of interest (in addition to scalars and perhaps fancier tensors). We refer to these vectors as **Kinematic Vectors**, examples being the position **r** of some particle of the apparatus, or the velocity **v** or acceleration **a** of that particle, or some electric field **E** at point **r**.

If the entire apparatus is forward-rotated by R about the Frame S origin, all these Kinematic Vectors transform in the usual manner $(V')_i = R_{ij}V_i$ or **V'** = R**V**. For example, a point that was located at position **r** in the apparatus (in Frame S) is now located at a new position **r'** = R**r** in Frame S. The Frame S **Basis Vectors** $\mathbf{e}_n$ do not move, only the apparatus moves. We call this the **Active View** of rotation. The vector **V'** is a new vector in Frame S that is different from **V**. There are no vectors $\mathbf{e'}_n$.

Alternatively, suppose the apparatus stays put, but the basis vectors $\mathbf{e}_n$ are back-rotated about the Frame S origin into new Frame S' basis vectors $\mathbf{e'}_n$ such that $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$. In this case, the components of

vector **V** (which were V_i in Frame S) become $(V)'_i = R_{ij}V_j$ in Frame S'. There is no vector **V'**. We call this the **Passive View** of rotation. [We have no clean way to write $(V)'_i = R_{ij}V_j$ in vector notation. One can try something like $(\mathbf{V})' = R\mathbf{V}$, but it is not very useful. But see below.]

The Kinematic Vectors and the Basis Vectors are in two disjoint classes: no vector is in both classes. In the Active View, all the Kinematic Vectors forward-rotate while all the Basis Vectors $\mathbf{e}_n$ stay put. In the Passive View the Kinematic Vectors all stay put, while the Basis Vectors e_n are back-rotated into new basis vectors $\mathbf{e}'_n$ which define Frame S'.

This subject can be extremely annoying, so we roll out a simple example in an attempt to exterminate a recurring confusion.

Example: The vector $\hat{\mathbf{x}}$ is a Basis Vector so one has $\hat{\mathbf{x}}' = R^{-1}\hat{\mathbf{x}}$ in the Passive View. If the vector **r** is a point in the Apparatus, one has $\mathbf{r}' = R\mathbf{r}$ in the Active View. There is no equation which says $\hat{\mathbf{x}}' = R\hat{\mathbf{x}}$ analogous to $\mathbf{r}' = R\mathbf{r}$ because $\hat{\mathbf{x}}$ is not an Kinematic Vector, it is a Basis Vector.

Now suppose in Frame S some point in the apparatus *happens to be* located at $\mathbf{r} = \hat{\mathbf{x}}$. This is potentially confusing since **r** is an Kinematic Vector and $\hat{\mathbf{x}}$ is a Basis Vector and in Frame S these vectors happen to be equal.

In the Active View, the vector **r** maps into $\mathbf{r}' = R\mathbf{r}$ but the Basis Vector $\hat{\mathbf{x}}$ stays put. There is no Basis Vector $\hat{\mathbf{x}}'$. There is no equation that says $\hat{\mathbf{x}}' = R\hat{\mathbf{x}}$ or $\hat{\mathbf{x}}' = R^{-1}\hat{\mathbf{x}}$.

In the Passive View, the Apparatus Vector **r** stays put while the Basis Vector $\hat{\mathbf{x}}$ maps into $\hat{\mathbf{x}}' = R^{-1}\hat{\mathbf{x}}$. The components of the vector **r** as seen in Frame S' are given by $(r)'_i = R_{ij}r_j$. There is no vector **r'**. We do not have $\mathbf{r}' = \hat{\mathbf{x}}'$. There is no equation which says $\hat{\mathbf{x}}' = R\hat{\mathbf{x}}$ because $\hat{\mathbf{x}}$ is not an Kinematic Vector.

Dot Products and Scalars

Suppose **a** and **b** are two apparatus kinematic vectors in the Special Case 4 scenario described above. In the Active View we then have $\mathbf{a}' = R\mathbf{a}$ and $\mathbf{b}' = R\mathbf{b}$. We know using (1.1.38) that

$$\mathbf{a} \bullet \mathbf{b} = [R\mathbf{a}] \bullet [R\mathbf{b}] = \mathbf{a}' \bullet \mathbf{b}'.$$

As an example, one then has $\mathbf{a} \bullet \mathbf{a} = \mathbf{a}' \bullet \mathbf{a}'$ which says $|\mathbf{a}|^2 = |\mathbf{a}'|^2$. Thus a real orthogonal transformation R is one which preserves the length of a vector. Note that both regular rotations (detR = 1) and reflections (detR = -1) have this property.

Since the dot product $\mathbf{a} \bullet \mathbf{b}$ has the same value in Frame S as in Frame S', it is a "rotational scalar", as distinct from a "scalar" which sometimes just means a 1-tuple.

Instead of dealing with two Kinematic Vectors, we can look at the dot product of two Basis Vectors. We have already seen in (1.1.1) that

$$\mathbf{e}_n \bullet \mathbf{e}_m = \mathbf{e}'_n \bullet \mathbf{e}'_m = \delta_{nm} .$$

Although this dot product is the same in both Frames, it is not a scalar. It is a rank-2 tensor which is known a metric tensor. From (1.1.3) and (1.1.8) the other possible Basis Vector dot product is

$\mathbf{e}_n \bullet \mathbf{e'}_m = (\mathbf{e'}_m)_n = \mathbf{e'}_m \bullet \mathbf{e}_n = (\mathbf{e}_n)'_m = R_{mn} = R'_{mn}$ // last equality is (1.1.35)

which is the rotation matrix. One can regard R_{mn} as a trivial rank-2 tensor since it transforms as $T' = RTR^{-1}$ for $T = R$. This dot product has "one foot in each Frame" so it makes no sense to ask if it is a scalar which has the same value in each Frame.

What about the dot product of a Kinematric Vector with a Basis Vector? We find

$\mathbf{a} \bullet \mathbf{e}_n = (a)_n$ $\mathbf{a} \bullet \mathbf{e'}_n = (a)'_n$ $\mathbf{a'} \bullet \mathbf{e}_n = (a')_n$ $\mathbf{a'} \bullet \mathbf{e'}_n = (a')'_n$.

These dot products are neither scalars nor tensors. They are all some kind of vector components. If **a** happens to be a Special Case #4 kinematic vector so **a'** = R**a**, we can use (1.1.38) to write

$(a')_n = \mathbf{a'} \bullet \mathbf{e}_n = [R\mathbf{a}] \bullet [R\mathbf{e'}_n] = \mathbf{a} \bullet \mathbf{e'}_n = (a)'_n \equiv a'_n$

In this special case we end up with $(a')_n = (a)'_n$ which we could then call a'_n without ambiguity. But as shown in (1.2.5), we are still stuck with $(a')'_n = R_{nm}(a')_m = R_{nm}a'_m = (R\mathbf{a}')_n = (R^2\mathbf{a})_n \neq a_n$.

Soon we shall be dealing with the Fig 1 equation **r'** = **r** - **b**. Since this is *not* of the form **r'** ≡ R**r** , we may *not* dispense with the parentheses, and we expect that $(r')_i$ and $(r)'_i$ will be different.

The fact that $(a)'_n \equiv a'_n$ means that $(V)'_i = (V')_i$ in our Passive View paragraph above. Thus, in place of the rule there that $(V)'_i = R_{ij}V_j$ we could create a new vector **V'** ≡ R**V** in Frame S and then display our rule $(V)'_i = R_{ij}V_j$ in vector form as **V'** ≡ R**V** which says $(V')_i = R_{ij}V_j$. This is then the same rule which appears in the Active View paragraph.

If both Frame S and Frame S' are static (so they are inertial frames), then both force **F** and acceleration **a** transform as kinematic variables so **F'** = R**F** and **a'** = R**a**. In this case Newton's Law **F** = m**a** in Frame S becomes **F'** = m**a'** in Frame S'. Since this equation has the same form in both Frames, it is said to be covariant under rotations. All valid laws of physics must be covariant under rotations, and also under the velocity boost transformations of special relativity.

1.4 When are two vectors equal?

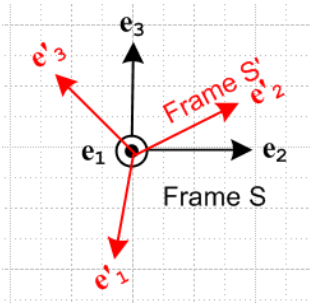
This topic will probably seem strange and unnecessary, but it has been a constant annoyance to the author so here are some words on the subject.

When we say two vectors **A** and **B** are the same or are equal, we mean that the two vectors have the same components in the same coordinate system and we write **A** = **B**. This does not require that vectors **A** and **B** coincide. It might be that **B** is a translated copy of **A**. To be really fussy, we could define a stronger equality $\mathbf{A} \triangleq \mathbf{B}$ to mean that not only do the vectors have the same components in the sense of **A** = **B,** but the vectors actually coincide with each other. We shall have no use for $\mathbf{A} \triangleq \mathbf{B}$ in this document. For us, two vectors are "the same" even if translated from one another.

In light of this interpretation of vectors being equal, we can examine the meaning of certain statements. For example, we normally say "a particle is located at **r** in Frame S ". This really means the particle is at *point* **r** in Frame S which has coordinates (x,y,z). What this means in terms of the graphic *vector* **r** is that *if* the vector **r** is translated so that its tail is at the origin of Frame S, *then* its tip will be at

the particle location. The vector **r** can be drawn anywhere in a picture. It describes the *displacement* of a particle in Frame S from the origin in Frame S.

Example 3: When we say $\mathbf{e}_n = R\ \mathbf{e'}_n$ as in (1.1.29), it is understood that the tails of all vectors involved (the $\mathbf{e}_n$ and the $\mathbf{e'}_n$) are at a common location, as in this picture



(1.4.1)

even though, in our application of Fig 1, the $\mathbf{e}_n$ are drawn with their tails at the origin of Frame S while the $\mathbf{e'}_n$ are drawn with their tails at the origin of Frame S'.

Example 4: In the expansion $\mathbf{r} = (r)_i \mathbf{e}_i$ we normally think of vector **r** having its tail at the origin of Frame S, while in the expansion $\mathbf{r} = (r')'_i \mathbf{e'}_i$ one would be inclined to think of vector **r** as having its tail at the origin of Frame S'. In our stricter sense of coincidence noted above, we might say $(r)_i\ \mathbf{e}_i \neq (r')'_i\ \mathbf{e'}_i$ but this is not of interest. What we care about is that $(r)_i\ \mathbf{e}_i = (r')'_i\ \mathbf{e'}_i$ in the sense **A** = **B** above and we don't care if the vectors **A** and **B** are translated relative to one another. What we care about is that the vectors have the same components in any given Frame.

1.5 The small rotation of a vector about an axis

We do this first algebraically and second geometrically. In the algebraic (linear algebra) approach, the reader must accept a few facts about rotation matrices. The 3x3 matrix which rotates a vector by angle φ about rotation axis $\hat{\mathbf{n}}$ according to the right hand rule is obtained by exponentiating another 3x3 matrix,

$$R_{\hat{\mathbf{n}}}(\varphi) = \exp(-i\,\varphi\,\hat{\mathbf{n}} \bullet \mathbf{J})\ , \tag{1.5.1}$$

where the $(J)_k$ are 3x3 matrices known as the rotation generator matrices:

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad J_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad J_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{1.5.2a}$$

The numbers in these three matrices can be summarized in this single statement,

$$(J_k)_{ij} = -\,i\,\varepsilon_{kij} \tag{1.5.2b}$$

where ε is the totally antisymmetric permutation tensor defined by

$\varepsilon_{abc} = +1$ if abc is an obtained from 123 by an *even* number of pairwise swaps (such as 312)
$\varepsilon_{abc} = -1$ if abc is an obtained from 123 by an *odd* number of pairwise swaps (such as 213)
$\varepsilon_{abc} = 0$ otherwise (ie, when two or more indices have the same value such as 122 or 333) (1.5.3)

Note that $\varepsilon_{abc} = -\varepsilon_{bac}$ regardless of index value. A cross product component can be expressed in terms of the permutation tensor as $[\mathbf{a} \times \mathbf{b}]_i = \varepsilon_{ijk} a_j b_k$ with implied sums on j and k. (cyclic $\varepsilon_{abc} = \varepsilon_{bca} = \varepsilon_{cab}$)

It is convenient to define a *vector* rotation angle in this manner

$$\boldsymbol{\varphi} \equiv \varphi\, \hat{\mathbf{n}} \tag{1.5.4}$$

and then the rotation (1.5.1) may be written in these new ways,

$$R_{\hat{\mathbf{n}}}(\varphi) = R(\boldsymbol{\varphi}) = \exp(-i\, \boldsymbol{\varphi} \bullet \mathbf{J}) \quad . \tag{1.5.5}$$

For a small rotation $d\boldsymbol{\varphi} = d\varphi\, \hat{\mathbf{n}}$, this may be approximated as (using $e^x = 1 + x + ...$ but applied to x = matrix)

$$R(d\boldsymbol{\varphi}) = \exp(-i\, d\boldsymbol{\varphi} \bullet \mathbf{J}) \approx 1 - i\, d\boldsymbol{\varphi} \bullet \mathbf{J} \quad . \tag{1.5.6}$$

With these preliminary remarks out of the way, we can consider the rotation of a vector **a** by a small amount as we move from time t to time t + dt ,

$$\mathbf{a}(t+dt) = R(d\boldsymbol{\varphi})\, \mathbf{a}(t) \tag{1.5.7}$$

where $d\boldsymbol{\varphi} \equiv d\varphi\, \hat{\mathbf{n}}$ is in some arbitrary direction $\hat{\mathbf{n}}$ which is unrelated to the direction of the vector **a**(t) . The change in vector **a** is given by

$$d\mathbf{a} = \mathbf{a}(t+dt) - \mathbf{a}(t) = R(d\boldsymbol{\varphi})\, \mathbf{a}(t) - \mathbf{a}(t) \approx [1 - i\, d\boldsymbol{\varphi} \bullet \mathbf{J}]\, \mathbf{a}(t) - \mathbf{a}(t) = -i\, d\boldsymbol{\varphi} \bullet \mathbf{J}\, \mathbf{a}(t) = -i\, d\varphi_k\, J_k\, \mathbf{a}(t) \quad .$$

Taking the i^{th} component of the above equation we get

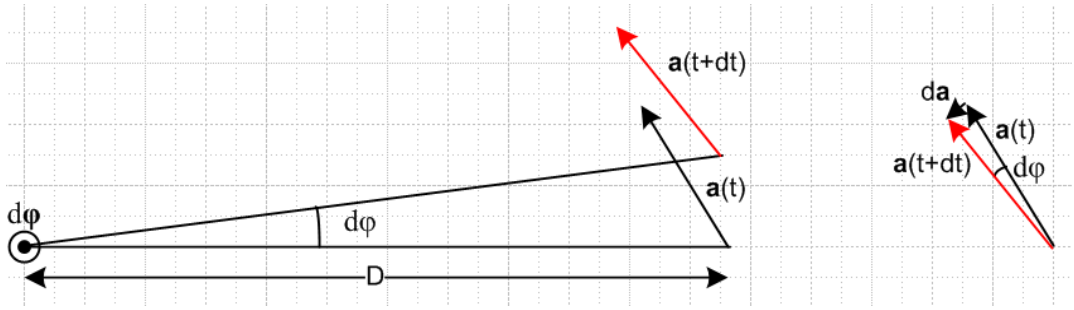
$$da_i = -i\, d\varphi_k [J_k \mathbf{a}]_i = -i\, d\varphi_k (J_k)_{ij} a_j = -i\, d\varphi_k (-i\, \varepsilon_{kij})\, a_j = -\varepsilon_{kij}\, d\varphi_k\, a_j = +\varepsilon_{ikj}\, d\varphi_k\, a_j$$

and going back to vector notation we find that

$$d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a} \tag{1.5.8}$$

which is our main result. It tells us the change in a vector **a** under a small rotation $d\boldsymbol{\varphi}$.

Here then is a graphical derivation of this same fact for a limited geometry. Consider this picture



(1.5.9)

where the small rotation vector d**φ** points out of the plane of paper, and where **a**(t) happens to lie in the plane of paper (hence this picture does not cover the most general case). Using the right hand rule for cross products, we can see that d**a** (shown on the right) lies in the direction of d**φ** x **a**, so we can write that d**a** = C d**φ** x **a** where C is some constant. We can determine C from the equation |d**a**| = C |d**φ** x **a**| . Since d**φ** and **a** are at right angles, we know that |d**φ** x **a**| = |d**φ**| |**a**| = dφ a. But from the picture on the right it seems quite clear that |d**a**| ≈ a dφ. Therefore

$$|d\mathbf{a}| = C\ |d\boldsymbol{\varphi} \times \mathbf{a}| \qquad \Rightarrow \qquad d\varphi\ a = C\ d\varphi\ a \qquad \Rightarrow C = 1 \tag{1.5.10}$$

so we end up with d**a** = d**φ** x **a** which agrees with (1.5.8). In the case that **a** does not lie in the plane of paper, the geometric derivation takes more work, and in this case we just rely on the algebraic result. We have made free use of the notion that translated vectors are "equal".

Notice the fact that d**a** = d**φ** x **a** does not depend on the *distance* D between the rotation axis and the tail of vector **a**! The result is true even if this distance is 0 so that the tail of vector **a** lies right on the rotation axis. In this case the pair of arrows in the left picture coincides with the pair of arrows in the right picture.

Footnote: The equation $R(\boldsymbol{\varphi}) = \exp(-i\ \boldsymbol{\varphi} \bullet \mathbf{J})$ can be interpreted as a rotation in N dimensions with a set of three appropriate NxN generator matrices $(J_k)_{ij}$. Only when N = 3 is (1.5.2) valid or even meaningful. In group theoretic language, these NxN matrices J_k form an N-dimensional irreducible representation of the Lie Algebra so(N) which is $[J_a, J_b] = i\varepsilon_{abc}J_c$. When **J** is exponentiated as shown in (1.5.5), the rotations R(**φ**) are then elements of the Lie Group known as SO(N). The meaning is Special (det = +1), Orthogonal (as in $R^TR = 1$), and N dimensions. The vector **φ** then has N components. For N = 2, the generators are $J_k = \sigma_k/2$ where σ_k are the so-called 2x2 Pauli matrices associated with "spin 1/2". See Appendix G.*
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1.6 The time rate of change of a rotating vector

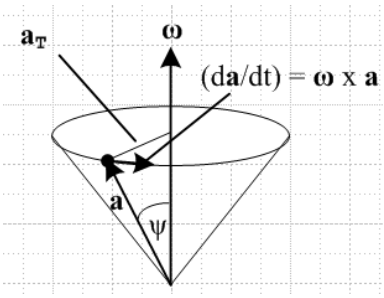
In the previous section we found that,

$$d\mathbf{a} = d\boldsymbol{\varphi} \times \mathbf{a} \quad . \tag{1.5.8}$$

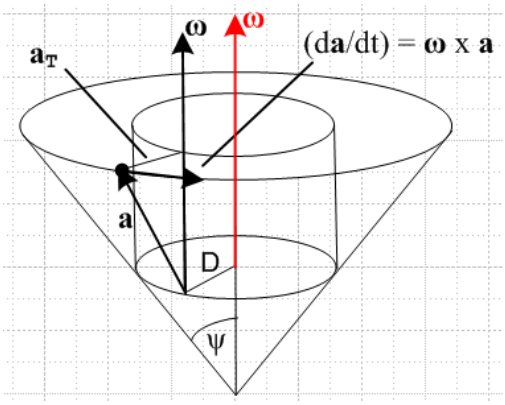
Dividing by dt gives

$$(d\mathbf{a}/dt) = \boldsymbol{\omega} \times \mathbf{a} \qquad \text{where } \boldsymbol{\omega} \equiv d\boldsymbol{\varphi}/dt \quad . \tag{1.6.1}$$

This equation is of fundamental importance in this document. It describes the rotation of vector **a** at rate ω about an axis parallel to **ω**. The equation can be represented by either of these "cone pictures" :



(a)



(b)

(1.6.2)

Comments:

1. In (b), the tip of vector **a** traverses a circular path and so does the tail. The tail is always distance D from the rotation axis. One usually sees the simpler picture (a) where D = 0. Recall from above that distance D of the tail from the rotation axis is irrelevant. The motion of the vector **a** is exactly the same in both these cone pictures. (Recall the comments in Section 1.4 about "when are two vectors equal".)

2. It is probably best to describe the rotating motion of vector **a** as a **conical motion** rather than a "circular motion", even though the *tip* of vector **a** travels in a circular motion. In the special case that $\psi = \pi/2$, meaning the tip of the cone has moved to the center of its circular end face (and $\boldsymbol{\omega} \bullet \mathbf{a} = 0$), *then* vector **a** would swing around in a true "circular motion".

3. Note that one might have $\boldsymbol{\omega} = \boldsymbol{\omega}(t)$ so the vector **ω** could be changing in both direction and magnitude as time progresses. But at time t we have a definite $\hat{\boldsymbol{\omega}}(t)$, and this is sometimes referred to as the **instantaneous** direction of rotation at time t, and **ω**(t) the instantaneous angular velocity. In everything below, we always think of **ω** in this instantaneous sense, even though we might draw guide circles and cones to show the instantaneous motion at some instant of time t.

4. Knowledge of vector **ω** does not in fact say *where* the rotation axis is located. Again we invoke the comments of Section 1.4 above. There are two translational degrees of freedom in that the rotation axis can be translated parallel to itself in two dimensions. Comparing (a) and (b) above, one sees an example of the same **ω** but two different rotation axes, one translated relative to the other. In (b) the red and black **ω** vectors are the same vector. One normally draws the vector **ω** on the rotation axis as done in red.

5. As noted, in (a) and (b) the conical motion of vector **a** is the same and is independent of the placement of the rotation axis as long as it is parallel to **ω**. *However*, in Fig 1 there is a large change in the physical relationship between Frame S and Frame S' if the rotation axis is translated and **ω** stays the same.

6. In the case that $\boldsymbol{\omega}$ = constant, a simple solution to (1.6.1) $\dot{\mathbf{a}} = \boldsymbol{\omega} \times \mathbf{a}$ can be obtained. This equation is a system of three coupled first-order linear ordinary differential equations. Writing $\mathbf{a} = a_x\hat{\mathbf{x}} + a_y\hat{\mathbf{y}} + a_z\hat{\mathbf{z}}$ and $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, the three equations become $\dot{a}_x = -\omega a_y$, $\dot{a}_y = \omega a_x$, and $\dot{a}_z = 0$. Thus $\ddot{a}_x = -\omega^2 a_x$ and we end up with this general solution,

$$\begin{aligned} a_x(t) &= A\cos(\omega t) - B\sin(\omega t) & a_x(0) &= A & // \; a_x^2 + a_y^2 &= A^2 + B^2 = \text{circle} \\ a_y(t) &= A\sin(\omega t) + B\cos(\omega t) & a_y(0) &= B \\ a_z(t) &= a_z(0) = C = \text{constant} & a_z(0) &= C \quad . \end{aligned} \tag{1.6.3}$$

The vector **a** starts out at some **a**(0) = (A,B,C) and does the conical motion drawn above, not just in the instantaneous sense, but in the full sense so that **a** really goes around the entire cone.

1.7 Rate of change of the basis vectors

We can apply our rate of change rule (1.6.1) to the basis vectors $\mathbf{e}'_n$ to obtain

$$(d\mathbf{e}'_n/dt) = \boldsymbol{\omega} \times \mathbf{e}'_n \quad .$$

We are implicitly computing this derivative while "standing" in Frame S. That is to say, looking at Fig 1 in the Introduction above, we can regard Frame S as being fixed to the paper and Frame S' is rotating and its basis vectors $\mathbf{e}'_n$ are changing as stated above. It is extremely important to make this fact explicit, so we now add a label S showing this fact,

$$(d\mathbf{e}'_n/dt)_S = \boldsymbol{\omega} \times \mathbf{e}'_n \quad . \tag{1.7.1}$$

Were we to compute this same derivative standing in Frame S', we would get

$$(d\mathbf{e}'_n/dt)_{S'} = 0 \tag{1.7.2}$$

because in Frame S' the basis vectors $\mathbf{e}'_n$ are not rotating at $\boldsymbol{\omega}$, but are just sitting there frozen. By the same argument, we know that

$$(d\mathbf{e}_n/dt)_S = 0 \; . \tag{1.7.3}$$

What about the fourth possible derivative $(d\mathbf{e}_n/dt)_{S'}$? We will show in (2.5) below that in fact

$$(d\mathbf{e}_n/dt)_{S'} = -\boldsymbol{\omega} \times \mathbf{e}_n \quad . \tag{1.7.4}$$

It seems at least reasonable that if the $\mathbf{e}'_n$ are rotating relative to the $\mathbf{e}_n$ by $\boldsymbol{\omega}$, then the $\mathbf{e}_n$ are rotating relative to the $\mathbf{e}'_n$ by $-\boldsymbol{\omega}$.

Main Conclusion: We thus arrive at the notion that, when dealing with multiple frames of reference and vectors represented in terms of their respective basis vectors, we absolutely must indicate with a label the frame in which a time derivative is being calculated.

1.8 Notations for the many time derivatives of vectors r, r', b and L

In the Overview we described **r** and **r'** as the position vectors of a Particle with respect to Frames S and S'. We can associate with these two vectors *four* different time derivatives.

$$(d\mathbf{r}/dt)_{\mathbf{S}} \qquad (d\mathbf{r}/dt)_{\mathbf{S}'} \qquad (d\mathbf{r}'/dt)_{\mathbf{S}} \qquad (d\mathbf{r}'/dt)_{\mathbf{S}'} \qquad (1.8.1)$$

In the usual manner, we represent a time derivative of a vector by an over-dot, and a second time derivative by an over-double-dot. The first time derivative of position **r** is velocity **v**, and the second is acceleration **a**. The same for **r'**, **v'** and **a'** , so

$$\mathbf{v} = \dot{\mathbf{r}} \qquad \mathbf{v}' = \dot{\mathbf{r}}'$$
$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} \qquad \mathbf{a}' = \dot{\mathbf{v}}' = \ddot{\mathbf{r}}' \quad . \qquad (1.8.2)$$

In order to save space, we can define these two *operators*

$$\partial_{\mathbf{S}} \equiv (d/dt)_{\mathbf{S}} \qquad \partial_{\mathbf{S}'} \equiv (d/dt)_{\mathbf{S}'} \qquad (1.8.3)$$

Although the ∂ symbol is used in partial differentiation, our use here is exactly as defined above and so our ∂ is not a partial derivative (except in a very abstract sense).

The list of derivatives above can be written now in several ways. Within each column below, all objects are exactly the same thing, just written in different notations: (**r** in Frame S appears as **r'** in Frame S')

Table of first derivatives of **r**

$(d\mathbf{r}/dt)_S$	$(d\mathbf{r}/dt)_{S'}$
$\dot{\mathbf{r}}_S \equiv \dot{\mathbf{r}}$	$\dot{\mathbf{r}}_{S'}$
$\mathbf{v}_S \equiv \mathbf{v}$	$\mathbf{v}_{S'}$
$\partial_S \mathbf{r}$	$\partial_{S'} \mathbf{r}$

Table of first derivatives of **r'**

$(d\mathbf{r}'/dt)_S$	$(d\mathbf{r}'/dt)_{S'}$
$\dot{\mathbf{r}}'_S$	$\dot{\mathbf{r}}'_{S'} \equiv \dot{\mathbf{r}}'$
$\mathbf{v}'_S$	$\mathbf{v}'_{S'} \equiv \mathbf{v}'$
$\partial_S \mathbf{r}'$	$\partial_{S'} \mathbf{r}'$. (1.8.4)

For further clutter reduction, we have added the four new notations shown in red according to the following rule: When a vector and all its derivatives (here only one) are all in (or associated with) the same frame, we suppress the frame subscript and just let the prime or lack of it "do the talking", and refer to such as object as being "natural". There is no notational ambiguity in so doing.

The velocities shown in the center two columns above, $\mathbf{v}_{S'} = (d\mathbf{r}/dt)_{S'}$ and $\mathbf{v}'_S = (d\mathbf{r}'/dt)_S$, shall be referred to as "cross velocities", as distinct from the two "natural velocities".

What about second derivatives? Things are more complicated now because vector **r** can have *four* distinct second derivatives, and so can vector **r'**. Just as done above, we construct a table for each, and within each column of each table, all objects are the same object : (**r** in Frame S appears as **r'** in Frame S')

Table of second derivatives of **r**

$\partial_S \partial_S \mathbf{r}$	$\partial_S \partial_{S'} \mathbf{r}$	$\partial_{S'} \partial_S \mathbf{r}$	$\partial_{S'} \partial_{S'} \mathbf{r}$
$\ddot{\mathbf{r}}_{SS} \equiv \ddot{\mathbf{r}}_S \equiv \ddot{\mathbf{r}}$	$\ddot{\mathbf{r}}_{SS'}$	$\ddot{\mathbf{r}}_{S'S}$	$\ddot{\mathbf{r}}_{S'S'} \equiv \ddot{\mathbf{r}}_{S'}$
$\dot{\mathbf{v}}_{SS} \equiv \dot{\mathbf{v}}_S \equiv \dot{\mathbf{v}}$	$\dot{\mathbf{v}}_{SS'}$	$\dot{\mathbf{v}}_{S'S}$	$\dot{\mathbf{v}}_{S'S'} \equiv \dot{\mathbf{v}}_{S'}$
$\mathbf{a}_{SS} \equiv \mathbf{a}_S \equiv \mathbf{a}$	$\mathbf{a}_{SS'}$	$\mathbf{a}_{S'S}$	$\mathbf{a}_{S'S'} \equiv \mathbf{a}_{S'}$ (1.8.5)

Table of second derivatives of **r'**

$\partial_S \partial_S \mathbf{r}'$	$\partial_S \partial_{S'} \mathbf{r}'$	$\partial_{S'} \partial_S \mathbf{r}'$	$\partial_{S'} \partial_{S'} \mathbf{r}'$
$\ddot{\mathbf{r}}'_{SS} \equiv \ddot{\mathbf{r}}'_S$	$\ddot{\mathbf{r}}'_{SS'}$	$\ddot{\mathbf{r}}'_{S'S}$	$\ddot{\mathbf{r}}'_{S'S'} \equiv \ddot{\mathbf{r}}'_{S'} \equiv \ddot{\mathbf{r}}'$
$\dot{\mathbf{v}}'_{SS} \equiv \dot{\mathbf{v}}'_S$	$\dot{\mathbf{v}}'_{SS'}$	$\dot{\mathbf{v}}'_{S'S}$	$\dot{\mathbf{v}}'_{S'S'} \equiv \dot{\mathbf{v}}'_{S'} \equiv \dot{\mathbf{v}}'$
$\mathbf{a}'_{SS} \equiv \mathbf{a}'_S$	$\mathbf{a}'_{SS'}$	$\mathbf{a}'_{S'S}$	$\mathbf{a}'_{S'S'} \equiv \mathbf{a}'_{S'} \equiv \mathbf{a}'$ (1.8.6)

Here we again use the rule mentioned above that when a vector and all its derivatives are in the same frame, we let the overall prime or lack of it do the talking. We have introduced a second rule as well, which says that whenever both frame subscripts are the same, we suppress one of them to save space. In the tables above we then have two "natural" accelerations **a** and **a'**, and six "cross accelerations".

So now we have the following "natural" vectors having minimal (that is, no) frame subscript clutter:

$\dot{\mathbf{r}}$	$\mathbf{v}$	$\ddot{\mathbf{r}}$	$\dot{\mathbf{v}}$	$\mathbf{a}$	natural in Frame S
$\dot{\mathbf{r}}'$	$\mathbf{v}'$	$\ddot{\mathbf{r}}'$	$\dot{\mathbf{v}}'$	$\mathbf{a}'$	natural in Frame S' (1.8.7)

Recall from the Introduction that the vector **b** connects the origins of two frames. We can make a table of first and second derivatives for this vector as well. Although $\dot{\mathbf{b}}$ is a velocity and $\ddot{\mathbf{b}}$ is an acceleration, we shall not make up separate symbol names for these objects (though some authors do). Also, note that there is no vector in Fig 1 called **b'**, we just have **r** = **r'** + **b**. Here are the corresponding tables for first and second derivatives of **b**, where we use only the second rule above that when two frame subscripts are the same we suppress one of them.

Table of first derivatives of **b** . (Items are the same within each column.)

$(d\mathbf{b}/dt)_S$	$(d\mathbf{b}/dt)_{S'}$
$\dot{\mathbf{b}}_S$	$\dot{\mathbf{b}}_{S'}$
$\partial_S \mathbf{b}$	$\partial_{S'} \mathbf{b}$

(1.8.8)

Table of second derivatives of **b**

$\partial_S \partial_S \mathbf{b}$	$\partial_S \partial_{S'} \mathbf{b}$	$\partial_{S'} \partial_S \mathbf{b}$	$\partial_{S'} \partial_{S'} \mathbf{b}$
$\ddot{\mathbf{b}}_{SS} \equiv \ddot{\mathbf{b}}_S$	$\ddot{\mathbf{b}}_{SS'}$	$\ddot{\mathbf{b}}_{S'S}$	$\ddot{\mathbf{b}}_{S'S'} \equiv \ddot{\mathbf{b}}_{S'}$

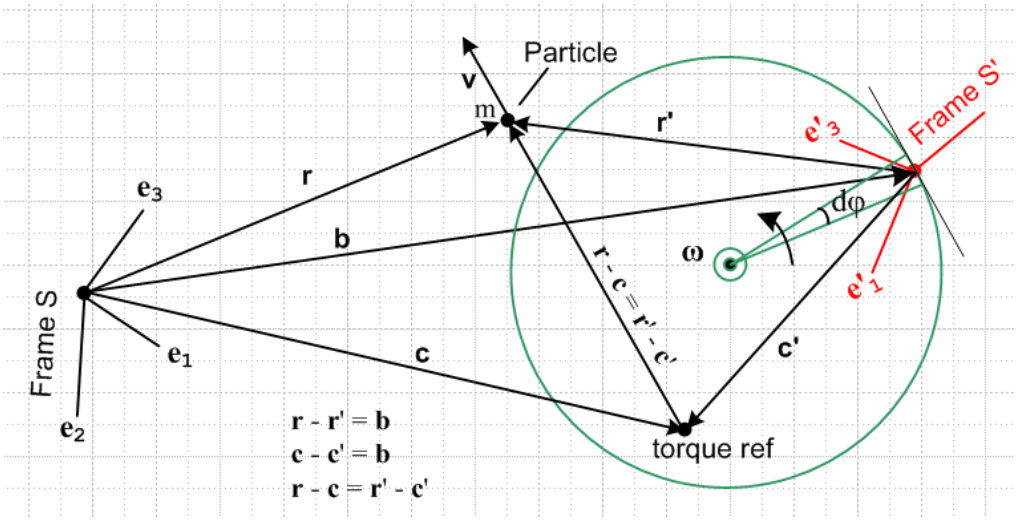
(1.8.9)

Comment: Just as a reminder, *any* derivative in the above section can be expanded onto either Frame S or Frame S' basis vectors, so *any* derivative has Frame S and Frame S' components. This statement would be true for any vector and we just remind the reader that the notation like $(d\mathbf{r}/dt)_S$ does not mean the components are evaluated only in Frame S. Just as an example, this first derivative has these two expansions where the expansion coefficients (components) are shown on the right,

$$(d\mathbf{r}/dt)_S = [(d\mathbf{r}/dt)_S]_i \, \mathbf{e}_i \qquad [(d\mathbf{r}/dt)_S]_i = (d\mathbf{r}/dt)_S \bullet \mathbf{e}_i$$
$$(d\mathbf{r}/dt)_S = [(d\mathbf{r}/dt)_S]'_i \, \mathbf{e}'_i \qquad [(d\mathbf{r}/dt)_S]'_i = (d\mathbf{r}/dt)_S \bullet \mathbf{e}'_i \quad . \tag{1.8.10}$$

1.9 Angular momentum

The drawing of interest is a reoriented Fig 1, to which we have added arbitrary points **c** and **c'**,



(1.9.1)

Whereas the linear momentum **p** = m**v** of a particle does not require a reference point, angular momentum **L** does require such a reference point. For example, the Particle in the above figure has many different values of **L** in Frame S, some of which we might denote as follows,

$$\begin{aligned}
\mathbf{L}^{(0)} &= \mathbf{r} \times m\mathbf{v} && \text{// } \mathbf{L} \text{ in Frame S with respect to Frame S origin} \\
\mathbf{L}^{(c)} &= (\mathbf{r}-\mathbf{c}) \times m\mathbf{v} && \text{// } \mathbf{L} \text{ in Frame S with respect to Frame S point } \mathbf{c} \\
\mathbf{L}^{(b)} &= (\mathbf{r}-\mathbf{b}) \times m\mathbf{v} && \text{// } \mathbf{L} \text{ in Frame S with respect to Frame S point } \mathbf{b} \\
\mathbf{L}^{(b)} &= \mathbf{r}' \times m\mathbf{v} \ . && \text{// } \mathbf{L} \text{ in Frame S with respect to Frame S' origin}
\end{aligned} \tag{1.9.2}$$

The last two lines are exactly the same since **b** is a vector between the two origins so **r'**= **r-b**.

Using the general form $\mathbf{L}^{(c)}$, we may identify the following two "natural" angular momenta in Frames S and S',

$$\mathbf{L}^{(c)}{}_S \equiv (\mathbf{r}-\mathbf{c}) \times m\mathbf{v}_S = (\mathbf{r}-\mathbf{c}) \times m\mathbf{v} = (\mathbf{r}-\mathbf{c}) \times \mathbf{p} \equiv \mathbf{L}^{(c)}$$

$$\mathbf{L}'^{(c')}{}_{S'} \equiv (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}'_{S'} = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}' = (\mathbf{r}'-\mathbf{c}') \times \mathbf{p}' \equiv \mathbf{L}'^{(c')} \ . \tag{1.9.3}$$

These definitions are analogous to **p** ≡ m**v** and **p'** ≡ m**v'** in the linear momentum world.

The time derivative of the first of these objects is given by

$$\begin{aligned}
\dot{\mathbf{L}}^{(c)} &= \dot{\mathbf{L}}^{(c)}{}_S = \partial_S \mathbf{L}^{(c)}{}_S = \partial_S [\, (\mathbf{r}-\mathbf{c}) \times m\mathbf{v}_S] = (\mathbf{v}_S - \dot{\mathbf{c}}_S) \times m\mathbf{v}_S + (\mathbf{r}-\mathbf{c}) \times \partial_S(m\mathbf{v}_S) \\
&= -\dot{\mathbf{c}}_S \times m\mathbf{v}_S + (\mathbf{r}-\mathbf{c}) \times m\mathbf{a}_S \\
&= -\dot{\mathbf{c}} \times m\mathbf{v} + (\mathbf{r}-\mathbf{c}) \times m\mathbf{a} \ .
\end{aligned}$$

A similar result is obtained by priming everything on the above line, so we end up with these four equations:

$$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times m\mathbf{v} \tag{1.9.4}$$

$$\dot{\mathbf{L}}^{(c)} = (\mathbf{r}-\mathbf{c}) \times m\mathbf{a} - \dot{\mathbf{c}} \times m\mathbf{v} \tag{1.9.5}$$

$$\mathbf{L}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}' \tag{1.9.6}$$

$$\dot{\mathbf{L}}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{a}' - \dot{\mathbf{c}}' \times m\mathbf{v}' \quad . \tag{1.9.7}$$

If one thinks of $\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times m\mathbf{v}$ just as a cross product of two vectors to get a third vector, one can apply our Theorem (A.20) to write $(\mathbf{L}^{(c)})' = (\mathbf{r}'-\mathbf{c}') \times m\mathbf{v}'$ and therefore $(\mathbf{L}^{(c)})' = \mathbf{L}'^{(c')}$. It is just another notation.

The discussion of angular momentum continues in Section 11.

1.10 No frame label is needed for d/dt of a scalar function

If one is differentiating a specific *component* of a vector, such as $a_i(t) = 3t^2$, there is no need to add the frame label since the derivative of this function is 6t no matter what frame it is computed in. This is true when differentiating any component of any tensor. That is to say, if $T_{ij..}(t)$ is a component of a tensor, then

$$(dT_{ij..}(t)/dt)_S = (dT_{ij..}(t)/dt)_{S'} = (dT_{ij..}(t)/dt) \quad . \tag{1.10.1}$$

We would like simply to say that the d/dt derivative of any scalar function does not need a label S or S', but the word "scalar" has multiple meanings. In one meaning, any single function f(t) is a scalar function since it is a 1-tuple of functions, but in another meaning (tensorial scalar), only a function which is a rotational scalar is a scalar function, and this would rule out the component of a vector as being a scalar function. We refer to the first meaning in the title of this subsection.

1.11 When do operations d/dt and taking a component "commute" ?

We claim the following theorem (to be proved below) :

Commutation Theorem: (1.11.1)

If the time derivative ($\partial_{\mathbf{X}}$) is computed in the same frame (Frame X) in which components are taken, then the operations of taking the time derivative and taking the component can be done in either order with the same result, so these operations are said to commute. Otherwise the operations do not commute. Thus

$$(\partial_{\mathbf{S}}\mathbf{a})_j = \partial_{\mathbf{S}}[(a)_j] \quad \text{but} \quad (\partial_{\mathbf{S}}\mathbf{a})'_j \neq \partial_{\mathbf{S}}(a)'_j$$
$$(\partial_{\mathbf{S}'}\mathbf{a})'_j = \partial_{\mathbf{S}'}[(a)'_j] \qquad\qquad (\partial_{\mathbf{S}'}\mathbf{a})_j \neq \partial_{\mathbf{S}'}(a)_j \ .$$

In each of these four equations, on the left we compute $\partial_{\mathbf{X}}\mathbf{a}$ and then we take a component, while on the right side we take a component and then compute $\partial_{\mathbf{X}}$ on that component.

Goldstein makes a point about the non-commuting cases as a "word of caution" on the bottom of page 133 with an example on the top of page 134. In Goldstein, Poole and Safko the caution is stated on page 173 below (4.86), but the example has been removed.

As a preliminary to proving the theorem, we note that for a scalar u and vector **v** one can write,

$$(d[u\mathbf{v}]/dt)_{\mathbf{S}} = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_{\mathbf{S}} \qquad \text{S}$$
$$(d[u\mathbf{v}]/dt)_{\mathbf{S}'} = (du/dt)\,\mathbf{v} + u\,(d\mathbf{v}/dt)_{\mathbf{S}'} \qquad \text{S'}$$

or

$$\partial_{\mathbf{S}}(u\mathbf{v}) = (\partial_{\mathbf{S}}u)\,\mathbf{v} + u\,(\partial_{\mathbf{S}}\mathbf{v}) \qquad \text{S}$$
$$\partial_{\mathbf{S}'}(u\mathbf{v}) = (\partial_{\mathbf{S}'}u)\,\mathbf{v} + u\,(\partial_{\mathbf{S}'}\mathbf{v}) \qquad \text{S'} \qquad (1.11.2)$$

where recall that $\partial_{\mathbf{S}} = \partial_{\mathbf{S}'} = \partial_{\mathbf{t}}$ when applied to a scalar like u. Within either frame this is just the calculus Leibniz product rule applied to as function u(t) times a vector function **v**(t),

$$\partial_{\mathbf{t}}(u\mathbf{v}) = (\partial_{\mathbf{t}}u)\,\mathbf{v} + u\,(\partial_{\mathbf{t}}\mathbf{v}) \ ,$$

so (1.11.2) is just a statement of this known rule in the two frames.

One can trivially generalize (1.11.2) to a sum of the form $u_i\mathbf{v}_i$ (implied sum on i) to get

$$\partial_{\mathbf{S}}(u_i\mathbf{v}_i) = (\partial_{\mathbf{S}}u_i)\,\mathbf{v}_i + u_i\,(\partial_{\mathbf{S}}\mathbf{v}_i) \qquad \text{S}$$
$$\partial_{\mathbf{S}'}(u_i\mathbf{v}_i) = (\partial_{\mathbf{S}'}u_i)\,\mathbf{v}_i + u_i\,(\partial_{\mathbf{S}'}\mathbf{v}_i) \qquad \text{S'} \qquad (1.11.3)$$

Reader Exercise: Show that $\partial_{\mathbf{X}}(\mathbf{a} \times \mathbf{b}) = (\partial_{\mathbf{X}}\mathbf{a}) \times \mathbf{b} + \mathbf{a} \times (\partial_{\mathbf{X}}\mathbf{b})$ for X = S or S' (more Leibniz). (1.11.4)

Proof of the Commutation Theorem: In the proof we use all the following facts developed above and gathered below for convenience. Notice that it is *not* assumed that **a'** = R**a** and in fact the vector **a'** does not appear anywhere.

$(R^{-1})_{ij} = (R^{T})_{ij} = R_{ji}$ rotation is real orthogonal *(1.1.2)* (1.11.5)

$R_{ij}(t)$ = function of time so $(\partial_t R_{ij}(t)) \neq 0$ and $(\partial_t (R^{-1})_{ij}(t)) \neq 0$ $(\boldsymbol{\omega} \neq 0)$ "rotating frames"

$\mathbf{a} = (a)_i \, \mathbf{e}_i = (a)'_i \, \mathbf{e}'_i$ expansions in the two frames *(1.2.2)*

$(\mathbf{e}_n)_i = \delta_{n,i}$ $(\mathbf{e}'_n)'_i = \delta_{n,i}$
$(\mathbf{e}_n)'_i = (R^{-1})_{ni}$ $(\mathbf{e}'_n)_i = R_{ni}$ basis vector components *(1.1.8)*

$(a)'_i = R_{ij}(a)_j$. components of **a** in the two frames *(1.3.1)*

$\partial_S \mathbf{e}_n = 0$ and $\partial_{S'} \mathbf{e}'_n = 0$ frame definitions *(1.7.3), (1.7.2)*

$\partial_S f(t) = \partial_{S'} f(t) = \partial_t f(t)$ time derivative of a component *(1.10.1)*

$\partial_S(u_i \mathbf{v}_i) = (\partial_S u_i) \, \mathbf{v}_i + u_i \, (\partial_S \mathbf{v}_i)$ product rule in frame S
$\partial_{S'}(u_i \mathbf{v}_i) = (\partial_{S'} u_i) \, \mathbf{v}_i + u_i \, (\partial_{S'} \mathbf{v}_i)$ product rule in frame S' *(1.11.3)*

Here then is the detailed proof of the above theorem. First, consider ∂_S with $\mathbf{a} = (a)_i \, \mathbf{e}_i$:

$(\partial_S \mathbf{a})_j = (\partial_S [(a)_i \mathbf{e}_i])_j = (\partial_S (a)_i) \, (\mathbf{e}_i)_j = (\partial_t (a)_i) \, \delta_{i,j} = \partial_t (a)_j$
$\partial_S (a)_j = \partial_t (a)_j$
$\Rightarrow \partial_S (a)_j = (\partial_S \mathbf{a})_j$ // commutes (∂_S with Frame S components)

$(\partial_S \mathbf{a})'_j = (\partial_S [(a)_i \mathbf{e}_i])'_j = (\partial_S (a)_i) \, (\mathbf{e}_i)'_j = (\partial_t (a)_i) \, R_{ji} = R_{ji} (\partial_t (a)_i)$
$(\partial_S (a)'_j) = \partial_S [R_{ji} (a)_i] = R_{ji} (\partial_S (a)_i) + (\partial_S R_{ji}) \, (a)_i = R_{ji} (\partial_t (a)_i) + (\partial_t R_{ji}) \, (a)_i$
$\Rightarrow (\partial_S (a)'_j) = (\partial_S \mathbf{a})'_j + (\partial_t R_{ji}) \, (a)_i$
$\Rightarrow (\partial_S (a)'_j) \neq (\partial_S \mathbf{a})'_j$ // does not commute (∂_S with Frame S' components)

Next consider $\partial_{S'}$ with $\mathbf{a} = (a)'_i \, \mathbf{e}'_i$:

$(\partial_{S'} \mathbf{a})_j = (\partial_{S'} [(a)'_i \mathbf{e}'_i])_j = (\partial_{S'} (a)'_i) \, (\mathbf{e}'_i)_j = (\partial_{S'} (a)'_i) \, R_{ij} = (R^{-1})_{ji} \, (\partial_t (a)'_i)$
$(\partial_{S'} (a)_j) = \partial_{S'} [(R^{-1})_{ji} (a)'_i] = (\partial_t (R^{-1})_{ji}) (a)'_i + (R^{-1})_{ji} (\partial_t (a)'_i)$
$\Rightarrow (\partial_{S'} (a)_j) = (\partial_{S'} \mathbf{a})_j + (\partial_t (R^{-1})_{ji}) (a)'_i$
$\Rightarrow (\partial_{S'} (a)_j) \neq (\partial_{S'} \mathbf{a})_j$ // does not commute ($\partial_{S'}$ with Frame S components)

$(\partial_{S'} \mathbf{a})'_j = (\partial_S [(a)'_i \mathbf{e}'_i])'_j = (\partial_{S'} (a)'_i) \, (\mathbf{e}_i')'_j = (\partial_{S'} (a)'_i) \, \delta_{i,j} = (\partial_{S'} (a)'_j) = (\partial_t (a)'_j)$
$\partial_{S'} (a)'_j = \partial_t (a)'_j$
$\Rightarrow \partial_{S'} (a)'_j = (\partial_{S'} \mathbf{a})'_j$ // commutes ($\partial_{S'}$ with Frame S' components)