Section 1 Formalism and Rank-2 Tensors PhL 3.11.17

I want to argue that in the Passive View a rank-2 tensor transforms as follows

$$(T)' = RTR^{T} \qquad (1a)$$

which is a shorthand for this more precise statement

$$(T)'_{ij} = R_{ia}T_{ab}(R^{T})_{bj} \qquad (1b)$$

In the Passive View, there is no new tensor (T') as there is in the Active View.

Here is the Dirac version of (2)

$$(T)'_{ij} = \langle \mathbf{e}'_i | T | \mathbf{e}'_j \rangle = \langle \mathbf{e}'_i | \mathbf{e}_n \rangle\langle \mathbf{e}_n | T | \mathbf{e}_m \rangle\langle \mathbf{e}_m | \mathbf{e}'_j \rangle = R_{in}(T)_{nm}R_{jm} = [RTR^{T}]_{ij} \qquad (2)$$

It is hard to deny this Dirac derivation of (1b)

Question #1. What then can be said about a new object (T') which is defined as follows

$$(T') \equiv RTR^{T} \qquad (3a)$$

or

$$(T')_{ij} = R_{ia}T_{ab}(R^{T})_{bj} \qquad (3b)$$

Comparing the right sides of (1b) and (3b) one would have to conclude that

$$(T')_{ij} = (T)'_{ij}$$

If you are free to define (T') as in (3a) due to no name conflicts, then yes, this is the expected result.

Example: Consider $T = J_k$, one of the rotation generators. Does this transform as a passive rank-2 tensor, yes or no? If yes, we would expect to have

$$(J_k)' = RJ_kR^{T} \qquad (4a)$$

$$(J_k)'_{ij} = R_{ia}(J_k)_{ab}(R^{T})_{bj} \qquad (4b)$$

If no, we might instead expect to have $(J_k)'_{ij} = (J_k)_{ij}$. In this case we might vaguely think of J_k as just a matrix of constants which is the same in any Frame. In some sense it is a scalar.

How does one answer the question: which is it? Yes or no? More nubbins questions. One method is to assume one answer and find a contraction. The usual issue is having clean definitions of things. Let's try one of the solutions and see where it leads.

Hypothesis: J_k does not transform as a passive rank-2 tensor. It is just a matrix of constants. Then

$(J_k)' = RJ_kR^T$ not true

$(J_k)'_{ij} = R_{ia}(J_k)_{ab}(R^T)_{bj}$ not true

$(J_k)'_{ij} = (J_k)_{ij}$ true

In the context of this hypothesis, we *could* define a new object as follows

$(J'_k) \equiv RJ_kR^T$

or

$(J'_k)_{ij} \equiv R_{ia}(J_k)_{ab}(R^T)_{bj}$ // $= (R^{-1})_{kn}(J_n)_{ij}$ by Theorem 1

This is a matrix whose elements are functions of angles like θ,φ,ψ.

Claim: You could say (as in tensor doc) that the pair of objects $\{ J_k, (J'_k)\}$ transform as an active rank-2 tensor. But in the passive view, the object (J'_k) does not exist. Only $(J_k)'$ exists. You cannot just define $(J'_k) \equiv (J_k)'$ because (J'_k) is already defined as RJ_kR^T and this is different from $(J_k)' = (J)_k$.

Fact: Nothing stops you from defining $(J'_k) \equiv RJ_kR^T$ if you want to do that. But you cannot make any statement about $\{ J_k, (J'_k)\}$ transforming as a passive tensor.

Review of Active and Passive Rank-2 tensors

In the Active View, you have the original tensor T in Frame S and you create a new tensor T' also in Frame S such that $(T') = RTR^T$, The pair {T,T'} transforms as an active rank-2 tensor. There is no Frame S', there is no new basis e'_n for Frame S'

In the Passive View, you have the original tensor T in Frame S and there is no tensor T'. You define Frame S' with rotated basis vectors e'_n and the rank-2 transformation rule is then $(T)' = RTR^T$.

Application to object A of the ω calculation.

In this calculation I have the following matrix $A = a_iJ_i$ as follows:

$$A \equiv -[(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)]$$

The matrix elements are

$$A_{ij} \equiv -[(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1)_{ij} + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2)_{ij} + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)_{ij}]$$

According to the Hypothesis above, since the J_k do NOT transform as passive rank-2 tensors, we must conclude that the object A_{ij} also does NOT transform as a passive rank-2 tensor. Furthermore, since we know that $(J_k)'_{ij} = (J_k)_{ij}$ we also know that $(A)'_{ij} = A_{ij}$.

We could if we wanted define the object (A') = RTR^T . It turns out that when we do the ω calculation we encounter the object B ≡ R^TAR so we cannot associate this (A') with B. But suppose we were to give the matrix A a different name like this

$$A' \equiv -[(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1) + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2) + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)]$$

$$(A')_{ij} \equiv -[(\dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi)(iJ_1)_{ij} + (\dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi)(iJ_2)_{ij} + (\dot{\psi} + \dot{\varphi}\cos\theta)(iJ_3)_{ij}]$$

Then we would end up with

$$B = R^TA'R \qquad \Rightarrow \qquad (A') = RBR^T$$

and in this case we could identify B with A where {A, (A') } form an active rank-2 tensor.

Does this add any clarity to the calculation? I think it does NOT add clarity, because then you have to mention active rank-2 tensor in a passive view world of frames doc. So don't bring this up, and just use A and B!!!

Well, in Frames Doc I do write things like T' = RTR^T. Is that active or passive?

OK, I have now made Sections 1.1,2,3 more consistent by always saying (T)' = RTR^T . I am trying to be totally consistent. In doing this in a separate file, I found various spelling errors!

Question: What should one make of this development ?

$$(T')_{mn} \equiv \langle \mathbf{e}_m|\, \mathcal{T}' \,|\mathbf{e}_n\rangle = \langle \mathbf{e}_m|\, \mathcal{R}\mathcal{T}\mathcal{R}^{-1} \,|\mathbf{e}_n\rangle = \langle R^{-1}\mathbf{e}_m|\mathcal{T}\,|R^{-1}\mathbf{e}_n\rangle = \langle \mathbf{e'}_m|\, \mathcal{T} \,|\mathbf{e'}_n\rangle = (T)'_{mn}$$

Well, here I defined a new tensor $\mathcal{T}' \equiv \mathcal{R}\mathcal{T}\mathcal{R}^{-1}$ or (T') ≡ RTR^{-1} . If you can do this with no naming conflicts, then yes, you will get $(T')_{mn} = (T)'_{mn}$. Same as what happens with vectors. You have (**V**') = R**V** and if you can define **V**' ≡ R**V** with no meaning conflict, you get $(V')_i = (V)'_i$.