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1. Notation, important role of the Prime Symbol, and other Preliminaries

Note: When an equation is repeated, its equation number is put in italics.

1.1 The basis vectors $\mathbf{e}_n$ and $\mathbf{e}'_n$ and two ways in which they are related

Frame S has Cartesian basis unit vectors $\mathbf{e}_n$, while Frame S' has Cartesian basis unit vectors $\mathbf{e}'_n$. The two sets of basis vectors are related by some rotation we shall call R (repeated indices have implied sums),

$$\mathbf{e}_n = R\,\mathbf{e}'_n \qquad n = 1,2,3 \qquad \text{or} \qquad (\mathbf{e}_n)_i = R_{ij}(\mathbf{e}'_n)_j \qquad (1.1.1)$$

where on the right the component subscripts refer to components in Frame S.

The above relation between $\mathbf{e}_n$ and $\mathbf{e}'_n$ can also be written in this manner

$$\mathbf{e}_n = (R^{-1})_{nm}\,\mathbf{e}'_m \qquad \text{or} \qquad \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \qquad (1.1.2)$$

a fact proven below in a short series of steps. Notice that the right equation in (1.1.1) involves a sum of basis vector components, whereas the equations in (1.1.2) involve a sum of basis vectors.

Proof of (1.1.2):

Step 1: The orthonormal Cartesian basis vectors have these properties

$$\delta_{n,k} = \mathbf{e}_n \bullet \mathbf{e}_k = \mathbf{e}'_n \bullet \mathbf{e}'_k \qquad \text{and} \qquad (\mathbf{e}_n)_k = \delta_{n,k} \qquad (1.1.3)$$

where $(\mathbf{e}_n)_k = \mathbf{e}_k \bullet \mathbf{e}_n$ denotes the kth component of $\mathbf{e}_n$ in Frame S. The basis vectors $\mathbf{e}_n$ are axis-aligned in Frame S, while the $\mathbf{e}'_n$ are axis-aligned in Frame S'. The dot product is $\mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \equiv a_i b_i$.

Step 2: Note that

$$(\mathbf{e}'_n)_m = \mathbf{e}_m \bullet \mathbf{e}'_n \overset{(1.1.1)}{=} \mathbf{e}_m \bullet [R^{-1}\mathbf{e}_n] = (\mathbf{e}_m)_i\,(R)^{-1}_{ij}(\mathbf{e}_n)_j \overset{(1.1.3)}{=} \delta_{m,i}\,(R)^{-1}_{ij}\,\overset{(1.1.3)}{\delta_{n,j}}$$

$$= (R^{-1})_{mn} = (R^T)_{mn} = R_{nm} \qquad (1.1.4)$$

where in the last step we use the fact that $R^{-1} = R^T$ (or $RR^T = R^TR = 1$) for any rotation. Such a matrix R is said to be "real orthogonal" (T means transpose).

Step 3: We wish to show (1.1.2) that $\mathbf{e}'_n = R_{nm}\mathbf{e}_m$. We shall verify this by showing that all three components of each side are equal in Frame S, and to do this we dot both sides into $\mathbf{e}_k$:

$$\text{LHS} = \mathbf{e}_k \bullet \mathbf{e}'_n = (\mathbf{e}'_n)_k = R_{nk} \qquad \text{// using (1.1.4)}$$

RHS = $\mathbf{e}_k \bullet [R_{nm}\,\mathbf{e}_m] = R_{nm}\,\mathbf{e}_k \bullet \mathbf{e}_m = R_{nm}\,\delta_{k,m} = R_{nk}$. QED

Time dependence of the R_{ij}

If Frame S is fixed and Frame S' is rotating, then we really have $\mathbf{e}_n = R(t)\mathbf{e}'_n(t)$ where the R matrix is a function of time, so we have $R_{ij}(t)$. Similarly, if Frame S is moving and Frame S' is fixed, $\mathbf{e}_n(t) = R(t)\,\mathbf{e}'_n$ and again one has $R_{ij}(t)$. Only in the case where there is no rotation between the frames are the R_{ij} independent of time. This means that $\boldsymbol{\omega} = 0$ in Fig 1. Since our document is about "rotating frames of reference" we exclude this no-rotation case from consideration.

Comment: In the above it was convenient to represent all three Cartesian unit vectors in Frame S and Frame S' by $\mathbf{e}_i$ and $\mathbf{e}'_i$. Later when we do particular problems, we shall often revert to the following more standard physics notation:

$$x_1, x_2, x_3 = x, y, z \qquad \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 = \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}} \qquad \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3 = \hat{\mathbf{x}}', \hat{\mathbf{y}}', \hat{\mathbf{z}}' \qquad (1.1.5)$$

Then for example $\delta_{n,k} = \mathbf{e}_n \bullet \mathbf{e}_k = \mathbf{e}'_n \bullet \mathbf{e}'_k$ for n = 1 and k = 2 says $0 = \hat{\mathbf{x}} \bullet \hat{\mathbf{y}} = \hat{\mathbf{x}}' \bullet \hat{\mathbf{y}}'$.

1.2 Expansions of a vector and use of primes and parentheses

Note: We write $(a)_i$ as a component of vector $\mathbf{a}$, but $(\mathbf{e}_n)_i$ as a component of $\mathbf{e}_n$. In the first case the a in $(a)_i$ is not bolded, but since $\mathbf{e}_n$ is decorated with a label n, it gets bolded. It is just our convention.

Any vector $\mathbf{a}$ can be expanded on either set of basis vectors so that, with implied summation on i,

$$\mathbf{a} = a_i\,\mathbf{e}_i = a'_i\,\mathbf{e}'_i \quad . \qquad a_i = \mathbf{a} \bullet \mathbf{e}_i \qquad a'_i = \mathbf{a} \bullet \mathbf{e}'_i \quad . \qquad (1.2.1)$$

If some other vector named $\mathbf{a}'$ is lurking in the wings, one might want to be more careful labeling components. A safe method would be this:

$$\begin{aligned} &\mathbf{a} = (a)_i\,\mathbf{e}_i = (a)'_i\,\mathbf{e}'_i \qquad (a)_i = \mathbf{a} \bullet \mathbf{e}_i \qquad (a)'_i = \mathbf{a} \bullet \mathbf{e}'_i \\ &\mathbf{a}' = (a')_i\,\mathbf{e}_i = (a')'_i\,\mathbf{e}'_i \qquad (a')_i = \mathbf{a}' \bullet \mathbf{e}_i \qquad (a')'_i = \mathbf{a}' \bullet \mathbf{e}'_i \quad . \end{aligned} \qquad (1.2.2)$$

Here, a prime *inside* a parentheses is part of the vector name, whereas a prime *outside* a parentheses denotes a vector component in Frame S' (whereas no prime outside means a component in Frame S). Unless the relationship between vectors $\mathbf{a}$ and $\mathbf{a}'$ has a certain simple form, it is very likely that $(a')_i \neq (a)'_i$. In this case the notation a'_i would be ambiguous since one doesn't know whether it refers to $(a')_i$ or $(a)'_i$. It is true that the notation a_i could be unambiguously identified with $(a)_i$, but we shall maintain the parentheses just to be uniform.

Example 1. As an example of the four expansions above, let us consider $\mathbf{a} = \mathbf{e}_n$:

$$\mathbf{e}_n = (\mathbf{e}_n)_i\,\mathbf{e}_i \qquad \Rightarrow \qquad (\mathbf{e}_n)_i = \delta_{n,i} \qquad \text{// by inspection}$$

$\mathbf{e'}_n = (\mathbf{e'}_n)'_i\, \mathbf{e'}_i \quad \Rightarrow \quad (\mathbf{e'}_n)'_i = \delta_{n,i}$ // by inspection

$\mathbf{e}_n = (\mathbf{e}_n)'_i\, \mathbf{e'}_i \quad \Rightarrow \quad (\mathbf{e}_n)'_i = (R^{-1})_{ni}$ // using (1.1.2) that $\mathbf{e}_n = (R^{-1})_{ni}\, \mathbf{e'}_i$

$\mathbf{e'}_n = (\mathbf{e'}_n)_i\, \mathbf{e}_i \quad \Rightarrow \quad (\mathbf{e'}_n)_i = R_{ni}$. // using (1.1.2) that $\mathbf{e'}_n = R_{ni}\, \mathbf{e}_i$

We can now restate these results showing the dot products which represent each expansion coefficient, and in this way we obtain expressions for all the dot products,

$\mathbf{e}_n = (\mathbf{e}_n)_i\, \mathbf{e}_i \quad \Rightarrow \quad (\mathbf{e}_n)_i = \mathbf{e}_i \bullet \mathbf{e}_n = \delta_{n,i}$

$\mathbf{e'}_n = (\mathbf{e'}_n)'_i\, \mathbf{e'}_i \quad \Rightarrow \quad (\mathbf{e'}_n)'_i = \mathbf{e'}_i \bullet \mathbf{e'}_n = \delta_{n,i}$

$\mathbf{e}_n = (\mathbf{e}_n)'_i\, \mathbf{e'}_i \quad \Rightarrow \quad (\mathbf{e}_n)'_i = \mathbf{e'}_i \bullet \mathbf{e}_n = (R^{-1})_{ni} = R^{T}_{ni} = R_{in}$

$\mathbf{e'}_n = (\mathbf{e'}_n)_i\, \mathbf{e}_i \quad \Rightarrow \quad (\mathbf{e'}_n)_i = \mathbf{e}_i \bullet \mathbf{e'}_n = R_{ni}$. (1.2.3)

Notice that all these results are consistent with claims made earlier.

In passing, we note that if R is any rotation (so therefore $R^{T}R = 1$), then

$\mathbf{a} \bullet \mathbf{b} = [R\mathbf{a}] \bullet [R\mathbf{b}]$. (1.2.4)

<u>One line proof</u>:

$[R\mathbf{a}] \bullet [R\mathbf{b}] = [R\mathbf{a}]_k [R\mathbf{b}]_k = R_{ki} a_i R_{kj} b_j = (R^{T})_{jk} R_{ki} a_i b_j = (R^{T}R)_{ji} a_i b_j = \delta_{j,i} a_i b_j = a_i b_i = \mathbf{a} \bullet \mathbf{b}$

<u>Matrix Notation to show how components are related.</u>

Consider the following expansion of vector **a** on the basis vectors $\mathbf{e'}_j$,

$\mathbf{a} = (a)'_j\, \mathbf{e'}_j$ // (1.2.2)

$= (a)'_j \{ R_{ji}\, \mathbf{e}_i \}$ // (1.1.2)

$= (a)'_j \{ (R^{-1})_{ij}\, \mathbf{e}_i \}$ // $R = (R^{-1})^{T}$ real orthogonal rotation

$= \{ (R^{-1})_{ij} (a)'_j \) \mathbf{e}_i$. // reorder (1.2.5)

Comparing this to $\mathbf{a} = (a)_i\, \mathbf{e}_i$ of (1.2.2) we conclude that, since $\mathbf{e}_i$ is a complete basis,

$$(a)_i = (R^{-1})_{ij} (a)'_j \quad \Leftrightarrow \quad \begin{pmatrix} (a)_1 \\ (a)_2 \\ (a)_3 \end{pmatrix} = R^{-1} \begin{pmatrix} (a)'_1 \\ (a)'_2 \\ (a)'_3 \end{pmatrix} \qquad (1.2.6)$$

where R^{-1} is a 3x3 rotation matrix.

We can repeat the above discussion replacing **a** with **a'** with this result,

$$(a')_i = (R^{-1})_{ij}(a')'_j \qquad \Leftrightarrow \qquad \begin{pmatrix} (a')_1 \\ (a')_2 \\ (a')_3 \end{pmatrix} = R^{-1} \begin{pmatrix} (a')'_1 \\ (a')'_2 \\ (a')'_3 \end{pmatrix} . \tag{1.2.7}$$

These matrix equations are convenient for computing the components of a vector on the $\mathbf{e}_i$ basis if they are known in the $\mathbf{e}'_i$ basis (and vice versa) .

1.3 Special case where a'_i is unambiguous

We shall now examine the type of relationship between **a'** and **a** in which $(a')_i = (a)'_i$ and therefore we can use the notation a'_i without ambiguity. First, the components $(a)'_i$ and $(a)_i$ are related in the following simple manner, using (1.2.3),

$$(a)'_n = \mathbf{e}'_n \bullet \mathbf{a} = (\mathbf{e}'_n)_m (a)_m = R_{nm}(a)_m . \tag{1.3.1}$$

Now suppose we *define* a new vector **a'** in this way,

$$\mathbf{a}' \equiv R\mathbf{a} . \tag{1.3.2}$$

If a vector **a** transforms into **a'** according to (1.3.2), we say it is a "vector under rotations" which means it "transforms as a vector under rotations".

When written in Frame S components this says

$$(a')_n = R_{nm}(a)_m . \tag{1.3.3}$$

Comparison of (1.3.1) and (1.3.3) shows that

$$(a)'_n = (a')_n \tag{1.3.4}$$

and therefore in this case we can use

$$a'_n \equiv (a)'_n = (a')_n . \tag{1.3.5}$$

Thus, if the vectors **a** and **a'** are related by $\mathbf{a}' = R\mathbf{a}$ where R is the rotation appearing in $\mathbf{e}_n = R\,\mathbf{e}'_n$, then we can dispense with the parentheses as shown in (1.3.4). We still have $(a')'_i$ which requires parentheses.

Example 2: Consider equation (1.1.1) ,

$$\mathbf{e}_n = R\,\mathbf{e}'_n .$$

Since this is *not* of the form $\mathbf{a}' \equiv R\mathbf{a}$, we may not dispense with the parentheses. In fact from (1.2.3) we have

$$(\mathbf{e}_n)'_i = R_{in}$$
$$(\mathbf{e}'_n)_i = R_{ni} \qquad (1.3.6)$$

and these are not the same because rotation matrices are not symmetric. Unlike normal vectors for which one writes the transform $\mathbf{a}' = R\mathbf{a}$, the basis vectors are "back-rotated" so $\mathbf{e}'_n = (R^{-1})\mathbf{e}_n$.

Active and Passive

One can think of $\mathbf{a}' = R\mathbf{a}$ as an **active** rotation of vector $\mathbf{a}$ into another vector $\mathbf{a}'$ within Frame S. In this case, the components of $\mathbf{a}'$ are $(a')_i$. The alternative is to think of vector $\mathbf{a}$ as not moving at all in Frame S, but the basis vectors are back-rotated from $\mathbf{e}_n$ to $\mathbf{e}'_n$ taking us to Frame S'. In this back-rotated basis the components of $\mathbf{a}$ are $(a)'_i$. This is the **passive** view of a rotation and is in fact the view we take in most of this document because we want to *observe* activities in Frame S from Frame S' and vice versa. Each view has its usefulness and we have just shown that if $\mathbf{a}' = R\mathbf{a}$, then $(a)'_n = (a')_n$.

In the active view, the "apparatus" is rotated and the axes stay put, while in the passive view the apparatus stays put but the axes are back-rotated. There is a third view in which the apparatus and the axes are both rotated in the same direction, and this view is useful in the discussion of covariance of equations in Lucht *Tensor*. One can regard the three views as three "experiments" one might perform. As noted, we shall work with the passive view exclusively in this document.

Example 3: Soon we shall be dealing with the Fig 1 equation $\mathbf{r}' = \mathbf{r} - \mathbf{b}$. Since this is *not* of the form $\mathbf{r}' \equiv R\mathbf{r}$, we may *not* dispense with the parentheses, and we expect that $(r')_i$ and $(r)'_i$ will be different.

Footnote: More generally, if R is the linearized version of some general transformation $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ at a point $\mathbf{x}$, so that $d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x}$, then (1.3.2) that $\mathbf{a}' = R\mathbf{a}$ says that $\mathbf{a}$ "transforms as a contravariant vector with respect to the underlying transformation $\mathbf{F}$ ". In general $R(\mathbf{x})$ is a combination of rotation and stretch and is a function of location. In our current document we deal only with R(x) = R = a rotation that is the same at all points in space. It turns out however that the notation a'_i is unambiguous in the general case as well as we shall now show. Lucht *Tensor* uses a different notation for basis vectors, and to make the connection between our current document and *Tensor* one must take

$$\{\mathbf{e}_n, \mathbf{e}'_n\} \rightarrow \{\mathbf{u}_n, \mathbf{e}_n\} \qquad \text{Frame S} = \{\mathbf{e}_n\} \rightarrow \text{Frame S} = \{\mathbf{u}_n\}$$
$$\mathbf{e}_n = R\,\mathbf{e}'_n \rightarrow \mathbf{u}_n = R\mathbf{e}_i \qquad \text{Frame S'} = \{\mathbf{e}'_n\} \rightarrow \text{Frame S'} = \{\mathbf{e}_n\} \ .$$

Then in the language of *Tensor* where • is the "covariant dot product", one has (implied summations),

$$(a')_n = \mathbf{a}' \bullet \mathbf{u}_n = (R\mathbf{a}) \bullet (R\mathbf{e}_n) = (R\mathbf{a})_i (R\mathbf{e}_n)^i = R_i{}^j (a)_j \, R^i{}_k (\mathbf{e}_n)^k = (R_i{}^j R^i{}_k)\,(a)_j (\mathbf{e}_n)^k$$

$$= \delta^j{}_k (a)_j (\mathbf{e}_n)^k = (a)_j (\mathbf{e}_n)^j = \mathbf{a} \bullet \mathbf{e}_n = (a)'_n \ .$$

In this general case, $\mathbf{u_n}$ are still axis-aligned basis vectors, but $\mathbf{e_n}$ are generally not axis-aligned and are generally not unit vectors. For example, in spherical coordinates $\mathbf{e_1} = \hat{\mathbf{r}}$, $\mathbf{e_2} = r\,\hat{\boldsymbol{\theta}}$ and $\mathbf{e_3} = r\sin\theta\,\hat{\boldsymbol{\varphi}}$ as shown in (E.6.8).

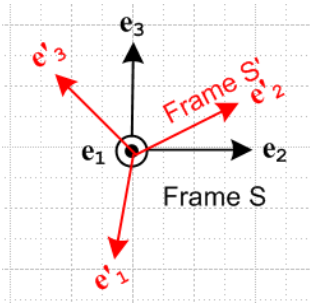
1.4 When are two vectors equal?

This topic will probably seem strange and unnecessary, but it has been a constant annoyance to the author so here are some words on the subject.

When we say two vectors **A** and **B** are the same or are equal, we mean that the two vectors have the same components in the same coordinate system and we write **A** = **B**. This does not require that vectors **A** and **B** coincide. It might be that **B** is a translated copy of **A**. To be really fussy, we could define a stronger equality $\mathbf{A} \triangleq \mathbf{B}$ to mean that not only do the vectors have the same components in the sense of **A** = **B,** but the vectors actually coincide with each other. We shall have no use for $\mathbf{A} \triangleq \mathbf{B}$ in this document. For us, two vectors are "the same" even if translated from one another.

In light of this interpretation of vectors being equal, we can examine the meaning of certain statements. For example, we normally say "a particle is located at **r** in Frame S ". This really means the particle is at *point* **r** in Frame S which has coordinates (x,y,z). What this means in terms of the graphic *vector* **r** is that *if* the vector **r** is translated so that its tail is at the origin of Frame S, *then* its tip will be at the particle location. The vector **r** can be drawn anywhere in a picture. It describes the *displacement* of a particle in Frame S from the origin in Frame S.

Example 4: When we say $\mathbf{e_n} = R\,\mathbf{e'_n}$ as in (1.1.1) above, it is understood that the tails of all vectors involved (the $\mathbf{e_n}$ and the $\mathbf{e'_n}$) are at a common location, as in this picture



(1.4.1)

even though, in our application of Fig 1, the $\mathbf{e_n}$ are drawn with their tails at the origin of Frame S while the $\mathbf{e'_n}$ are drawn with their tails at the origin of Frame S'.

Example 5: In the expansion $\mathbf{r} = (r)_i\mathbf{e_i}$ we normally think of vector **r** having its tail at the origin of Frame S, while in the expansion $\mathbf{r} = (r')'_i\mathbf{e'_i}$ one would be inclined to think of vector **r** as having its tail at the origin of Frame S'. In our stricter sense of coincidence noted above, we might say $(r)_i\,\mathbf{e_i} \triangleq (r')'_i\,\mathbf{e'_i}$ but this is not of interest. What we care about is that $(r)_i\,\mathbf{e_i} = (r')'_i\,\mathbf{e'_i}$ in the sense **A** = **B** above and we don't care if the vectors **A** and **B** are translated relative to one another. What we care about is that the vectors have the same components in any given Frame.