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1.2 Expansions of a vector and use of primes and parentheses

Note: We write $(a)_i$ as a component of vector $\mathbf{a}$, but $(\mathbf{e}_n)_i$ as a component of $\mathbf{e}_n$. In the first case the a in $(a)_i$ is not bolded, but since $\mathbf{e}_n$ is decorated with a label n, it gets bolded. It is just our convention.

Any vector $\mathbf{a}$ can be expanded on either set of basis vectors $\mathbf{e}_i$ (Frame S) or $\mathbf{e'}_i$ (Frame S') so that, with implied summation on i,

$$\mathbf{a} = a_i\,\mathbf{e}_i = a'_i\,\mathbf{e'}_i \quad . \qquad a_i = \mathbf{a}\bullet\mathbf{e}_i \qquad a'_i = \mathbf{a}\bullet\mathbf{e'}_i \quad . \qquad (1.2.1)$$

If some other vector named $\mathbf{a}'$ is lurking in the wings, one might want to be more careful labeling components. A safe method is this:

$$\begin{aligned} &\mathbf{a} = (a)_i\,\mathbf{e}_i = (a)'_i\,\mathbf{e'}_i \qquad (a)_i = \mathbf{a}\bullet\mathbf{e}_i \qquad (a)'_i = \mathbf{a}\bullet\mathbf{e'}_i \\ &\mathbf{a'} = (a')_i\,\mathbf{e}_i = (a')'_i\,\mathbf{e'}_i \qquad (a')_i = \mathbf{a'}\bullet\mathbf{e}_i \qquad (a')'_i = \mathbf{a'}\bullet\mathbf{e'}_i \quad . \end{aligned} \qquad (1.2.2)$$

Here, a prime *inside* a parentheses is part of the vector name, whereas a prime *outside* a parentheses denotes a vector component in Frame S' (whereas no prime outside means a component in Frame S). Unless the relationship between vectors $\mathbf{a}$ and $\mathbf{a}'$ has a certain simple form, it is very likely that $(a')_i \neq (a)'_i$. In this case the notation a'_i would be ambiguous since one doesn't know whether it refers to $(a')_i$ or $(a)'_i$. It is true that the notation a_i could be unambiguously identified with $(a)_i$, but we shall maintain the parentheses just to be uniform.

Matrix Notation to show how components are related.

Let R be the transformation appearing in the Basis Theorem (1.1.29) such that $\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$ and $\mathbf{e}_n = R\mathbf{e'}_n$. Consider the following expansion of vector $\mathbf{a}$ on the basis vectors $\mathbf{e'}_j$,

$$\begin{aligned} \mathbf{a} &= (a)'_j\,\mathbf{e'}_j && \text{// (1.2.2)} \\ &= (a)'_j\,\{\,R_{ji}\,\mathbf{e}_i\,\} && \text{// } \mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \text{ , linear combination of vectors} \\ &= (a)'_j\,\{\,(R^{-1})_{ij}\,\mathbf{e}_i\,\} && \text{// } R = (R^{-1})^{T} \text{ real orthogonal rotation} \\ &= \{\,(R^{-1})_{ij}(a)'_j\,)\mathbf{e}_i \quad . && \text{// reorder} \end{aligned} \qquad (1.2.3)$$

Comparing this to $\mathbf{a} = (a)_i\,\mathbf{e}_i$ of (1.2.2) we conclude that, since $\mathbf{e}_i$ is a complete basis,

$$(a)_i = (R^{-1})_{ij}(a)'_j \qquad \text{so} \qquad (a)'_i = R_{ij}(a)_i \qquad (1.2.4)$$

where R^{-1} is a 3x3 rotation matrix.

We can repeat the above discussion replacing **a** with **a'** with this result,

$$(a')_i = (R^{-1})_{ij}(a')'_j \,. \qquad \text{so} \qquad (a')'_i = R_{ij}(a')_i \tag{1.2.5}$$

These matrix equations are convenient for computing the components of a vector on the $\mathbf{e}_i$ basis if they are known in the $\mathbf{e}'_i$ basis (and vice versa) .

Dot Products

Consider two normal (normally transforming) vectors $\mathbf{a'} = R\mathbf{a}$ and $\mathbf{b'} = R\mathbf{b}$, We know using (1.1.38) that the quantity $\mathbf{a} \bullet \mathbf{b} = [R\mathbf{a}] \bullet [R\mathbf{b}] = \mathbf{a'} \bullet \mathbf{b'}$. As an example, one then has $\mathbf{a} \bullet \mathbf{a} = \mathbf{a'} \bullet \mathbf{a'}$ which says $|\mathbf{a}|^2 = |\mathbf{a}'|^2$. Thus a real orthogonal transformation R is one which preserves the length of a vector. Note that both regular rotations (detR=1) and reflections (detR=-1) have this property.

Since the dot product $\mathbf{a} \bullet \mathbf{b}$ has the same value in Frame S as in Frame S', it is a "rotational scalar", as distinct from a "scalar" which sometimes just means a 1-tuple. The basis vectors are not normal vectors because they are back rotated, meaning $\mathbf{e_n'} = R^{-1}\mathbf{e_n}$. We can of course "front rotate" a basis vector to get $\mathbf{q_n} = R\mathbf{e_n}$ but these $\mathbf{q_n}$ are not the basis vectors $\mathbf{e_n'}$. This means that the dot product $\mathbf{a} \bullet \mathbf{e_n}$ is not a rotational scalar, even though it is a dot product of two "vectors". So one will *not* have $\mathbf{a} \bullet \mathbf{e_n} = \mathbf{a'} \bullet \mathbf{e'_n}$. In fact

$$\mathbf{a} \bullet \mathbf{e_n} = a_n$$

$$\mathbf{a'} \bullet \mathbf{e'_n} = [R\mathbf{a}] \bullet [R^{-1}\mathbf{e_n}] = [R^2\mathbf{a}] \bullet \mathbf{e_n} = [R^2\mathbf{a}]_n \neq \mathbf{a} \bullet \mathbf{e_n} \,. \qquad \text{// (1.1.38)}$$

On the other hand, we still have

$$\mathbf{e_n} \bullet \mathbf{e_m} = [R\mathbf{e'_n}] \bullet [R\mathbf{e'_n}] = \mathbf{e'_n} \bullet \mathbf{e'_m} = \delta_{nm} \,.$$

Although this dot product is the same in both frames, it is not a rotational scalar because it is in fact a rank-2 tensor known as the metric tensor. The distinction is minor for rotations because in fact the dot product is the same number in both frames, either 0 or 1.

1.3 Special case where a'_i is unambiguous

We shall now examine the type of relationship between **a'** and **a** in which $(a')_i = (a)'_i$ and therefore we can use the notation a'_i without ambiguity. First, the components $(a)'_i$ and $(a)_i$ are related in the following simple manner, using (1.1.8),

$$(a)'_n = \mathbf{e'_n} \bullet \mathbf{a} = (\mathbf{e'_n})_m (a)_m = R_{nm}(a)_m \,. \tag{1.3.1}$$

Now suppose we *define* a new vector **a'** in this way,

$$\mathbf{a}' \equiv R\mathbf{a} \,. \tag{1.3.2}$$

If a vector **a** transforms into **a'** according to (1.3.2), we say it is a "vector under rotations" which means it "transforms as a vector under rotations".

When written in Frame S components this says

$$(a')_n = R_{nm}(a)_m \quad . \tag{1.3.3}$$

Comparison of (1.3.1) and (1.3.3) shows that

$$(a)'_n = (a')_n \tag{1.3.4}$$

and therefore in this case we can use

$$a'_n \equiv (a)'_n = (a')_n \quad . \tag{1.3.5}$$

Thus, if the vectors **a** and **a'** are related by **a'** = R**a** where R is the rotation appearing in $\mathbf{e}_n = R\,\mathbf{e}'_n$, then we can dispense with the parentheses as shown in (1.3.4). We still have $(a')'_i$ which requires parentheses.

Example 1: Consider the equation in the Basis Theorem (1.1.30) ,

$$\mathbf{e}'_n = R^{-1}\,\mathbf{e}_n \quad .$$

Since this is *not* of the form **a'** ≡ R**a**, we may not dispense with the parentheses. In fact from (1.1.8) we have

$$\begin{aligned} (\mathbf{e}_n)'_i &= R_{in} \\ (\mathbf{e}'_n)_i &= R_{ni} \end{aligned} \tag{1.3.6}$$

and these are not the same because rotation matrices are not symmetric. Unlike normal vectors for which one writes the transform **a'** = R**a**, the basis vectors are "back-rotated" as noted earlier so $\mathbf{e}'_n = (R^{-1})\mathbf{e}_n$.

Active and Passive

One can think of **a'** = R**a** as an **active** rotation of vector **a** into another vector **a'** within Frame S. In this case, the components of **a'** are $(a')_i$. The alternative is to think of vector **a** as not moving at all in Frame S, but the basis vectors are back-rotated from $\mathbf{e}_n$ to $\mathbf{e}'_n$ taking us to Frame S'. In this back-rotated basis the components of **a** are $(a)'_i$. This is the **passive** view of a rotation and is in fact the view we take in most of this document because we want to *observe* activities in Frame S from Frame S' and vice versa. Each view has its usefulness and we have just shown that if **a'** = R**a**, then $(a)'_n = (a')_n$.

In the active view, the "apparatus" is rotated and the axes stay put, while in the passive view the apparatus stays put but the axes are back-rotated. There is a third view in which the apparatus and the axes are both rotated in the same direction, and this view is useful in the discussion of covariance of equations (e.g. Lucht *Tensor* ***). One can regard the three views as three "experiments" one might perform.

Example 2: Soon we shall be dealing with the Fig 1 equation $\mathbf{r'} = \mathbf{r} - \mathbf{b}$. Since this is *not* of the form $\mathbf{r'} \equiv R\mathbf{r}$, we may *not* dispense with the parentheses, and we expect that $(r')_i$ and $(r)'_i$ will be different.

Footnote: More generally, if R is the linearized version of some general transformation $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ at a point $\mathbf{x}$, so that $d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x}$, then (1.3.2) that $\mathbf{a}' = R\mathbf{a}$ says that $\mathbf{a}$ "transforms as a vector with respect to the underlying transformation $\mathbf{F}$ ". In general $R(\mathbf{x})$ is a combination of rotation and stretch and is a function of location. In our current document we deal only with R(x) = R = a rotation that is the same at all points in space. It turns out however that the notation a'_i is unambiguous in the general case as well as we shall now show. Lucht *Tensor* uses a different notation for basis vectors, and to make the connection between our current document and *Tensor* one must take

$\{\mathbf{e}_n, \mathbf{e}'_n\} \rightarrow \{\mathbf{u}_n, \mathbf{e}_n\}$ Frame S = $\{\mathbf{e}_n\}$ → Frame S = $\{\mathbf{u}_n\}$

$\mathbf{e}_n = R\,\mathbf{e}'_n \rightarrow \mathbf{u}_n = R\mathbf{e}_i$ Frame S' = $\{\mathbf{e}'_n\}$ → Frame S' = $\{\mathbf{e}_n\}$.

Then in the language of *Tensor* where • is the "covariant dot product", one has (implied summations),

$$(a')_n = \mathbf{a}' \bullet \mathbf{u}_n = (R\mathbf{a}) \bullet (R\mathbf{e}_n) = (R\mathbf{a})_i (R\mathbf{e}_n)^i = R_i{}^j\, (a)_j\, R^i{}_k\, (\mathbf{e}_n)^k = (R_i{}^j R^i{}_k)\, (a)_j (\mathbf{e}_n)^k$$

$$= \delta^j{}_k (a)_j (\mathbf{e}_n)^k = (a)_j (\mathbf{e}_n)^j = \mathbf{a} \bullet \mathbf{e}_n = (a)'_n \text{ of this document.}$$

In this general case, within Frame S the $\mathbf{u}_n$ are still axis-aligned basis vectors but the $\mathbf{e}_n$ are generally not axis-aligned and are generally not unit vectors. For example, in spherical coordinates $\mathbf{e}_1 = \hat{\mathbf{r}}$, $\mathbf{e}_2 = r\,\hat{\boldsymbol{\theta}}$ and $\mathbf{e}_3 = r\sin\theta\,\hat{\boldsymbol{\varphi}}$ as shown in (E.6.8) below.