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1. Notation, important role of the Prime Symbol, and other Preliminaries

1.1 Basis vectors $\mathbf{e}_n$ and $\mathbf{e'}_n$, rotation R, and the Basis Rule

Unless otherwise specified, repeated indices have implicit sums. For example $(\mathbf{e'}_n)_i\mathbf{e}_i$ means $\Sigma_i(\mathbf{e'}_n)_i\mathbf{e}_i$. This is known as the Einstein convention.

We have in mind that we are operating in Euclidian space E^3, but everything in this section is generally valid in E^N.

Let Frame S have orthonormal basis vectors $\mathbf{e}_n$.
Let Frame S' have orthonormal basis vectors $\mathbf{e'}_n$. Thus,

$$\mathbf{e}_n \bullet \mathbf{e}_m = \delta_{n,m} \qquad \mathbf{e'}_n \bullet \mathbf{e'}_m = \delta_{n,m} \ . \tag{1.1.1}$$

Assume that the two basis vector sets are related in this manner:

$$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \qquad n = 1,2,3 \qquad RR^T = 1 \ . \tag{1.1.2}$$

This says that each basis vector of Frame S' is a certain *linear combination* of Frame S basis vectors. We shall assume that the matrix R of coefficients is real orthogonal ($R^{-1} = R^T$ or $RR^T = R^TR = 1$). Since $RR^T = 1$, we know that det(R) = ± 1. Real orthogonal matrices with det(R) = - 1 are combinations of regular rotations with a reflection, whereas for regular rotations one has det(R) = +1.

Notice that the n on $\mathbf{e}_n$ is a label and not a component index.

Using a notation described more in the next section, we expand each basis vector onto both bases:

expansions	projections
$\mathbf{e'}_n = (\mathbf{e}_i \bullet \mathbf{e'}_n)\,\mathbf{e}_i = (\mathbf{e'}_n)_i\,\mathbf{e}_i$	$(\mathbf{e'}_n)_i = (\mathbf{e}_i \bullet \mathbf{e'}_n)$
$\mathbf{e'}_n = (\mathbf{e'}_i \bullet \mathbf{e'}_n)\,\mathbf{e'}_i = (\mathbf{e'}_n)'_i\,\mathbf{e'}_i$	$(\mathbf{e'}_n)'_i = (\mathbf{e'}_i \bullet \mathbf{e'}_n)$
$\mathbf{e}_n = (\mathbf{e}_i \bullet \mathbf{e}_n)\,\mathbf{e}_i = (\mathbf{e}_n)_i\,\mathbf{e}_i$	$(\mathbf{e}_n)_i = (\mathbf{e}_i \bullet \mathbf{e}_n)$
$\mathbf{e}_n = (\mathbf{e'}_i \bullet \mathbf{e}_n)\,\mathbf{e'}_i = (\mathbf{e}_n)'_i\,\mathbf{e'}_i$	$(\mathbf{e}_n)'_i = (\mathbf{e'}_i \bullet \mathbf{e}_n)$.

(1.1.3)

Lines 2 and 3 are not very interesting because they just say what we already know,

$$\mathbf{e'}_n = \delta_{i,n}\,\mathbf{e'}_i = (\mathbf{e'}_n)'_i\,\mathbf{e'}_i \qquad (\mathbf{e'}_n)'_i = \delta_{i,n} \quad \text{projection}$$

$$\mathbf{e}_n = \delta_{i,n}\,\mathbf{e}_i = (\mathbf{e}_n)_i\,\mathbf{e}_i \qquad\qquad (\mathbf{e}_n)_i = \delta_{i,n} \quad \text{projection .} \tag{1.1.4}$$

Now dot (1.1.2) [$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$] first with $\mathbf{e}_i$ and then with $\mathbf{e'}_i$ and use the projections in (1.1.3) to get

$$(\mathbf{e'}_n)_i = R_{nm}\,(\mathbf{e}_m)_i \qquad \text{Frame S components}$$

$$(\mathbf{e'}_n)'_i = R_{nm}\,(\mathbf{e}_m)'_i \qquad \text{Frame S' components .} \tag{1.1.5}$$

Rewrite the first as

$$(\mathbf{e'}_n)_i = R_{nm}\,\delta_{m,i} = R_{ni} \,. \tag{1.1.6}$$

For the second, one has

$$\delta_{n,i} = R_{nm}\,(\mathbf{e}_m)'_i \;\Rightarrow\; R^T_{kn}\delta_{n,i} = R^T_{kn}R_{nm}\,(\mathbf{e}_m)'_i \;\Rightarrow\; R^T_{ki} = (R^TR)_{km}\,(\mathbf{e}_m)'_i = \delta_{k,m}(\mathbf{e}_m)'_i = (\mathbf{e}_k)'_i$$

and therefore

$$(\mathbf{e}_k)'_i = R^T_{ki} = R_{ik} \qquad \Rightarrow \qquad (\mathbf{e}_n)'_i = R_{in} \,. \tag{1.1.7}$$

So now we know all about the components of the basis vectors in each Frame :

$$(\mathbf{e'}_n)_i = R_{ni} \qquad (\mathbf{e}_n)_i = \delta_{n,i} \qquad \text{Frame S components}$$

$$(\mathbf{e}_n)'_i = R_{in} \qquad (\mathbf{e'}_n)'_i = \delta_{n,i} \qquad \text{Frame S' components} \tag{1.1.8}$$

Equation (1.1.1) says that either set of basis vectors is orthonormal. It is also true that each set of basis vectors is "complete". In Frame S this means that *any* vector **a** can be expanded as $\mathbf{a} = a_n\mathbf{e}_n$. As in (1.1.3) one can then write

$$\mathbf{a} = a_n\mathbf{e}_n \quad \text{where } a_n = (\mathbf{e}_n \bullet \mathbf{a}) \qquad \Rightarrow \qquad \mathbf{a} = (\mathbf{e}_n \bullet \mathbf{a})\mathbf{e}_n \quad .$$

Writing this last equation out in Frame S components, one gets

$$a_j = (\,(\mathbf{e}_n)_i a_i\,)(\mathbf{e}_n)_j \qquad \text{or} \qquad a_j = [(\mathbf{e}_n)_i(\mathbf{e}_n)_j]\,a_i$$

In order that this last equation be valid for *any* **a**, it must be true that (implicit sum on n) ,

$$(\mathbf{e}_n)_i(\mathbf{e}_n)_j = \delta_{i,j} \qquad\qquad (\mathbf{e'}_n)_i(\mathbf{e'}_n)_j = \delta_{i,j} \quad . \quad \text{// completeness} \tag{1.1.9}$$

This is the formal statement that the $\mathbf{e}_n$ are complete. By starting instead expanding **a** onto the $\mathbf{e'}_n$, one gets the statement on the right above that the $\mathbf{e'}_n$ are also complete. If we write out these two statements using (1.1.8), the above completeness statements become

$\delta_{n,i}\,\delta_{n,j} = \delta_{i,j}$ $\qquad\qquad$ $R_{ni}\,R_{nj} = \delta_{i,j}$.

The equation on the left is obvious. On the right we know that $R_{ni}\,R_{nj} = R^T_{in}R_{nj} = (R^TR)_{ij}$ and we know that this equals $\delta_{i,j}$ because R is real orthogonal $R^TR = 1$. Thus the completeness statements are "nothing new".

We are now going to prove a crucial "rule" which seems to have no official name:

The Basis Rule: This simple rule states that (proof below),

$$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \quad \Leftrightarrow \quad \mathbf{e}_n = R\mathbf{e'}_n \ . \tag{1.1.10}$$

The left equation involves a linear combination of basis vectors, while the right equation is a statement that the vector $\mathbf{e}_n$ is the same as the vector $[R\mathbf{e'}_n]$. One can take components of $\mathbf{e}_n = R\mathbf{e'}_n$ to get,

$$\begin{aligned} &(\mathbf{e}_n)_i = R_{ij}\,(\mathbf{e'}_n)_j && \text{Frame S components} \\ &(\mathbf{e}_n)'_i = R'_{ij}\,(\mathbf{e'}_n)'_j && \text{Frame S' components} \end{aligned} \tag{1.1.11}$$

The matrix R_{ij} is a matrix in the Frame S basis, while R'_{ij} is in the Frame S' basis. It turns out that these two matrices are the same R' = R, but if R were not a rotation, they likely would *not* be the same, see below.

Once proven, both equations in (1.1.10) can be inverted to give

$$\mathbf{e}_m = (R^{-1})_{nm}\,\mathbf{e'}_n \qquad \text{and} \qquad \mathbf{e'}_n = R^{-1}\mathbf{e}_n \ . \tag{1.1.12}$$

Each $\mathbf{e'}_n$ is thus a "back-rotated" version of $\mathbf{e}_n$ by R^{-1}. Later we shall discuss "normal vectors" which have the property $\mathbf{a}' = R\mathbf{a}$ where $\mathbf{a}$ is rotated into $\mathbf{a}'$ by R. Since $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$ is an exception to this normal transformation, we say it is back-rotated. More on this later.

Proof $\Rightarrow$: Show that $\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \ \Rightarrow \ \mathbf{e}_n = R\mathbf{e'}_n$

We prove that $\mathbf{e}_n = R\mathbf{e'}_n$ by showing that both sides have the same components in Frame S:

$\mathbf{e}_n = R\mathbf{e'}_n$	?	
$(\mathbf{e}_n)_i = (R\mathbf{e'}_n)_i$	?	
$\delta_{n,i} = R_{ij}\,(\mathbf{e'}_n)_j$	?	
$\delta_{n,i} = R_{ij}\,R_{nj}$	?	// from (1.1.8) which is based on $\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$
$\delta_{n,i} = R_{ij}\,R^T_{jn}$	?	
$\delta_{n,i} = (RR^T)_{in}$	yes !	// from (1.1.2) that $1 = RR^T$

Thus our proof here consists of just reversing the above steps. QED

Proof $\Leftarrow$: Show that $\mathbf{e}_n = R\mathbf{e}'_n \;\Rightarrow\; \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m$

Each $\mathbf{e}'_n$ must be some linear combination of the $\mathbf{e}_n$ since a basis is by definition complete. So we write

$$\mathbf{e}'_n = A_{nm}\,\mathbf{e}_m \ .$$

We know that $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$ so then

$$R^{-1}\mathbf{e}_n = A_{nm}\,\mathbf{e}_m \ .$$

Taking components in Frame S

$$(R^{-1}\mathbf{e}_n)_i = A_{nm}\,(\mathbf{e}_m)_i$$

$$(R^{-1})_{ij}(\mathbf{e}_n)_j = A_{nm}\,(\mathbf{e}_m)_i$$

$$R_{ji}\,\delta_{n,j} = A_{nm}\,\delta_{m,i} \qquad // \ R^{-1} = R^T$$

$$R_{ni} = A_{ni} \ .$$

Therefore $A_{nm} = R_{nm}$ and so $\mathbf{e}'_n = R_{nm}\,\mathbf{e}_m$. QED

Alternate shorthand notations

$$\mathbf{e}_n = R\mathbf{e}'_n \qquad \Leftrightarrow \qquad (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = R\,(\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3) \tag{1.1.13}$$

$$\mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \qquad \Leftrightarrow \qquad \begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = R \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} . \tag{1.1.14}$$

The first alternate notation implies for example that $\mathbf{e}_1 = R\mathbf{e}'_1$. This is *not* implied by the second alternate notation which is meant to say $\mathbf{e}'_1 = R_{11}\mathbf{e}_1 + R_{12}\mathbf{e}_2 + R_{13}\mathbf{e}_3$ = linear combination of vectors. These alternate notations are useful when the basis vectors have names like $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ or $\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}$.

In passing, we note that if R is any rotation (so therefore $R^TR = 1$), then

$$\mathbf{a} \bullet \mathbf{b} = [R\mathbf{a}] \bullet [R\mathbf{b}] \ . \tag{1.1.15}$$

Proof:

$[R\mathbf{a}] \bullet [R\mathbf{b}] = [R\mathbf{a}]_k[R\mathbf{b}]_k = R_{ki}a_iR_{kj}b_j = (R^T)_{jk}\,R_{ki}a_ib_j = (R^TR)_{ji}a_ib_j = \delta_{j,i}a_ib_j = a_ib_i = \mathbf{a} \bullet \mathbf{b}$

Rank-2 tensors, Dirac notation and Completeness

Whereas a normal vector transforms under rotation R as $\mathbf{a}' = R\mathbf{a}$, a rank-2 tensor transforms as

$$T' = RTR^{-1} \quad . \tag{1.1.16}$$

See e.g. Lucht *Tensor* ***. If we stick T = R into this formula, we get $R' = RRR^{-1} = R$. So for the tensor R which defines the rotation of interest, we get this simplification. For other tensors this will generally not be the case, unless the tensor T happens to commute with R, [T,R] = TR-RT = 0.

We find the Dirac Notation helpful in understanding equations like (1.1.16). In this notation, basis vectors like $\mathbf{e}_n$ are written $|\mathbf{e}_n\rangle$ and $\mathbf{e}_n^T = \langle\mathbf{e}_n|$. Matrices like T' and T above are regarded as matrices of the same abstract operator $\mathcal{T}$ (the tensor) evaluated in the two different bases $\mathbf{e}'_n$ and $\mathbf{e}_n$. In Dirac notation statements above appear as follows:

$$\mathbf{e}_n \bullet \mathbf{e}_m = \delta_{n,m} \quad (= \mathbf{e}_n^T \mathbf{e}_m) \qquad \mathbf{e}'_n \bullet \mathbf{e}'_m = \delta_{n,m} \ . \qquad \text{// orthonormality} \qquad \textit{(1.1.1)}$$
$$\langle \mathbf{e}_n \mid \mathbf{e}_m \rangle = \delta_{n,m} \qquad \langle \mathbf{e}'_n \mid \mathbf{e}'_m \rangle = \delta_{n,m} \tag{1.1.17}$$

$$(\mathbf{e}_n)_i (\mathbf{e}_n)_j = \delta_{ij} \qquad (\mathbf{e}'_n)_i (\mathbf{e}'_n)_j = \delta_{ij} \qquad \text{// completeness} \qquad \textit{(1.1.9)}$$
$$|\mathbf{e}_n\rangle\langle\mathbf{e}_n| = 1 \qquad |\mathbf{e}'_n\rangle\langle\mathbf{e}'_n| = 1 \tag{1.1.18}$$

The last line equations do seem unusual, but the "1" is the unity operator in the space in which $\mathcal{T}$ lives. To verify $|\mathbf{e}'_n\rangle\langle\mathbf{e}'_n| = 1$ one can "close" each side as follows

$$\langle \mathbf{e}_i |\mathbf{e}'_n\rangle\langle\mathbf{e}'_n| \mathbf{e}_j\rangle = \langle \mathbf{e}_i| \ 1 \ | \ \mathbf{e}_j\rangle = \langle \mathbf{e}_i| \ \mathbf{e}_j\rangle$$

or

$$(\mathbf{e}'_n)_i (\mathbf{e}'_n)_j = \delta_{ij} \quad .$$

Finally one can see then that

$$\langle \mathbf{e}_n \mid \mathbf{e}'_m\rangle = \mathbf{e}_n \bullet \mathbf{e}'_m = (\mathbf{e}'_m)_n = R_{mn} = \mathbf{e}'_m \bullet \mathbf{e}_n = \langle \mathbf{e}'_m \mid \mathbf{e}_n\rangle$$

so then

$$\langle \mathbf{e}_n \mid \mathbf{e}'_m\rangle = \langle \mathbf{e}'_m \mid \mathbf{e}_n\rangle = R_{mn} \ . \tag{1.1.19}$$

One says that the matrix R_{mn} is the "basis change matrix" between Frame S and Frame S' coordinates.

We now interpret the matrices T and T' in the following manner,

$$T_{ij} = \langle \mathbf{e}_i \mid \mathcal{T} \mid \mathbf{e}_j \rangle$$

$$T'_{ij} = \langle \mathbf{e}'_i \mid \mathcal{T} \mid \mathbf{e}'_j \rangle \ . \tag{1.1.20}$$

Matrix T is the Dirac sandwich of the rank-2 tensor $\mathcal{T}$ in the Frame S basis.
Matrix T' is the Dirac sandwich of the rank-2 tensor $\mathcal{T}$ in the Frame S' basis.

These sandwich objects are referred to as "matrix elements of the operator $\mathcal{T}$" in quantum mechanics.

Consider now,

$$(RTR^{-1})_{in} = R_{ij}\, T_{jk}\, (R^{-1})_{kn} = R_{ij} \langle \mathbf{e}_j \,|\, \mathcal{T} \,|\, \mathbf{e}_k \rangle R_{nk} = \langle \mathbf{e}'_i \,|\, \mathbf{e}_j \rangle \langle \mathbf{e}_j \,|\, \mathcal{T} \,|\, \mathbf{e}_k \rangle \langle \mathbf{e}_k \,|\, \mathbf{e}'_n \rangle$$

$$= \langle \mathbf{e}'_i \,|\, \mathbf{e}_j \rangle \langle \mathbf{e}_j \,|\, \mathcal{T} \,|\, \mathbf{e}_k \rangle \langle \mathbf{e}_k \,|\, \mathbf{e}'_n \rangle = \langle \mathbf{e}'_i \,|\, 1\, \mathcal{T}\, 1 \,|\, \mathbf{e}'_n \rangle = \langle \mathbf{e}'_i \,|\, \mathcal{T} \,|\, \mathbf{e}'_n \rangle = T'_{in}\,,$$

Thus we provide a derivation of sorts for the claim (1.1.16) that $T' = RTR^{-1}$.

In this same language we may write

$$R_{ij} = \langle \mathbf{e}_i \,|\, \mathcal{R} \,|\, \mathbf{e}_j \rangle = \langle \mathbf{e}_i \,|\, R\mathbf{e}_j \rangle \overset{(1.1.15)}{=} \mathbf{e}_i \bullet [R\mathbf{e}_j] \overset{(1.1.12)}{=} [R^{-1}\mathbf{e}_i] \bullet \mathbf{e}_j = \mathbf{e}'_i \bullet \mathbf{e}_j = \langle \mathbf{e}'_i \,|\, \mathbf{e}_j \rangle$$

$$R'_{ij} = \langle \mathbf{e}'_i \,|\, \mathcal{R} \,|\, \mathbf{e}'_j \rangle = \langle \mathbf{e}'_i \,|\, R\mathbf{e}'_j \rangle = \mathbf{e}'_i \bullet [R\mathbf{e}'_j] \overset{(1.1.10)}{=} \mathbf{e}'_i \bullet \mathbf{e}_j = \langle \mathbf{e}'_i \,|\, \mathbf{e}_j \rangle \qquad (1.1.21)$$

and we recover the fact that R = R' alluded to earlier.

Time dependence of the R_{ij}

If Frame S is fixed and Frame S' is rotating, then we really have $\mathbf{e}_n = R(t)\mathbf{e}'_n(t)$ where the R matrix is a function of time, so we have $R_{ij}(t)$. Similarly, if Frame S is moving and Frame S' is fixed, $\mathbf{e}_n(t) = R(t)\, \mathbf{e}'_n$ and again one has $R_{ij}(t)$. Only in the case where there is no rotation between the frames are the R_{ij} independent of time. This means that $\boldsymbol{\omega} = 0$ in Fig 1. Since our document is about "rotating frames of reference" we exclude this no-rotation case from consideration.

Multiple Transformation Pitfalls

Consider two rotation transformations involving R and S, each of which has a Basis Rule (1.1.10),

$$\mathbf{e}'_n = R_{nm}\, \mathbf{e}_m \quad \Leftrightarrow \quad \mathbf{e}_n = R\, \mathbf{e}'_n \qquad R_{ij} = R'_{ij} \qquad R = R'$$

$$\mathbf{e}''_n = S'_{nm}\, \mathbf{e}'_m \quad \Leftrightarrow \quad \mathbf{e}'_n = S\, \mathbf{e}''_n \qquad S'_{ij} = S''_{ij} \qquad S' = S''$$

On the right we are showing on the second line that the matrix S' in the Frame S' basis is the same as the matrix S" in the Frame S" basis. We can combine the two transformations to get

$$\mathbf{e}''_n = S'_{nm}R'_{nk}\, \mathbf{e}_k = (S'R')_{nk}\, \mathbf{e}_k \qquad \text{and} \qquad \mathbf{e}_n = (RS)\, \mathbf{e}''_n\,. \qquad (1.1.22)$$

On the other hand, we can apply the Basis Rule (1.1.10) to the combined transformation RS

$$\mathbf{e}''_n = (RS)_{nm}\, \mathbf{e}_m \quad \Leftrightarrow \quad \mathbf{e}_n = (RS)\mathbf{e}''_n \qquad (1.1.23)$$

We then obtain two statements which seem at odds with each other

$\mathbf{e''}_n = (S'R')_{nm}\mathbf{e}_m$ *(1.1.22)*

$\mathbf{e''}_n = (RS)_{nm}\,\mathbf{e}_m$ *(1.1.23)* (1.1.24)

These would be the same if we knew that S'R' = RS. Since R' = R, this says S'R = RS or S' = RSR^{-1}. But this is just the way in which the two tensors S' and S are related relative to the R transformation, and that is why both statements in (1.1.24) are valid. In an application, we are likely to know the matrix S, and to not know the matrix S', so the second form in (1.1.24) is more useful. In our alternate notation this second form would read

$$\begin{pmatrix} \mathbf{e''}_1 \\ \mathbf{e''}_2 \\ \mathbf{e''}_3 \end{pmatrix} = RS \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \qquad (1.1.25)$$

Similar comments apply to the triple Euler angle transformations encountered in Appendix G.

1.2 Expansions of a vector and use of primes and parentheses

Note: We write $(a)_i$ as a component of vector **a** , but $(\mathbf{e}_n)_i$ as a component of $\mathbf{e}_n$. In the first case the a in $(a)_i$ is not bolded, but since $\mathbf{e}_n$ is decorated with a label n, it gets bolded. It is just our convention.

Any vector **a** can be expanded on either set of basis vectors so that, with implied summation on i,

$\mathbf{a} = a_i\,\mathbf{e}_i = a'_i\,\mathbf{e}'_i$. $a_i = \mathbf{a} \bullet \mathbf{e}_i$ $a'_i = \mathbf{a} \bullet \mathbf{e}'_i$. (1.2.1)

If some other vector named **a'** is lurking in the wings, one might want to be more careful labeling components. A safe method would be this:

$\mathbf{a} = (a)_i\,\mathbf{e}_i = (a)'_i\,\mathbf{e}'_i$ $(a)_i = \mathbf{a} \bullet \mathbf{e}_i$ $(a)'_i = \mathbf{a} \bullet \mathbf{e}'_i$

$\mathbf{a'} = (a')_i\,\mathbf{e}_i = (a')'_i\,\mathbf{e}'_i$ $(a')_i = \mathbf{a'} \bullet \mathbf{e}_i$ $(a')'_i = \mathbf{a'} \bullet \mathbf{e}'_i$. (1.2.2)

Here, a prime *inside* a parentheses is part of the vector name, whereas a prime *outside* a parentheses denotes a vector component in Frame S' (whereas no prime outside means a component in Frame S). Unless the relationship between vectors **a** and **a'** has a certain simple form, it is very likely that $(a')_i \neq (a)'_i$. In this case the notation a'_i would be ambiguous since one doesn't know whether it refers to $(a')_i$ or $(a)'_i$. It is true that the notation a_i could be unambiguously identified with $(a)_i$, but we shall maintain the parentheses just to be uniform.

Matrix Notation to show how components are related.

Let R be the transformation appearing in (1.1.10) such that $\mathbf{e}'_n = R_{nm}\,\mathbf{e}_m$ and $\mathbf{e}_n = R\mathbf{e}'_n$.
Consider the following expansion of vector **a** on the basis vectors $\mathbf{e}'_j$,

$\mathbf{a} = (a)'_j\,\mathbf{e}'_j$ // (1.2.2)

$= \ (a)'_j \ \{ \ R_{ji} \ \mathbf{e}_i \ \}$ // $\mathbf{e}'_n = R_{nm} \ \mathbf{e}_m$

$= (a)'_j \ \{ \ (R^{-1})_{ij} \ \mathbf{e}_i \ \}$ // $R = (R^{-1})^T$ real orthogonal rotation

$= \ \{ \ (R^{-1})_{ij} (a)'_j \ ) \mathbf{e}_i \ .$ // reorder (1.2.3)

Comparing this to $\mathbf{a} = (a)_i \ \mathbf{e}_i$ of (1.2.2) we conclude that, since $\mathbf{e}_i$ is a complete basis,

$(a)_i = \ (R^{-1})_{ij} (a)'_j$ so $(a)'_i \ = \ R_{ij} (a)_i$ (1.2.4)

where R^{-1} is a 3x3 rotation matrix.

We can repeat the above discussion replacing **a** with **a**' with this result,

$(a')_i = \ (R^{-1})_{ij} (a')'_j \ .$ so $(a')'_i \ = \ R_{ij} (a')_i$ (1.2.5)

These matrix equations are convenient for computing the components of a vector on the $\mathbf{e}_i$ basis if they are known in the $\mathbf{e}'_i$ basis (and vice versa) .

Dot Products

Consider two normal (normally transforming) vectors $\mathbf{a}' = R\mathbf{a}$ and $\mathbf{b}' = R\mathbf{b}$, We know using (1.1.15) that the quantity $\mathbf{a} \bullet \mathbf{b} \ = \ [R\mathbf{a}] \bullet [R\mathbf{b}] \ = \mathbf{a}' \bullet \mathbf{b}'$. As an example, one then has $\mathbf{a} \bullet \mathbf{a} = \ \mathbf{a}' \bullet \mathbf{a}'$ which says $|\mathbf{a}|^2 = |\mathbf{a}'|^2$. Thus a real orthogonal transformation R is one which preserves the length of a vector. Note that both regular rotations (detR=1) and reflections (detR=-1) have this property.

Since the dot product $\mathbf{a} \bullet \mathbf{b}$ has the same value in Frame S as in Frame S', it is a "rotational scalar", as distinct from a "scalar" which sometimes just means a 1-tuple. The basis vectors are not normal vectors because they are back rotated, meaning $\mathbf{e}_n' = R^{-1}\mathbf{e}_n$. We can of course "front rotate" a basis vector to get $\mathbf{q}_n = R\mathbf{e}_n$ but these $\mathbf{q}_n$ are not the basis vectors $\mathbf{e}_n'$. This means that the dot product $\mathbf{a} \bullet \mathbf{e}_n$ is not a rotational scalar, even though it is a dot product of two "vectors". So one will *not* have $\mathbf{a} \bullet \mathbf{e}_n = \mathbf{a}' \bullet \mathbf{e}'_n$. In fact

$\mathbf{a} \bullet \mathbf{e}_n \ = a_n$

$\mathbf{a}' \bullet \mathbf{e}'_n = \ [R\mathbf{a}] \bullet [R^{-1}\mathbf{e}_n] = [R^2\mathbf{a}] \bullet \mathbf{e}_n = \ [R^2\mathbf{a}]_n \ \neq \ \mathbf{a} \bullet \mathbf{e}_n$ // (1.1.15)

On the other hand, we still have

$\mathbf{e}_n \bullet \mathbf{e}_m = \ [R\mathbf{e}'_n] \bullet \ [R\mathbf{e}'_n] = \mathbf{e}'_n \bullet \mathbf{e}'_m \ = \delta_{n,m}$

Although this dot product is the same in both frames, it is not a rotational scalar because it is in fact a rank-2 tensor known as the metric tensor. The distinction is minor for rotations because in fact the dot product is the same number in both frames, either 0 or 1.

1.3 Special case where a'_i is unambiguous

We shall now examine the type of relationship between **a**' and **a** in which $(a')_i = (a)'_i$ and therefore we can use the notation a'_i without ambiguity. First, the components $(a)'_i$ and $(a)_i$ are related in the following simple manner, using (1.1.8),

$$(a)'_n = \mathbf{e}'_n \bullet \mathbf{a} = (\mathbf{e}'_n)_m (a)_m = R_{nm}(a)_m \, . \tag{1.3.1}$$

Now suppose we *define* a new vector **a**' in this way,

$$\mathbf{a}' \equiv R\mathbf{a} \quad . \tag{1.3.2}$$

If a vector **a** transforms into **a**' according to (1.3.2), we say it is a "vector under rotations" which means it "transforms as a vector under rotations".

When written in Frame S components this says

$$(a')_n = R_{nm}(a)_m \quad . \tag{1.3.3}$$

Comparison of (1.3.1) and (1.3.3) shows that

$$(a)'_n = (a')_n \tag{1.3.4}$$

and therefore in this case we can use

$$a'_n \equiv (a)'_n = (a')_n \quad . \tag{1.3.5}$$

Thus, if the vectors **a** and **a**' are related by **a**' = R**a** where R is the rotation appearing in $\mathbf{e}_n = R\,\mathbf{e}'_n$, then we can dispense with the parentheses as shown in (1.3.4). We still have $(a')'_i$ which requires parentheses.

Example 1: Consider the equation in the Basis Rule (1.1.10) ,

$$\mathbf{e}_n = R\,\mathbf{e}'_n \quad .$$

Since this is *not* of the form **a**' ≡ R**a**, we may not dispense with the parentheses. In fact from (1.1.8) we have

$$\begin{aligned} (\mathbf{e}_n)'_i &= R_{in} \\ (\mathbf{e}'_n)_i &= R_{ni} \end{aligned} \tag{1.3.6}$$

and these are not the same because rotation matrices are not symmetric. Unlike normal vectors for which one writes the transform **a**' = R**a**, the basis vectors are "back-rotated" so $\mathbf{e}'_n = (R^{-1})\mathbf{e}_n$.

Active and Passive

One can think of $\mathbf{a}' = R\mathbf{a}$ as an **active** rotation of vector **a** into another vector **a'** within Frame S. In this case, the components of **a'** are $(a')_i$. The alternative is to think of vector **a** as not moving at all in Frame S, but the basis vectors are back-rotated from $\mathbf{e}_n$ to $\mathbf{e}'_n$ taking us to Frame S'. In this back-rotated basis the components of **a** are $(a)'_i$. This is the **passive** view of a rotation and is in fact the view we take in most of this document because we want to *observe* activities in Frame S from Frame S' and vice versa. Each view has its usefulness and we have just shown that if $\mathbf{a}' = R\mathbf{a}$, then $(a)'_n = (a')_n$.

In the active view, the "apparatus" is rotated and the axes stay put, while in the passive view the apparatus stays put but the axes are back-rotated. There is a third view in which the apparatus and the axes are both rotated in the same direction, and this view is useful in the discussion of covariance of equations (e.g. Lucht *Tensor* ***). One can regard the three views as three "experiments" one might perform.

Example 2: Soon we shall be dealing with the Fig 1 equation $\mathbf{r}' = \mathbf{r} - \mathbf{b}$. Since this is *not* of the form $\mathbf{r}' \equiv R\mathbf{r}$, we may *not* dispense with the parentheses, and we expect that $(r')_i$ and $(r)'_i$ will be different.

Footnote: More generally, if R is the linearized version of some general transformation $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ at a point **x**, so that $d\mathbf{x}' = R(\mathbf{x})\, d\mathbf{x}$, then (1.3.2) that $\mathbf{a}' = R\mathbf{a}$ says that **a** "transforms as a vector with respect to the underlying transformation **F** ". In general $R(\mathbf{x})$ is a combination of rotation and stretch and is a function of location. In our current document we deal only with R(x) = R = a rotation that is the same at all points in space. It turns out however that the notation a'_i is unambiguous in the general case as well as we shall now show. Lucht *Tensor* uses a different notation for basis vectors, and to make the connection between our current document and *Tensor* one must take

$$\{\mathbf{e}_n, \mathbf{e}'_n\} \quad \rightarrow \quad \{\mathbf{u}_n, \mathbf{e}_n\} \qquad\qquad \text{Frame S} = \{\mathbf{e}_n\} \rightarrow \text{Frame S} = \{\mathbf{u}_n\}$$
$$\mathbf{e}_n = R\,\mathbf{e}'_n \quad \rightarrow \quad \mathbf{u}_n = R\mathbf{e}_i \qquad\qquad \text{Frame S'} = \{\mathbf{e}'_n\} \rightarrow \text{Frame S'} = \{\mathbf{e}_n\}\ .$$

Then in the language of *Tensor* where • is the "covariant dot product", one has (implied summations),

$$(a')_n = \mathbf{a}' \bullet \mathbf{u}_n = (R\mathbf{a}) \bullet (R\mathbf{e}_n) = (R\mathbf{a})_i (R\mathbf{e}_n)^i = R_i{}^j\ (a)_j\ R^i{}_k\ (\mathbf{e}_n)^k = (R_i{}^j R^i{}_k)\,(a)_j (\mathbf{e}_n)^k$$

$$= \delta^j{}_k (a)_j (\mathbf{e}_n)^k = (a)_j (\mathbf{e}_n)^j = \mathbf{a} \bullet \mathbf{e}_n = (a)'_n \text{ of this document.}$$

In this general case, within Frame S the $\mathbf{u}_n$ are still axis-aligned basis vectors but the $\mathbf{e}_n$ are generally not axis-aligned and are generally not unit vectors. For example, in spherical coordinates $\mathbf{e}_1 = \hat{\mathbf{r}}$, $\mathbf{e}_2 = r\,\hat{\boldsymbol{\theta}}$ and $\mathbf{e}_3 = r\sin\theta\,\hat{\boldsymbol{\varphi}}$ as shown in (E.6.8).

1.4 When are two vectors equal?

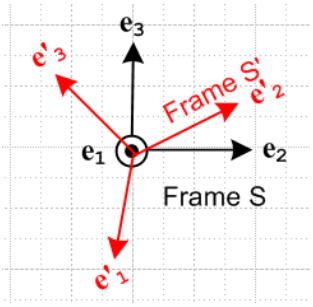
This topic will probably seem strange and unnecessary, but it has been a constant annoyance to the author so here are some words on the subject.

When we say two vectors **A** and **B** are the same or are equal, we mean that the two vectors have the same components in the same coordinate system and we write **A** = **B**. This does not require that vectors **A** and **B** coincide. It might be that **B** is a translated copy of **A**. To be really fussy, we could define a stronger

equality $\mathbf{A} \overset{\bullet}{=} \mathbf{B}$ to mean that not only do the vectors have the same components in the sense of $\mathbf{A} = \mathbf{B}$, but the vectors actually coincide with each other. We shall have no use for $\mathbf{A} \overset{\bullet}{=} \mathbf{B}$ in this document. For us, two vectors are "the same" even if translated from one another.

In light of this interpretation of vectors being equal, we can examine the meaning of certain statements. For example, we normally say "a particle is located at **r** in Frame S ". This really means the particle is at *point* **r** in Frame S which has coordinates (x,y,z). What this means in terms of the graphic *vector* **r** is that *if* the vector **r** is translated so that its tail is at the origin of Frame S, *then* its tip will be at the particle location. The vector **r** can be drawn anywhere in a picture. It describes the *displacement* of a particle in Frame S from the origin in Frame S.

Example 3: When we say $\mathbf{e}_n = R\ \mathbf{e'}_n$ as in (1.1.10), it is understood that the tails of all vectors involved (the $\mathbf{e}_n$ and the $\mathbf{e'}_n$) are at a common location, as in this picture



(1.4.1)

even though, in our application of Fig 1, the $\mathbf{e}_n$ are drawn with their tails at the origin of Frame S while the $\mathbf{e'}_n$ are drawn with their tails at the origin of Frame S'.

Example 4: In the expansion $\mathbf{r} = (r)_i \mathbf{e}_i$ we normally think of vector **r** having its tail at the origin of Frame S, while in the expansion $\mathbf{r} = (r')'_i \mathbf{e'}_i$ one would be inclined to think of vector **r** as having its tail at the origin of Frame S'. In our stricter sense of coincidence noted above, we might say $(r)_i\ \mathbf{e}_i \overset{\bullet}{\neq} (r')'_i\ \mathbf{e'}_i$ but this is not of interest. What we care about is that $(r)_i\ \mathbf{e}_i = (r')'_i\ \mathbf{e'}_i$ in the sense $\mathbf{A} = \mathbf{B}$ above and we don't care if the vectors **A** and **B** are translated relative to one another. What we care about is that the vectors have the same components in any given Frame.