$$\mathbf{e}_n = |\mathbf{e}_n\rangle = \begin{pmatrix} (\mathbf{e}_n)_1 \\ (\mathbf{e}_n)_2 \\ (\mathbf{e}_n)_3 \end{pmatrix} \quad = \text{"vector"}$$

$$\mathbf{e}_n^{\mathrm{T}} = \langle \mathbf{e}_n| = ((\mathbf{e}_n)_1,\ (\mathbf{e}_n)_2,\ (\mathbf{e}_n)_3) = \text{"transpose vector"}$$

**********

$$= R_{ni}R_{nj} = (R^{\mathrm{T}}R)_{ij} = \delta_{ij}$$

Here is how orthogonality (1.1.1) and completeness (1.1.9) appear in Dirac notation

$\langle \mathbf{e}_n|\mathbf{e}_m\rangle = \delta_{nm}$ orthogonality

$I = |\mathbf{e}_i\rangle\langle \mathbf{e}_i|$ completeness

The second notation may seem odd, but consider

Notice how the dot product (scalar product, inner product) works in the following examples, where on the left we use expressions from (1.1.8),

$$\delta_{n,m} = (\mathbf{e}_m)_n = \mathbf{e}_n \bullet \mathbf{e}_m = \mathbf{e}_n^{\mathrm{T}}\mathbf{e}_m = ((\mathbf{e}_n)_1,\ (\mathbf{e}_n)_2,\ (\mathbf{e}_n)_3)\begin{pmatrix} (\mathbf{e}_m)_1 \\ (\mathbf{e}_m)_2 \\ (\mathbf{e}_m)_3 \end{pmatrix} = \langle \mathbf{e}_n|\mathbf{e}_m\rangle$$

$$\delta_{n,m} = (\mathbf{e}'_m)'_n = \mathbf{e}'_n \bullet \mathbf{e}'_m = \mathbf{e}'^{\mathrm{T}}_n\mathbf{e}'_m = ((\mathbf{e}'_n)_1,\ (\mathbf{e}'_n)_2,\ (\mathbf{e}'_n)_3)\begin{pmatrix} (\mathbf{e}'_m)_1 \\ (\mathbf{e}'_m)_2 \\ (\mathbf{e}'_m)_3 \end{pmatrix} = \langle \mathbf{e}'_n|\mathbf{e}'_m\rangle$$

$$R_{mn} = (\mathbf{e}'_m)_n = \mathbf{e}_n \bullet \mathbf{e}'_m = \mathbf{e}_n^{\mathrm{T}}\mathbf{e}'_m = ((\mathbf{e}_n)_1,\ (\mathbf{e}_n)_2,\ (\mathbf{e}_n)_3)\begin{pmatrix} (\mathbf{e}'_m)_1 \\ (\mathbf{e}'_m)_2 \\ (\mathbf{e}'_m)_3 \end{pmatrix} = \langle \mathbf{e}_n|\mathbf{e}'_m\rangle$$

Since our space $\mathcal{H}$ is real (not complex), we know that

We now imagine that $\mathcal{T}$ is some operator in $\mathcal{H}$ . We can associate various matrices with $\mathcal{T}$ as follows

$T_{nm} = \langle \mathbf{e}_n|\mathcal{T}|\mathbf{e}_m\rangle = \langle \mathbf{e}_n| T\mathbf{e}_m\rangle$ $\quad$ $|T\mathbf{e}_m\rangle$ is a certain vector in $\mathcal{H}$

$T'_{nm} = \langle \mathbf{e}'_n|\mathcal{T}|\mathbf{e}'_m\rangle = \langle \mathbf{e}'_n| T\mathbf{e}'_m\rangle$ $\quad$ $|T\mathbf{e}'_m\rangle$ is some other vector in $\mathcal{H}$

$T''_{nm} = \langle \mathbf{e}''_n|\mathcal{T}|\mathbf{e}''_m\rangle = \langle \mathbf{e}''_n| T\mathbf{e}''_m\rangle$ $\quad$ $|T\mathbf{e}''_m\rangle$ is still some other vector in $\mathcal{H}$

In this Dirac language, there is only one operator $\mathcal{T}$ but it is associated with three different matrices, one for each basis used to span the space $\mathcal{H}$. Formally $\mathcal{T}$ is a rank-2 tensor, which is represented by three different matrices in the three different bases. One can expand $\mathcal{T}$ on the $\mathbf{e}_n$ basis in this way

$$\mathcal{T} = T_{ij}\,|\mathbf{e}_i\rangle\langle\mathbf{e}_j| = T_{ij}\,\mathbf{e}_i\mathbf{e}_j^T \quad \Rightarrow \quad \langle \mathbf{e}_n|\mathcal{T}|\mathbf{e}_m\rangle = T_{ij}\langle \mathbf{e}_n|\mathbf{e}_i\rangle\langle e_j|\mathbf{e}_m\rangle = T_{ij}\delta_{ni}\delta_{jm} = T_{nm}$$

Notice that, whereas $\mathbf{e}_i^T\mathbf{e}_j = \delta_{ij}$ is a number, $\mathbf{e}_i\mathbf{e}_j^T = |\mathbf{e}_i\rangle\langle e_j| =$ is a 3x3 matrix which is an operator in the space $\mathcal{H}$. One can write the above line in the other bases, such as $\mathcal{T} = T'_{ij}\,|\mathbf{e}'_i\rangle\langle\mathbf{e}'_j|$.

One operator in $\mathcal{H}$ of special interest of special interest is the unity operator $\mathcal{1}$ such that $\mathcal{1}\,|\mathbf{a}\rangle = |\,1\mathbf{a}\rangle = |\mathbf{a}\rangle$ for any vector $|\mathbf{a}\rangle$ in $\mathcal{H}$. In the Dirac notation one writes

$$\mathcal{1} = |\mathbf{e}_i\rangle\langle\mathbf{e}_i| = |\mathbf{e}'_i\rangle\langle\mathbf{e}'_i| = |\mathbf{e}''_i\rangle\langle\mathbf{e}''_i|$$

Each of these is just a statement of completeness as in (1.1.9). For example,

$$\langle \mathbf{e}_n|\,\mathcal{1}\,|e_m\rangle = \langle \mathbf{e}_n|\,\mathbf{e}'_i\rangle\langle\mathbf{e}'_i|\,e_m\rangle = (\mathbf{e}'_i)_n\,(\mathbf{e}_m)'_i$$

Then for example

$$\langle \mathbf{e}_n|\,\mathcal{1}\,|e_m\rangle = \langle \mathbf{e}_n|\,\mathbf{e}'_i\rangle\langle\mathbf{e}'_i|\,e_m\rangle = (\mathbf{e}'_i)_n\,(\mathbf{e}_m)'_i = R_{in}R_{mi} = (R^TR)_{mn} = \delta_{mn}$$

**********

We have shown that the vector $[T\mathbf{e}'_n] = \mathcal{T}\,|\mathbf{e}'_n\rangle$ is associated with three different matrices in the three bases $\mathbf{e}_n$, $\mathbf{e}'_n$ and $\mathbf{e}''_n$ :

$T_{mn} = \langle \mathbf{e}_m|\mathcal{T}\,|\mathbf{e}_n\rangle = \langle \mathbf{e}_m| T\mathbf{e}_n\rangle$

$T'_{mn} = \langle \mathbf{e}'_m|\mathcal{T}\,|\mathbf{e}'_n\rangle = \langle \mathbf{e}'_m| T\mathbf{e}'_n\rangle$

$T''_{mn} = \langle \mathbf{e}''_m|\mathcal{T}\,|\mathbf{e}''_n\rangle = \langle \mathbf{e}''_m| T\mathbf{e}''_n\rangle$

> We have three matrices R, R' and R'' associated with this operator $\mathcal{R}$,

$R_{mn} = \langle \mathbf{e}_m|\mathcal{R}\,|\mathbf{e}_n\rangle = \langle \mathbf{e}_m| R\mathbf{e}_n\rangle$

$R'_{mn} = \langle \mathbf{e}'_m|\mathcal{R}\,|\mathbf{e}'_n\rangle = \langle \mathbf{e}'_m| R\mathbf{e}'_n\rangle$

$R''_{mn} = \langle e''_m | \mathcal{R} | e''_n \rangle = \langle e''_m | Re''_n \rangle$

So we have hopefully clarified the idea that one can have multiple matrices R, R' and R" associated with the same vector $|Re_n\rangle$ .

**********

Consider now the following object

$Re'_n$

This object is a certain vector whose name is $Re'_n$. Suppose we had Frames S, S' and S" each having their own bases $e_n$, $e'_n$ and $e''_n$. We would express the components of the vector $Re'_n$ in each of these three coordinate systems as follows

$[Re'_n]_i = R_{ij}(e'_n)_j$ Frame S components
$[Re'_n]'_i = R'_{ij}(e'_n)'_j$ Frame S' components
$[Re'_n]''_i = R''_{ij}(e'_n)''_j$ Frame S" components

On the right sides we see three *matrices* which we would write as R, R' and R". All three of these matrices (which in general are not all the same) are associated with the symbol R in vector $Re'_n$. This symbol R in the name $Re'_n$ is not itself a matrix, but in each basis it is *represented* by a cetain matrix.

One way to clarify this seeming ambiguity in the meaning of symbol R is to use the Dirac Notation which makes explicit the fact that there is an underlying "Hilbert Space" $\mathcal{H}$ which is spanned by any of the three sets of basis vectors shown above, and which is inhabited by "operators". We feel that this is the only "bulletproof" way to understand the meaning of the casual notation $Re'_n$ in an environment where there are multiple coordinate systems of interest (such as in discussing rotating frames of reference). WE shall now digress momentarily to discuss this Dirac notation.

**********

for some constants $A_{nm}$. Close with <

Then

$|e'_i\rangle\langle e'_i| \mathcal{R} |e'_n\rangle = |e_n\rangle$

$|e'_i\rangle R_{in}$

We now compute

$R'_{nm} = \langle e'_n | \mathcal{R} | e'_m \rangle = \langle e'_n | \mathcal{1R1} | e'_m \rangle = \langle e'_n | e_i \rangle\langle e_i | \mathcal{R} | e_j \rangle\langle e_j | e'_m \rangle$

$= R_{ni} \langle \mathbf{e}_i | \mathcal{R} | \mathbf{e}_j \rangle R_{mj} = R_{ni} R_{ij} R_{mj} = R_{ni} R_{ij} (R^{-1})_{jm} = R_{ni} (RR^{-1})_{im} = R_{ni} \delta_{im} = R_{nm}$

Thus we find that the two matrices R and R' associated with the operator $\mathcal{R}$ are the same. This is consistent with ** which says

$T' = RTR^{-1} \qquad \Rightarrow \qquad R' = RRR^{-1} = R$

Assume now ** which states

$|\mathbf{e}'_n\rangle = R_{nm} |\mathbf{e}_m\rangle$

Apply $\mathcal{R}$ to get

$\mathcal{R}|\mathbf{e}'_n\rangle = R_{nm} \mathcal{R}|\mathbf{e}_m\rangle$

Since I am trying to prove that $\mathcal{R}|\mathbf{e}'_n\rangle = |\mathbf{e}_n\rangle$ is true, I have to show that

$R_{nm} \mathcal{R}|\mathbf{e}_m\rangle = |\mathbf{e}_n\rangle$

How do I show this???

$R_{nm} \mathcal{R}|\mathbf{e}_m\rangle = R_{nm} |e_i\rangle\langle e_i|\mathcal{R}|\mathbf{e}_m\rangle = R_{nm} R_{im} |e_i\rangle = \delta_{ni}$

Now consider

$\mathcal{R} \, |\mathbf{e}'_n\rangle = |R\mathbf{e}'_n\rangle$

so then

$|R\mathbf{e}'_n\rangle = \mathcal{R} \, |\mathbf{e}'_n\rangle = |\mathbf{e}'_m\rangle\langle\mathbf{e}'_m| \, \mathcal{R} \, |\mathbf{e}'_n\rangle = |\mathbf{e}'_m\rangle R'_{mn} = |\mathbf{e}'_m\rangle R_{mn}$

Then

$\mathcal{R} \, |\mathbf{e}'_n\rangle = |R\mathbf{e}'_n\rangle$

so

$\langle \mathbf{e}_m | \, \mathcal{R} \, | \mathbf{e}_n \rangle = \langle \mathbf{e}'_m \, | R\mathbf{e}'_n \rangle$

Using 1.1.8 and ** we can write this as

$\langle \mathbf{e}'_n | \, \mathbf{e}_m \rangle = \langle \mathbf{e}_n | R\mathbf{e}_m \rangle$

want to show that $|R\mathbf{e}'_m\rangle = |\mathbf{e}_m\rangle$

want to show that $\mathcal{R} \, |\mathbf{e}'_m\rangle = |\mathbf{e}_m\rangle$

We can write this as

$$|\mathbf{e'}_n\rangle = |\mathbf{e}_m\rangle\langle\mathbf{e}_m| \mathcal{R} |\mathbf{e}_n\rangle$$

We now finally return to our discussion of the meaning of $R\mathbf{e'}_n$ back in **. It is this

$$R\mathbf{e'}_n = |R\mathbf{e}_n\rangle = \mathcal{R}|\mathbf{e}_n\rangle$$

where $\mathcal{R}$ is the rotation operator in the Hilbert Space $\mathcal{H}$.

So far, we have used R or $\mathcal{R}$ to represent an arbitrary rotation and $\mathbf{e}_n$, $\mathbf{e'}_n$ and $\mathbf{e''}_n$ were three arbitrary bases. We now consider R and $\mathcal{R}$ to define the specific relationship between $\mathbf{e}_n$ and $\mathbf{e'}_n$ such that

$$\mathbf{e}_n = R\mathbf{e'}_n \qquad \text{or} \qquad |\mathbf{e}_n\rangle = \mathcal{R}\,|\mathbf{e'}_n\rangle$$

In this situation

In this situation the matrices R and R' are the same, since according to **,

$$R' = RRR^{-1} = R$$

so

$$\langle\mathbf{e}_m|\,\mathcal{R}\,|\mathbf{e}_n\rangle = \langle\mathbf{e'}_m|\,\mathcal{R}\,|\mathbf{e'}_n\rangle$$

We now have a simple thoerem of interest:

**The Basis Rule:** This simple rule states that (proof below),

$$\mathbf{e}_n = R\mathbf{e'}_n \qquad \Leftrightarrow \qquad \mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \tag{1.1.10}$$

The right equation involves a linear combination of basis vectors. The left equation is a statement that the vector $\mathbf{e}_n$ is the same as the vector $R\mathbf{e'}_n$. In Dirac notation the Basis Rule is made clearer,

$$|\mathbf{e}_n\rangle = |R\mathbf{e'}_n\rangle = \mathcal{R}|\mathbf{e'}_n\rangle \qquad \Leftrightarrow \qquad |\mathbf{e'}_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle$$

Proof $\Rightarrow$

$$|\mathbf{e}_n\rangle = \mathcal{R}|\mathbf{e'}_n\rangle = \mathit{1}\mathcal{R}|\mathbf{e'}_n\rangle = |\mathbf{e'}_m\rangle\langle\mathbf{e'}_m|\mathcal{R}|\mathbf{e'}_n\rangle$$

$$|\mathbf{e'}_n\rangle = \mathit{1}\,|\mathbf{e'}_n\rangle = |\mathbf{e}_m\rangle\langle\mathbf{e}_m|\mathbf{e'}_n\rangle = |\mathbf{e}_m\rangle R_{nm} = R_{nm}\,|\mathbf{e}_m\rangle$$

STOP. Easy to show that all three matrices are the same, so what was the point of all this effort?? You have to have a situation where |e'n> = R |en> and |e"n> = S |e'n>

$T'_{mn} \equiv \langle \mathbf{e}'_m | \mathcal{T} | \mathbf{e}'_n \rangle$ so then $[T\mathbf{e}'_n]'_m = T'_{mn}$

$T''_{mn} \equiv \langle \mathbf{e}''_m | \mathcal{T} | \mathbf{e}''_n \rangle$ so then $[T\mathbf{e}''_n]''_m = T''_{mn}$

Although the matrices change, the symbol T says put inside $[T\mathbf{e}_n]$ because this means

Trying to get to this: $R\mathbf{e}'_n$

Here the matrix $T_{mn}$ represents the operator $\mathcal{T}$ in the $\mathbf{e}_n$ basis.

We now imagine that $\mathcal{T}$ is some operator in $\mathcal{H}$ . We can associate various matrices with $\mathcal{T}$ as follows

$T_{nm} = \langle \mathbf{e}_n | \mathcal{T} | \mathbf{e}_m \rangle = \langle \mathbf{e}_n | T\mathbf{e}_m \rangle$ $|T\mathbf{e}_m\rangle$ is a certain vector in $\mathcal{H}$

$T'_{nm} = \langle \mathbf{e}'_n | \mathcal{T} | \mathbf{e}'_m \rangle = \langle \mathbf{e}'_n | T\mathbf{e}'_m \rangle$ $|T\mathbf{e}'_m\rangle$ is some other vector in $\mathcal{H}$

$T''_{nm} = \langle \mathbf{e}''_n | \mathcal{T} | \mathbf{e}''_m \rangle = \langle \mathbf{e}''_n | T\mathbf{e}''_m \rangle$ $|T\mathbf{e}''_m\rangle$ is still some other vector in $\mathcal{H}$

In the Dirac language, there is only one operator $\mathcal{T}$ but it is associated with three different matrices, one for each basis used to span the space $\mathcal{H}$. Formally $\mathcal{T}$ is a rank-2 tensor, which is represented by three different matrices in the three different bases. One can expand $\mathcal{T}$ on the $\mathbf{e}_n$ basis in this way,

$$\mathcal{T} = T_{ij} |\mathbf{e}_i\rangle\langle\mathbf{e}_j| = T_{ij}\, \mathbf{e}_i \mathbf{e}_j^{\mathrm{T}} \quad \Rightarrow \quad \langle \mathbf{e}_n | \mathcal{T} | \mathbf{e}_m \rangle = T_{ij} \langle \mathbf{e}_n | \mathbf{e}_i \rangle\langle \mathbf{e}_j | \mathbf{e}_m \rangle = T_{ij} \delta_{ni} \delta_{jm} = T_{nm}$$

For the $\mathbf{e}'_n$ basis one would write $\mathcal{T} = T'_{ij} |\mathbf{e}'_i\rangle\langle\mathbf{e}'_j|$ .

One operator in $\mathcal{H}$ of special interest is the unity operator $\mathbf{1}$ such that $\mathbf{1} |\mathbf{a}\rangle = | 1\mathbf{a}\rangle = |\mathbf{a}\rangle$ for any vector $|\mathbf{a}\rangle$ in $\mathcal{H}$. In the Dirac notation one can write, for example in the $\mathbf{e}'_n$ basis,

$\mathbf{1} = |\mathbf{e}'_n\rangle\langle\mathbf{e}'_n|$ // implied sum on n !!

This is in fact the statement of completeness in the $\mathbf{e}'_n$ basis. If we "close" with $\langle \mathbf{e}_i|$ on the left and $|\mathbf{e}_j\rangle$ on the right, we find

$$\delta_{ij} = \langle \mathbf{e}_i | \mathbf{e}_j \rangle = \langle \mathbf{e}_i | \mathbf{1} | \mathbf{e}_j \rangle = \langle \mathbf{e}_i | \mathbf{e}'_n \rangle\langle \mathbf{e}'_n | \mathbf{e}_j \rangle = (\mathbf{e}'_n)_i (\mathbf{e}'_n)_j$$

and this replicates the completeness statement (1.1.9).

Consider now the object

$$<\mathbf{a} \mid \mathcal{T} \mid \mathbf{b}> \;=\; <\mathbf{a} \mid T\mathbf{b}>$$

where $\mathcal{T}$ is an operator in $\mathcal{H}$ . The meaning of $\mid T\mathbf{b}>$ is that it is the vector in $\mathcal{H}$ obtained by applying the operator $\mathcal{T}$ to the vector

We can return now to our discussion of the meaning

We are now going to prove a crucial "rule" which seems to have no official name:

**The Basis Rule:** This simple rule states that (proof below),

$$\mathbf{e}_n = R\mathbf{e}'_n \qquad \Leftrightarrow \qquad \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \tag{1.1.10}$$

The right equation involves a linear combination of basis vectors and its meaning us unambiguous. The left equation is a statement that the vector $\mathbf{e}_n$ is the same as the vector $[R\mathbf{e}'_n]$. In this notation $[R\mathbf{e}'_n]$ the object R is not a matrix, it is part of the name of the vector $R\mathbf{e}'_n$ . However, to find out

We can learn about this vector by taking its components in Frame S or in Frame S':

$$[R\mathbf{e}'_n]_i \;= R_{ij}(\mathbf{e}'_n)_j \qquad \text{Frame S components}$$
$$[R\mathbf{e}'_n]'_i \;= R'_{ij}(\mathbf{e}'_n)'_j \qquad \text{Frame S' components}$$

In the Frame S world, the object R in $R\mathbf{e}'_n$ is represented by a matrix

One can take components of $\mathbf{e}_n = R\mathbf{e}'_n$ to get,

$$(\mathbf{e}_n)_i = R_{ij}\,(\mathbf{e}'_n)_j \qquad \text{Frame S components}$$
$$(\mathbf{e}_n)'_i = R'_{ij}\,(\mathbf{e}'_n)'_j \qquad \text{Frame S' components} \tag{1.1.11}$$

The matrix $R_{ij}$ is a matrix in the Frame S basis, while $R'_{ij}$ is in the Frame S' basis. It turns out that these two matrices are the same R' = R (see below), but if R were not a rotation, they likely would *not* be the same, see below.

We wish to stress the fact that in $\mathbf{e}'_n = R_{nm}\,\mathbf{e}_m$ the implied sum on m creates a linear combination of basis vectors, whereas the implied sums on j in (1.1.11) are sums over the component indices of basis vectors. One must not accidentally confuse these two kinds of equations (it is easy to do).

Once the Basis Rule is proven, both equations in (1.1.10) can be inverted to give

$$\mathbf{e'}_n = R^{-1}\mathbf{e'}_n \qquad \Leftrightarrow \qquad \mathbf{e}_n = (R^{-1})_{nm}\ \mathbf{e'}_m \quad . \tag{1.1.12}$$

Each $\mathbf{e'}_n$ is thus a "back-rotated" version of $\mathbf{e}_n$ by $R^{-1}$. Later we shall discuss "normal vectors" which have the property $\mathbf{a'} = R\mathbf{a}$ where $\mathbf{a}$ is rotated into $\mathbf{a'}$ by R. Since $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$ is an exception to this normal transformation, we say it is back-rotated. More on this later.

Proof $\Rightarrow$ : Show that $\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \ \Rightarrow\ \mathbf{e}_n = R\mathbf{e'}_n$

We prove that $\mathbf{e}_n = R\mathbf{e'}_n$ by showing that both sides have the same components in Frame S:

| | | |
|---|---|---|
| $\mathbf{e}_n = R\mathbf{e'}_n$ | ? | |
| $(\mathbf{e}_n)_i = (R\mathbf{e'}_n)_i$ | ? | |
| $\delta_{n,i} = R_{ij}\,(\mathbf{e'}_n)_j$ | ? | |
| $\delta_{n,i} = R_{ij}\,R_{nj}$ | ? | // from (1.1.8) which is based on $\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$ |
| $\delta_{n,i} = R_{ij}\,R^T_{jn}$ | ? | |
| $\delta_{n,i} = (RR^T)_{in}$ | yes ! | // from (1.1.2) that $1 = RR^T$ |

Thus our proof here consists of just reversing the above steps. QED

Proof $\Leftarrow$ : Show that $\mathbf{e}_n = R\mathbf{e'}_n \ \Rightarrow\ \mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$

Each $\mathbf{e'}_n$ must be some linear combination of the $\mathbf{e}_n$ since a basis is by definition complete. So we write

$$\mathbf{e'}_n = A_{nm}\,\mathbf{e}_m \ .$$

We know that $\mathbf{e'}_n = R^{-1}\mathbf{e}_n$ so then

$$R^{-1}\mathbf{e}_n = A_{nm}\,\mathbf{e}_m \ .$$

Taking components in Frame S

$$(R^{-1}\mathbf{e}_n)_i = A_{nm}\,(\mathbf{e}_m)_i$$

$$(R^{-1})_{ij}(\mathbf{e}_n)_j = A_{nm}\,(\mathbf{e}_m)_i$$

$$R_{ji}\,\delta_{n,j} = A_{nm}\,\delta_{m,i} \qquad // \ R^{-1} = R^T$$

$$R_{ni} = A_{ni} \quad .$$

Therefore $A_{nm} = R_{nm}$ and so $\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$. QED

vvv

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Rank-2 tensors, Dirac notation and Completeness

Whereas a normal vector transforms under rotation R as $\mathbf{a}' = R\mathbf{a}$ , a rank-2 tensor transforms as

$$T' = RTR^{-1} \ . \qquad (1.1.16)$$

See e.g. Lucht *Tensor* ***. If we stick T = R into this formula, we get $R' = RRR^{-1} = R$. So for the tensor R which defines the rotation of interest, we get this simplification. For other tensors this will generally not be the case, unless the tensor T happens to commute with R, [T,R] = TR-RT = 0.

We find the Dirac Notation helpful in understanding equations like (1.1.16). In this notation, basis vectors like $\mathbf{e}_n$ are written $|\mathbf{e}_n\rangle$ and $\mathbf{e}_n^T = \langle\mathbf{e}_n|$. Matrices like T' and T above are regarded as matrices of the same abstract operator $\mathcal{T}$ (the tensor) evaluated in the two different bases $\mathbf{e}'_n$ and $\mathbf{e}_n$. In Dirac notation statements above appear as follows:

$$\mathbf{e}_n \bullet \mathbf{e}_m = \delta_{n,m} \quad (= \mathbf{e}_n^T\mathbf{e}_m) \qquad \mathbf{e}'_n \bullet \mathbf{e}'_m = \delta_{n,m} \ . \qquad \text{// orthonormality} \qquad (1.1.1)$$

$$\langle\mathbf{e}_n \mid \mathbf{e}_m\rangle = \delta_{n,m} \qquad \langle\mathbf{e}'_n \mid \mathbf{e}'_m\rangle = \delta_{n,m} \qquad (1.1.17)$$

$$(\mathbf{e}_n)_i(\mathbf{e}_n)_j = \delta_{ij} \qquad (\mathbf{e}'_n)_i(\mathbf{e}'_n)_j = \delta_{ij} \qquad \text{// completeness} \qquad (1.1.9)$$

$$|\mathbf{e}_n\rangle\langle\mathbf{e}_n| = 1 \qquad |\mathbf{e}'_n\rangle\langle\mathbf{e}'_n| = 1 \qquad (1.1.18)$$

The last line equations do seem unusual, but the "1" is the unity operator in the space in which $\mathcal{T}$ lives. To verify $|\mathbf{e}'_n\rangle\langle\mathbf{e}'_n| = 1$ one can "close" each side as follows

$$\langle\mathbf{e}_i\,|\mathbf{e}'_n\rangle\langle\mathbf{e}'_n|\,\mathbf{e}_j\rangle = \langle\mathbf{e}_i|\ 1\ |\ \mathbf{e}_j\rangle = \langle\mathbf{e}_i|\ \mathbf{e}_j\rangle$$

or

$$(\mathbf{e}'_n)_i(\mathbf{e}'_n)_j \ = \ \delta_{ij} \ .$$

Finally one can see then that

$$\langle\mathbf{e}_n \mid \mathbf{e}'_m\rangle \ = \mathbf{e}_n \bullet \mathbf{e}'_m = (\mathbf{e}'_m)_n = R_{mn} = \ \mathbf{e}'_m \bullet \mathbf{e}_n \ = \langle\mathbf{e}'_m \mid \mathbf{e}_n\rangle$$

so then

$$\langle\mathbf{e}_n \mid \mathbf{e}'_m\rangle = \langle\mathbf{e}'_m \mid \mathbf{e}_n\rangle = R_{mn} \ . \qquad (1.1.19)$$

One says that the matrix $R_{mn}$ is the "basis change matrix" between Frame S and Frame S' coordinates.

We now interpret the matrices T and T' in the following manner,

$$T_{ij} = \langle\mathbf{e}_i \mid \mathcal{T} \mid \mathbf{e}_j\rangle$$

$$T'_{ij} = \langle \mathbf{e}'_i \,|\, \mathcal{T} \,|\, \mathbf{e}'_j \rangle \,. \tag{1.1.20}$$

Matrix T is the Dirac sandwich of the rank-2 tensor $\mathcal{T}$ in the Frame S basis.
Matrix T' is the Dirac sandwich of the rank-2 tensor $\mathcal{T}$ in the Frame S' basis.
These sandwich objects are referred to as "matrix elements of the operator $\mathcal{T}$" in quantum mechanics.

Consider now,

$$(RTR^{-1})_{in} = R_{ij}\, T_{jk}\, (R^{-1})_{kn} = R_{ij} \langle \mathbf{e}_j \,|\, \mathcal{T} \,|\, \mathbf{e}_k \rangle R_{nk} = \langle \mathbf{e}'_i \,|\, \mathbf{e}_j \rangle \langle \mathbf{e}_j \,|\, \mathcal{T} \,|\, \mathbf{e}_k \rangle \langle \mathbf{e}_k \,|\, \mathbf{e}'_n \rangle$$

$$= \langle \mathbf{e}'_i \,|\, \mathbf{e}_j \rangle \langle \mathbf{e}_j \,|\, \mathcal{T} \,|\, \mathbf{e}_k \rangle \langle \mathbf{e}_k \,|\, \mathbf{e}'_n \rangle = \langle \mathbf{e}'_i \,|\, 1\, \mathcal{T}\, 1 \,|\, \mathbf{e}'_n \rangle = \langle \mathbf{e}'_i \,|\, \mathcal{T} \,|\, \mathbf{e}'_n \rangle = T'_{in}\,,$$

Thus we provide a derivation of sorts for the claim (1.1.16) that $T' = RTR^{-1}$.

In this same language we may write

$$R_{ij} = \langle \mathbf{e}_i \,|\, \mathcal{R} \,|\, \mathbf{e}_j \rangle = \langle \mathbf{e}_i \,|\, R\mathbf{e}_j \rangle = \mathbf{e}_i \bullet [R\mathbf{e}_j] \overset{(1.1.15)}{=} [R^{-1}\mathbf{e}_i] \bullet \mathbf{e}_j \overset{(1.1.12)}{=} \mathbf{e}'_i \bullet \mathbf{e}_j = \langle \mathbf{e}'_i \,|\, \mathbf{e}_j \rangle$$

$$R'_{ij} = \langle \mathbf{e}'_i \,|\, \mathcal{R} \,|\, \mathbf{e}'_j \rangle = \langle \mathbf{e}'_i \,|\, R\mathbf{e}'_j \rangle = \mathbf{e}'_i \bullet [R\mathbf{e}'_j] \overset{(1.1.10)}{=} \mathbf{e}'_i \bullet \mathbf{e}_j = \langle \mathbf{e}'_i \,|\, \mathbf{e}_j \rangle \tag{1.1.21}$$

and we recover the fact that R = R' alluded to earlier.