This is the Title PhL 1.11.15

This is finally all figured out and written up in frames doc Section 1.1. Do not edit or mess here.

Note that page numbering is turned on in this template and view is 125%,
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$\mathbf{e'}_n \bullet \mathbf{e'}_m = \delta_{n,m}$ $\mathbf{e''}_n \bullet \mathbf{e''}_m = \delta_{n,m}$. (1.1.1)

$(\mathbf{e'}_n)'_i = \delta_{i,n}$ $(\mathbf{e''}_n)''_i = \delta_{i,n}$

$\mathbf{e''}_n = S'_{nm}\,\mathbf{e'}_m$ n = 1,2,3 $S'S'^{T} = 1$. (1.1.2)

$(\mathbf{e''}_n)'_i = S'_{nm}\,(\mathbf{e'}_m)'_i = S'_{nm}\,\delta_{mi} = S'_{ni}$ Frame S' components

$(\mathbf{e''}_n)''_i = S'_{nm}\,(\mathbf{e'}_m)''_i \Rightarrow$

$$S'^{T}_{kn}\,\delta_{in} = S'^{T}_{kn}S'_{nm}\,(\mathbf{e'}_m)''_i = (S'^{T}S')_{km}\,(\mathbf{e'}_m)''_i = \delta_{km}(\mathbf{e'}_m)''_i = (\mathbf{e'}_k)''_i$$

Therefore

$$S'^{T}_{ki} = S'_{ik} = (\mathbf{e'}_k)''_i$$

So then

$(\mathbf{e''}_n)'_i = S'_{ni}$ $(\mathbf{e'}_n)'_i = \delta_{i,n}$

$(\mathbf{e'}_n)''_i = S'_{in}$ $(\mathbf{e''}_n)''_i = \delta_{n,i}$ // the new (1.1.8)

$T''_{mn} = \langle \mathbf{e''}_m | \mathcal{T} | \mathbf{e''}_n \rangle = \langle \mathbf{e''}_m | \mathbf{e'}_i \rangle\langle \mathbf{e'}_i | \mathcal{T} | \mathbf{e'}_j \rangle\langle \mathbf{e'}_j | \mathbf{e''}_n \rangle$

$= S'_{mi}\,T'_{ij}\,S'_{nj} = S'_{mi}\,T'_{ij}\,S'^{T}_{jn} = (S'T'S'^{-1})_{mn}$

so

$T'' = S'T'S'^{-1}$ // the new tensor transformation rule!

Special case

$S'' = S'S'S'^{-1} = S'$

Also

$\langle \mathbf{e'}_m | \mathcal{T} | \mathbf{e'}_n \rangle = T'_{mn}$

$\langle \mathbf{e}''_m | \mathcal{T} | \mathbf{e}''_n \rangle = T''_{mn}$

The translation rules seem to be

$R \rightarrow S'$ $\quad R' \rightarrow R''$ $\quad T \rightarrow T'$ $\quad T' \rightarrow T''$

What happens now with the Basis Theorem ????

$\langle \mathbf{e}'_i | \mathcal{S}'^{-1} | \mathbf{e}'_m \rangle = (S'^{-1})_{im}$

$\mathbf{e}''_n = S'_{nm}\, \mathbf{e}'_m \quad \Rightarrow \quad |S'\mathbf{e}''_n\rangle = |\mathbf{e}'_n\rangle$ or $\mathcal{S}' |\mathbf{e}''_n\rangle = |\mathbf{e}'_n\rangle$ or $S'\mathbf{e}''_n = \mathbf{e}'_n$

So I don't want to say $S\mathbf{e}''_n = \mathbf{e}'_n$ because that leads to inconsistent notation!

The Basis Theorem:

$\mathbf{e}''_n = S'_{nm}\, \mathbf{e}'_m \quad \Leftrightarrow \quad S\mathbf{e}''_n = \mathbf{e}'_n$ // vector notation

$|\mathbf{e}''_n\rangle = S'_{nm}\, |\mathbf{e}'_m\rangle \quad \Leftrightarrow \quad |S\mathbf{e}''_n\rangle = |\mathbf{e}'_n\rangle$ // Dirac notation

$\mathcal{S} \,|\, \mathbf{e}''_n\rangle = |\mathbf{e}'_n\rangle$

The operator is $\mathcal{S}$ and not $\mathcal{S}'$. I then say

$S'_{ij} = \langle \mathbf{e}'_i | \mathcal{S} | \mathbf{e}'_j \rangle$

$S''_{ij} = \langle \mathbf{e}''_i | \mathcal{S} | \mathbf{e}''_j \rangle$

Multiple Transformation Pitfalls

Consider two rotation transformations involving R and S, each of which has a Basis Theorem ***

$\mathbf{e}_n = R\, \mathbf{e}'_n \quad \Leftrightarrow \quad \mathbf{e}'_n = R_{nm}\, \mathbf{e}_m \quad R_{ij} = R'_{ij}$

$\mathbf{e}'_n = S\, \mathbf{e}''_n \quad \Leftrightarrow \quad \mathbf{e}''_n = S'_{nm}\, \mathbf{e}'_m \quad S'_{ij} = S''_{ij}$

On the right we are showing on the second line that the matrix S' in the Frame S' basis is the same as the matrix S" in the Frame S" basis. We can combine the two transformations to get

$\mathbf{e}_n = (RS)\, \mathbf{e}''_n$ and $\mathbf{e}''_n = S'_{nm}R_{nk}\, \mathbf{e}_k = (S'R)_{nk}\, \mathbf{e}_k = (S'R)_{nk}\, \mathbf{e}_k$ (1.1.22)

On the other hand, we can apply the Basis Rule (1.1.10) to the combined transformation RS

$\mathbf{e}_n = (RS)\mathbf{e}''_n \quad \Leftrightarrow \quad \mathbf{e}''_n = (RS)_{nm}\, \mathbf{e}_m$ (1.1.23)

We then obtain two statements which seem at odds with each other

$\mathbf{e''}_n = (S'R')_{nm}\mathbf{e}_m$ *(1.1.22)*

$\mathbf{e''}_n = (RS)_{nm}\,\mathbf{e}_m$ *(1.1.23)* (1.1.24)

For these both to be valid I have to have

$S'R = RS$ or $S' = RSR^{-1}$

but this is just the tensor rule applied not to general T but to specific S.

but this looks very bad!!! So I will need to call upon my Dirac notation to resolve this returned Paradox!

Multiple Transformation Pitfalls

Consider two rotation transformations involving R and S, each of which has a Basis Theorem ***

$|\mathbf{e}_n\rangle = \mathcal{R}\,|\mathbf{e}'_n\rangle \quad \Leftrightarrow \quad |\mathbf{e}'_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \quad R_{ij} = R'_{ij}$

$|\mathbf{e}'_n\rangle = \mathcal{S}\,|\mathbf{e}''_n\rangle \quad \Leftrightarrow \quad |\mathbf{e}''_n\rangle = S'_{nm}\,|\mathbf{e}'_m\rangle \quad S'_{ij} = S''_{ij}$

On the right we are showing on the second line that the matrix S' in the Frame S' basis is the same as the matrix S'' in the Frame S'' basis. We can combine the two transformations to get

$|\mathbf{e}_n\rangle = (\mathcal{RS})\,|\mathbf{e}''_n\rangle$ and $|\mathbf{e}''_n\rangle = S'_{nm}R_{mk}\,|\mathbf{e}_k\rangle = (S'R)_{nk}\,|\mathbf{e}_k\rangle$ (1.1.22)

On the other hand, we can apply the Basis Rule (1.1.10) to the combined transformation RS'

$|\mathbf{e}_n\rangle = (\mathcal{RS})|\mathbf{e}''_n\rangle \quad \Leftrightarrow \quad |\mathbf{e}''_n\rangle = (RS')_{nm}\,|\mathbf{e}_m\rangle$ not sure of this (1.1.23)

We then obtain two statements which seem at odds with each other

$|\mathbf{e}''_n\rangle = (S'R)_{nm}|\mathbf{e}_m\rangle$ *(1.1.22)*

$|\mathbf{e}''_n\rangle = (RS')_{nm}\,|\mathbf{e}_m\rangle$ *(1.1.23)* (1.1.24)

For these both to be valid I have to have

$S'R = RS'$ or $S' = RS'R^{-1}$

but this looks very bad!!! So I will need to call upon my Dirac notation to resolve this returned Paradox!

Let $\mathcal{RS} = \mathcal{Q}$, then 1.1.23 says

$$|\mathbf{e}_n\rangle = (\mathcal{Q})\,|\mathbf{e}''_n\rangle \qquad \text{and} \qquad |\mathbf{e}''_n\rangle = Q_{nk}\,|\mathbf{e}_k\rangle \qquad Q = Q'' \tag{1.1.22}$$

Go calculate the matrix Q_{nk} !

$$Q_{nk} = \langle \mathbf{e}_n|\, \mathcal{Q} \,|\, \mathbf{e}_k\rangle = \langle \mathbf{e}_n|\, \mathcal{RS} \,|\, \mathbf{e}_k\rangle = \langle \mathbf{e}_n|\, \mathcal{R}\, |e_i\rangle\langle e_i|\mathcal{S} \,|\, \mathbf{e}_k\rangle$$

$$= R_{ni}\, S_{ik} = (RS)_{nk}$$

so my 1.1.23 is then wrong! The corrected version would have to be

$$|\mathbf{e}_n\rangle = (\mathcal{RS})|\mathbf{e}''_n\rangle \quad \Leftrightarrow \qquad |\mathbf{e}''_n\rangle = (RS)_{nm}\,|\mathbf{e}_m\rangle \tag{1.1.23}$$

THEN the comparison is this

$$|\mathbf{e}''_n\rangle = (S'R)_{nk}\,|\mathbf{e}_k\rangle$$

$$|\mathbf{e}''_n\rangle = (RS)_{nm}\,|\mathbf{e}_m\rangle$$

and this requires that

$$S'R = RS \quad \text{or} \qquad S' = RSR^{-1} \text{ and this is TRUE so we are rescued!~!!!}$$

These would be the same if we knew that S'R' = RS'. Since R' = R, this says S'R = RS' or $S' = RSR^{-1}$. But this is just the way in which the two tensors S' and S are related relative to the R transformation, and that is why both statements in (1.1.24) are valid. In an application, we are likely to know the matrix S, and to not know the matrix S', so the second form in (1.1.24) is more useful. In our alternate notation this second form would read

$$\begin{pmatrix} \mathbf{e}''_1 \\ \mathbf{e}''_2 \\ \mathbf{e}''_3 \end{pmatrix} = RS \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{1.1.25}$$

Similar comments apply to the triple Euler angle transformations encountered in Appendix G.

next day 2.22.17.

Let's now go for a triple transformation as encountered in my Euler stuff.

Multiple Transformation Pitfalls

Consider two rotation transformations involving R and S, each of which has a Basis Theorem ***

$|\mathbf{e}_n\rangle = \mathcal{R}\,|\mathbf{e}'_n\rangle \quad \Leftrightarrow \quad |\mathbf{e}'_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \quad R_{ij} = R'_{ij}$

$|\mathbf{e}'_n\rangle = \mathcal{S}\,|\mathbf{e}''_n\rangle \quad \Leftrightarrow \quad |\mathbf{e}''_n\rangle = S'_{nm}\,|\mathbf{e}'_m\rangle \quad S'_{ij} = S''_{ij}$

$|\mathbf{e}''_n\rangle = \mathcal{T}\,|\mathbf{e}'''_n\rangle \quad \Leftrightarrow \quad |\mathbf{e}'''_n\rangle = T''_{nm}\,|\mathbf{e}''_m \quad T''_{ij} = T'''_{ij}$

On the right we are showing on the second line that the matrix S' in the Frame S' basis is the same as the matrix S" in the Frame S" basis. We can combine the three transformations to get

$|\mathbf{e}_n\rangle = (\mathcal{RST})\,|\mathbf{e}''_n\rangle$ and $\quad |\mathbf{e}''_k\rangle = T''_{kn}S'_{nm}R_{mk}\,|\mathbf{e}_k\rangle = (T''S'R)_{nk}\,|\mathbf{e}_k\rangle$ (1.1.22)

On the other hand, we can apply the Basis Rule (1.1.10) to the combined transformation $\mathcal{RST}$

$|\mathbf{e}_n\rangle = (\mathcal{RST})|\mathbf{e}''_n\rangle \Leftrightarrow \quad |\mathbf{e}''_n\rangle = (RS')_{nm}\,|\mathbf{e}_m\rangle$ not sure of this (1.1.23)

We then obtain two statements which seem at odds with each other

$|\mathbf{e}''_n\rangle = (S'R)_{nm}|\mathbf{e}_m\rangle$ *(1.1.22)*

$|\mathbf{e}''_n\rangle = (RS')_{nm}\,|\mathbf{e}_m\rangle$ *(1.1.23)* (1.1.24)

For these both to be valid I have to have

$S'R = RS' \quad$ or $\quad S' = RS'R^{-1}$

Let $\mathcal{RST} = \mathcal{Q}$, then 1.1.23 says

$|\mathbf{e}_n\rangle = (\mathcal{Q})\,|\mathbf{e}''_n\rangle$ and $\quad |\mathbf{e}''_n\rangle = Q_{nk}\,|\mathbf{e}_k\rangle \quad Q = Q''$ (1.1.22)

Go calculate the matrix Q_{nk} !

$Q_{nk} = \langle\mathbf{e}_n|\,\mathcal{Q}\,|\,\mathbf{e}_k\rangle = \langle\mathbf{e}_n|\,\mathcal{RST}\,|\,\mathbf{e}_k\rangle = \langle\mathbf{e}_n|\,\mathcal{R}\,|e_i\rangle\langle e_i|\mathcal{S}\,|\,\mathbf{e}_j\rangle\langle e_j|\mathcal{T}\,|\,\mathbf{e}_k\rangle$

$= R_{ni}\,S_{ij}T_{jk} = (RST)_{nk}$

so my 1.1.23 is then wrong! The corrected version would have to be

$|\mathbf{e}_n\rangle = (\mathcal{RST})|\mathbf{e}''_n\rangle \Leftrightarrow \quad |\mathbf{e}''_n\rangle = (RST)_{nm}\,|\mathbf{e}_m\rangle$ (1.1.23)

THEN the comparison is this

$|\mathbf{e}'''_n\rangle = (T''S'R)_{nk}\,|\mathbf{e}_k\rangle$

$|\mathbf{e}'''_n\rangle = (RST)_{nm}\,|\mathbf{e}_m\rangle$

and this requires that

$$RST = T''S'R$$

Is this true?

I know this is true

$$S' = RSR^{-1} \qquad X' = RXR^{-1}$$

But that does not help much here. This is the tensor rule associated with ' to ". Each level probably has its own rule. I would guess that

$$\langle e'_n|e_i\rangle = R_{ni}$$

$$\langle e''_n|e'_i\rangle = S'_{nk}$$

$$\langle e'''_n|e''_i\rangle = T''_{nk} \qquad \text{correct, see below}$$

How again do I derive these? Each level has a Basis Theorem I think. I have already written them above. So Just close each one like so:

$$|\mathbf{e}'_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \quad \Rightarrow \quad \langle e_i|\mathbf{e}'_n\rangle = R_{nm}\langle e_i|\mathbf{e}_m\rangle = R_{ni} \qquad \text{check}$$

$$|\mathbf{e}''_n\rangle = S'_{nm}\,|\mathbf{e}'_m\rangle \quad \Rightarrow \quad \langle e'_i|\mathbf{e}''_n\rangle = S'_{nm}\langle e'_i|\mathbf{e}'_m\rangle = S'_{ni} \qquad \text{check}$$

$$|\mathbf{e}'''_n\rangle = T''_{nm}\,|\mathbf{e}''_m\rangle \Rightarrow \quad \langle e''_i|\mathbf{e}'''_n\rangle = T''_{nm}\langle e''_i|\mathbf{e}''_m\rangle = T''_{ni} \qquad \text{check}$$

So derive the tensor transform rules

$$\langle e'_i|\mathcal{X}|e'_j\rangle = \langle e'_i|e_n\rangle\langle e_n|\mathcal{X}|e_m\rangle\langle e_m|e'_j\rangle = R_{in}X_{nm}R^{T}_{mj} = (RXR^{T})_{ij}$$

$$\langle e''_i|\mathcal{X}|e''_j\rangle = \langle e''_i|e'_n\rangle\langle e'_n|\mathcal{X}|e'_m\rangle\langle e'_m|e''_j\rangle = S'_{in}X'_{nm}S'^{T}_{mj} = (S'X'S'^{T})_{ij}$$

$$\langle e''_i|\mathcal{X}|e'''_j\rangle = \langle e'''_i|e''_n\rangle\langle e''_n|\mathcal{X}|e''_m\rangle\langle e''_m|e'''_j\rangle = T''_{in}X''_{nm}T''^{T}_{mj} = (T''X''T''^{T})_{ij}$$

So our three rules are

$$X' = RXR^{T}$$

$$X'' = S'X'S'^{T}$$

$$X''' = T''X''T''^{T}$$

Good. Now try to verify that

RST = T''S'R

rhs = T''S'R = (S'T'S'^{T})S'R = S'T'R = (RSR^{T})(RTR^{T})R = R S T yes !!!!

Now how does this play in my Euler section?

top $(\hat{\xi},\hat{\eta},\hat{\zeta}) = R_z(\varphi)\,(\hat{x},\hat{y},\hat{z})$ where $\hat{\zeta}=\hat{z}$

left $(\hat{\xi}',\hat{\eta}',\hat{\zeta}') = R_\xi(\theta)\,(\hat{\xi},\hat{\eta},\hat{\zeta})$ where $\hat{\xi}'=\hat{\xi}$

right $(\hat{x}',\hat{y}',\hat{z}') = R_{\zeta'}(\psi)(\hat{\xi}',\hat{\eta}',\hat{\zeta}')$ where $\hat{z}'=\hat{\zeta}'$ (G.5.6)

What do I mean by these italicized rotations? They are in operator notation just temp here. They are different matrices depending on which basis you work in. If all in Frame S basis, then can replace with non-italic matrices. I will present them as non-italic but make that statement. Same here

$(\hat{\xi},\hat{\eta},\hat{\zeta}) = R_z(\varphi)\,(\hat{x},\hat{y},\hat{z})$ // each equation is like $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$

$(\hat{\xi}',\hat{\eta}',\hat{\zeta}') = R_\xi(\theta)\,(\hat{\xi},\hat{\eta},\hat{\zeta}) = R_\xi(\theta)\,R_z(\varphi)\,(\hat{x},\hat{y},\hat{z})$

$(\hat{x}',\hat{y}',\hat{z}') = R_{\zeta'}(\psi)\,(\hat{\xi}',\hat{\eta}',\hat{\zeta}') = R_{\zeta'}(\psi)\,R_\xi(\theta)\,(\hat{\xi},\hat{\eta},\hat{\zeta}) = R_{\zeta'}(\psi)\,R_\xi(\theta)R_z(\varphi)\,(\hat{x},\hat{y},\hat{z})$ (G.5.7)

Now the basis rule is

$\mathbf{e}'_n = A\mathbf{e}_n \quad \Leftrightarrow \quad \mathbf{e}'_n = (A^{-1})_{nm}\,\mathbf{e}_m$ or $\begin{pmatrix}\mathbf{e}'_1\\ \mathbf{e}'_2\\ \mathbf{e}'_3\end{pmatrix} = [A]^{-1}\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}$. (G.5.8)

This then leads to my desired results where all matrices are Frame S.