I think this should replace the existing Sections 1.1,2,3 where I am everywhere consistent with how rank-2 tensors transform. But hold off just a little while before installing. // OK it is installed

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1. Notation, important role of the Prime Symbol, and other Preliminaries

1.1 Basis vectors $\mathbf{e}_n$, $\mathbf{e'}_n$, rotation R, Dirac notation, the Basis Theorem, and Concatenation

Unless otherwise specified, repeated indices have implicit sums. For example $(\mathbf{e'}_n)_i \mathbf{e}_i$ means $\Sigma_i (\mathbf{e'}_n)_i \mathbf{e}_i$. This is known as the Einstein convention. We write δ_{ij} in place of the usual $\delta_{i,j}$. Both these conventions are efforts to reduce symbol clutter.

We have in mind that we are operating in Euclidian space E^3, but most everything in this section is generally valid in E^N.

Let Frame S have orthonormal basis vectors $\mathbf{e}_n$.
Let Frame S' have orthonormal basis vectors $\mathbf{e'}_n$. Thus,

$$\mathbf{e}_n \bullet \mathbf{e}_m = \delta_{nm} \qquad \mathbf{e'}_n \bullet \mathbf{e'}_m = \delta_{nm} \quad . \tag{1.1.1}$$

Assume that the two basis vector sets are related in this manner:

$$\mathbf{e'}_n = R_{nm}\, \mathbf{e}_m \qquad n = 1,2,3 \qquad RR^T = 1 \quad . \tag{1.1.2}$$

This says that each basis vector of Frame S' is a certain *linear combination* of Frame S basis vectors. We shall assume that the matrix R of coefficients is real orthogonal ($R^{-1} = R^T$ or $RR^T = R^T R = 1$). Since $RR^T = 1$, we know that det(R) = ± 1. Real orthogonal matrices with det(R) = - 1 are combinations of regular rotations with a reflection, whereas for regular rotations one has det(R) = +1.

Notice that the n on $\mathbf{e}_n$ and $\mathbf{e'}_n$ is a label and not a component index.

Using a notation described more in the Section 1.2, we expand each basis vector onto both bases:

expansions	projections
$\mathbf{e'}_n = (\mathbf{e}_i \bullet \mathbf{e'}_n)\, \mathbf{e}_i = (\mathbf{e'}_n)_i\, \mathbf{e}_i$	$(\mathbf{e'}_n)_i = (\mathbf{e}_i \bullet \mathbf{e'}_n)$
$\mathbf{e'}_n = (\mathbf{e'}_i \bullet \mathbf{e'}_n)\, \mathbf{e'}_i = (\mathbf{e'}_n)'_i\, \mathbf{e'}_i$	$(\mathbf{e'}_n)'_i = (\mathbf{e'}_i \bullet \mathbf{e'}_n)$
$\mathbf{e}_n = (\mathbf{e}_i \bullet \mathbf{e}_n)\, \mathbf{e}_i = (\mathbf{e}_n)_i\, \mathbf{e}_i$	$(\mathbf{e}_n)_i = (\mathbf{e}_i \bullet \mathbf{e}_n)$
$\mathbf{e}_n = (\mathbf{e'}_i \bullet \mathbf{e}_n)\, \mathbf{e'}_i = (\mathbf{e}_n)'_i\, \mathbf{e'}_i$	$(\mathbf{e}_n)'_i = (\mathbf{e'}_i \bullet \mathbf{e}_n)$.

(1.1.3)

Lines 2 and 3 are not very interesting because they just say what we already know,

$$\mathbf{e'}_n = \delta_{in}\ \mathbf{e'}_i = (\mathbf{e'}_n)'_i\ \mathbf{e'}_i \qquad (\mathbf{e'}_n)'_i = \delta_{in} \quad \text{projection}$$

$$\mathbf{e}_n = \delta_{in}\ \mathbf{e}_i = (\mathbf{e}_n)_i\ \mathbf{e}_i \qquad (\mathbf{e}_n)_i = \delta_{in} \quad \text{projection} . \qquad (1.1.4)$$

Now dot (1.1.2) [$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m$] first with $\mathbf{e}_i$ and then with $\mathbf{e'}_i$ and use the projections in (1.1.3) to get

$$(\mathbf{e'}_n)_i = R_{nm}\,(\mathbf{e}_m)_i \qquad \text{Frame S components}$$

$$(\mathbf{e'}_n)'_i = R_{nm}\,(\mathbf{e}_m)'_i \qquad \text{Frame S' components} . \qquad (1.1.5)$$

Using (1.1.1) write the first equation as

$$(\mathbf{e'}_n)_i = R_{nm}\,\delta_{mi} = R_{ni} . \qquad (1.1.6)$$

For the second equation, one has

$$\delta_{ni} = R_{nm}\,(\mathbf{e}_m)'_i \ \Rightarrow\ R^T_{kn}\delta_{ni} = R^T_{kn}R_{nm}\,(\mathbf{e}_m)'_i$$
$$\Rightarrow\ R^T_{ki} = (R^TR)_{km}\,(\mathbf{e}_m)'_i = \delta_{km}(\mathbf{e}_m)'_i = (\mathbf{e}_k)'_i$$

and therefore

$$(\mathbf{e}_k)'_i = R^T_{ki} = R_{ik} \qquad \Rightarrow \qquad (\mathbf{e}_n)'_i = R_{in} . \qquad (1.1.7)$$

So now we know all about the components of the basis vectors in each Frame :

$$(\mathbf{e'}_n)_i = R_{ni} \qquad (\mathbf{e}_n)_i = \delta_{ni} \qquad \text{Frame S components}$$

$$(\mathbf{e}_n)'_i = R_{in} \qquad (\mathbf{e'}_n)'_i = \delta_{ni} \qquad \text{Frame S' components} \qquad (1.1.8)$$

Equation (1.1.1) says that either set of basis vectors is **orthonormal**. It is also true that each set of basis vectors is **complete**. In Frame S this means that *any* vector **a** can be expanded as $\mathbf{a} = a_n\mathbf{e}_n$. As in (1.1.3) one can then write

$$\mathbf{a} = a_n\mathbf{e}_n \quad \text{where } a_n = (\mathbf{e}_n \bullet \mathbf{a}) \qquad \Rightarrow \qquad \mathbf{a} = (\mathbf{e}_n \bullet \mathbf{a})\mathbf{e}_n \quad .$$

Writing this last equation out in Frame S components, one gets

$$a_j = (\,(\mathbf{e}_n)_i a_i\,)(\mathbf{e}_n)_j \qquad \text{or} \qquad a_j = [(\mathbf{e}_n)_i(\mathbf{e}_n)_j]\,a_i \quad .$$

In order that this last equation be valid for *any* **a**, it must be true that (implicit sum on n) ,

$$(\mathbf{e}_n)_i(\mathbf{e}_n)_j = \delta_{ij} . \qquad \text{// completeness of the } \mathbf{e}_n \qquad (1.1.9a)$$

This is the formal statement that the $\mathbf{e}_n$ are complete. The equation is obvious since with (1.1.1) it just says $\delta_{ni}\delta_{nj} = \delta_{ij}$. Starting instead with $\mathbf{a} = (\mathbf{e'}_n \bullet \mathbf{a})\mathbf{e'}_n$ one finds $a_i = (\,(\mathbf{e'}_n)_i a_i\,)(\mathbf{e'}_n)_j$ and concludes that,

$$(\mathbf{e'}_n)_i(\mathbf{e'}_n)_j = \delta_{ij} \qquad \text{// completeness of the } \mathbf{e'}_n \qquad (1.1.9b)$$

From (1.1.8) this says $R_{ni}R_{jn} = \delta_{ij}$ which we know is true since $RR^T = 1$. Thus the completeness statements are "nothing new". As we shall see in the Dirac world, completeness is very useful tool.

The Notation Problem

We shall soon be pondering equations of the following form,

$$\mathbf{a} = T\mathbf{b} \ .$$

If we had only one basis $\mathbf{e}_n$ to worry about, we would simply state that T was a matrix and the meaning of the equation $\mathbf{a} = T\mathbf{b}$ is $a_i = T_{ij}b_j$ where a_i and b_i are Frame S components of **a** and **b**. However, when there are multiple bases involved (such as in dealing with "rotating frames of reference"), the meaning of the above equation is not so clear, especially when **a** and **b** are basis vectors in different bases. As we shall see, the existence of multiple bases implies the existence of tensors which are defined in terms of the transformation between those bases.

We have found (after a lifetime of pain regarding this subject) that the so-called Dirac notation described below always provides a clean, efficient and unambiguous meaning for expressions of the above type. In a sense, it is the Gold Standard, although we generally use simpler vector notations that have more ambiguity. Whenever an equation's meaning seems unclear, one should ask what that equation looks like in Dirac notation. For this reason, we ask the reader to absorb the following Dirac Notation digression.

The Dirac notation was invented for use in quantum mechanics by Paul Dirac (1947). It appears in most quantum mechanics texts (including Saxon, Schiff, Messiah and Shankar). Various Hilbert Spaces are associated with the notation in quantum mechanics applications (spin space, configuration space, momentum space, etc.) but we will only be concerned about the Hilbert Space E^3 whose operators can always be represented by 3x3 real matrices in any given basis.

The Dirac notation does not add any new math or physics, it just makes things clearer. For example, we will say things like the following,

$$(T)'_{ij} = \langle \mathbf{e'}_i | \mathcal{T} | \mathbf{e'}_j \rangle = (\mathbf{e'}_i)^T T (\mathbf{e'}_j) = (* \ * \ *)\begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}\begin{pmatrix} * \\ * \\ * \end{pmatrix}$$

$$= (\mathbf{e'}_i)^T{}_n T_{nm} (\mathbf{e'}_j)_m = R_{in}T_{nm}R_{jm} = (RTR^T)_{mn}$$

In $\langle \mathbf{e'}_i | \mathcal{T} | \mathbf{e'}_j \rangle$ we imagine the existence of an operator $\mathcal{T}$ whose matrix in the $\mathbf{e'}_n$ basis is $(T)'_{ij}$. As the vector notation on the right shows, this can all be done with normal vector/matrix notation and no "operator" is needed.

Various other notations have appeared in the literature from time to time to express the above idea. For example,

$$\langle \mathbf{e'}_i | \mathscr{T} | \mathbf{e'}_j \rangle = (\mathbf{e'}_i)^{\mathrm{T}}_n T_{nm} (\mathbf{e'}_j)_m = \mathbf{e'}_i \bullet (T\mathbf{e'}_j) = \mathbf{e'}_i \bullet T \bullet \mathbf{e'}_j = \mathbf{e'}_i \overleftrightarrow{T} \mathbf{e'}_j .$$

Often the Dirac notation is made even more compact by writing

$$\langle \mathbf{e'}_i | \mathscr{T} | \mathbf{e'}_j \rangle = \langle i' | \mathscr{T} | j' \rangle \qquad \text{or} \qquad \mathbf{1} = |\mathbf{e'}_j\rangle\langle \mathbf{e'}_j| = |j'\rangle\langle j'| \quad \text{(completeness)}$$

where only the minimal necessary information is displayed. We shall not take things this far.

Dirac Notation

In this notation, one writes

$$\mathbf{a} = |\mathbf{a}\rangle = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \text{"vector"} \qquad \text{// known as a "ket"}$$

$$\mathbf{a}^{\mathrm{T}} = \langle \mathbf{a}| = (a_1, a_2, a_3) = \text{"transpose vector"} \qquad \text{// known as a "bra" .} \tag{1.1.10}$$

In formal language $|\mathbf{a}\rangle$ is a vector in the space $\mathscr{H}$ while $\langle \mathbf{a}|$ is a corresponding vector in the "dual space" $\mathscr{H}^*$ (sometimes $\langle \mathbf{a}|$ called a covector). Notice how the dot product (scalar product, inner product) works in the following example,

$$\mathbf{a} \bullet \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b} = (a_1, a_2, a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \langle \mathbf{a}|\mathbf{b}\rangle = a_1b_1 + a_2b_2 + a_3b_3 = \text{a number} \quad . \tag{1.1.11}$$

On the other hand, one writes

$$|\mathbf{a}\rangle\langle\mathbf{b}| = \mathbf{a}\mathbf{b}^{\mathrm{T}} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} (b_1, b_2, b_3) = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} = \text{a 3x3 matrix} \tag{1.1.12}$$

where the vector components are implicitly in the $\mathbf{e}_n$ basis.

Comments:

1. Sometimes $\mathbf{a}\mathbf{b}^{\mathrm{T}}$ is written $\mathbf{a}\mathbf{b}$ and is called a "dyadic product" or a "dyad". Since $[\mathbf{a}\mathbf{b}]_{ij} = a_i b_j$, as (1.1.12) shows, the dyad is really just the "outer product" of two vectors, so $\mathbf{a}\mathbf{b}^{\mathrm{T}} = \mathbf{a}\mathbf{b} = \mathbf{a}\otimes\mathbf{b} = |\mathbf{a}\rangle\langle\mathbf{b}|$ in four different notations!

2. A Hilbert Space is basically a vector space with an inner product $\mathbf{a} \bullet \mathbf{b} = \langle \mathbf{a}|\mathbf{b}\rangle$. Using $|\mathbf{a}| = \sqrt{\mathbf{a} \bullet \mathbf{a}}$ and then $d(\mathbf{a},\mathbf{b}) = |\mathbf{a}-\mathbf{b}|$ this space has an implicit "natural" norm and metric.

Since our space $\mathscr{H}$ is real (not complex), we know that

$$\langle \mathbf{a}|\mathbf{b}\rangle = \langle \mathbf{b}|\mathbf{a}\rangle \quad \text{or} \quad \mathbf{a} \bullet \mathbf{b} = \mathbf{b} \bullet \mathbf{a} \; . \tag{1.1.13}$$

We can of course let **a** and **b** be any of the basis vectors $\mathbf{e}_n$, $\mathbf{e'}_n$. For example, using (1.1.8),

$$\delta_{nm} = (\mathbf{e}_m)_n = \mathbf{e}_n \bullet \mathbf{e}_m = \mathbf{e}_n{}^T \mathbf{e}_m = \langle \mathbf{e}_n|\mathbf{e}_m\rangle = \langle \mathbf{e}_m|\mathbf{e}_n\rangle$$

$$R_{mn} = (\mathbf{e'}_m)_n = \mathbf{e}_n \bullet \mathbf{e'}_m = \mathbf{e}_n{}^T \mathbf{e'}_m = \langle \mathbf{e}_n|\mathbf{e'}_m\rangle = \langle \mathbf{e'}_m|\mathbf{e}_n\rangle \; . \tag{1.1.14}$$

We now imagine that $\mathcal{T}$ is some **operator** in $\mathcal{H}$, and $|\mathbf{a}\rangle$ is some vector in $\mathcal{H}$. We write,

$$\mathcal{T}\,|\mathbf{a}\rangle = |T\mathbf{a}\rangle \quad \text{where } |T\mathbf{a}\rangle \text{ is some new vector in } \mathcal{H}. \tag{1.1.15}$$

In particular, we can write for the basis vectors $\mathbf{e}_n$

$$\mathcal{T}\,|\mathbf{e}_n\rangle = |T\mathbf{e}_n\rangle \; . \tag{1.1.16}$$

The definition of $|T\mathbf{e}_n\rangle$ is that it is what you get by applying operator $\mathcal{T}$ to the vector $|\mathbf{e}_n\rangle$.

If we want to know the Frame S components of the vector $|T\mathbf{e}_n\rangle$, we calculate them :

$$\langle \mathbf{e}_m|\,\mathcal{T}\,|\mathbf{e}_n\rangle = \langle \mathbf{e}_m|\,T\mathbf{e}_n\rangle = [T\mathbf{e}_n]_m \; . \tag{1.1.17}$$

We then define the *matrix* T to be

$$T_{mn} \equiv \langle \mathbf{e}_m|\,\mathcal{T}\,|\mathbf{e}_n\rangle \quad \text{so then} \quad [T\mathbf{e}_n]_m = T_{mn} \; . \tag{1.1.18}$$

Repeating the above in Frame S' gives,

$$(T)'_{mn} \equiv \langle \mathbf{e'}_m|\,\mathcal{T}\,|\mathbf{e'}_n\rangle = \langle \mathbf{e'}_m|T\mathbf{e'}_n\rangle = [T\mathbf{e'}_n]'_m \tag{1.1.19}$$

where we have now found the Frame S' components of $T\mathbf{e'}_n = \mathcal{T}\,|\mathbf{e'}_n\rangle$.

Notice the distinction between the matrices T and (T)', and the symbol T in the vectors $[T\mathbf{e}_n]$ and $[T\mathbf{e'}_n]$. It is the same symbol T because these vectors are $\mathcal{T}\,|\mathbf{e}_n\rangle$ and $\mathcal{T}\,|\mathbf{e'}_n\rangle$ with the same operator $\mathcal{T}$. The symbol T in $[T\mathbf{e'}_n]$ is not itself a matrix, it is part of the name of the vector $[T\mathbf{e'}_n]$.

One operator in $\mathcal{H}$ of special interest is the unity operator $\mathcal{1}$ such that $\mathcal{1}\,|\mathbf{a}\rangle = |\,1\mathbf{a}\rangle = |\mathbf{a}\rangle$ for any vector $|\mathbf{a}\rangle$ in $\mathcal{H}$. In the Dirac notation one can write, for example in the $\mathbf{e'}_n$ basis,

$$\mathcal{1} = |\mathbf{e'}_n\rangle\langle \mathbf{e'}_n| \; . \qquad \text{// implied sum on n !!}$$

This is in fact the statement of completeness in the $\mathbf{e'}_n$ basis. "Closing" with $\langle \mathbf{e}_i|$ on the left and $|\mathbf{e}_j\rangle$ on the right, one gets

$$\delta_{ij} = \langle \mathbf{e}_i | \mathbf{e}_j \rangle = \langle \mathbf{e}_i | \boldsymbol{1} | \mathbf{e}_j \rangle = \langle \mathbf{e}_i | \mathbf{e}'_n \rangle \langle \mathbf{e}'_n | \mathbf{e}_j \rangle = (\mathbf{e}'_n)_i (\mathbf{e}'_n)_j$$

and this replicates the completeness statement (1.1.9b). Completeness is valid in any basis, so

$$\boldsymbol{1} = |\mathbf{e}_i\rangle\langle\mathbf{e}_i| = |\mathbf{e}'_i\rangle\langle\mathbf{e}'_i| = |\mathbf{e}''_i\rangle\langle\mathbf{e}''_i| \qquad \text{completeness in Frames S, S' and S''} \ . \tag{1.1.20}$$

One might wonder how the matrices T and T' are related. Consider,

$$(T)'_{mn} \equiv \langle \mathbf{e}'_m | \mathcal{T} | \mathbf{e}'_n \rangle = \langle \mathbf{e}'_m | \boldsymbol{1}\mathcal{T}\boldsymbol{1} | \mathbf{e}'_n \rangle = \langle \mathbf{e}'_m | \mathbf{e}_i \rangle \langle \mathbf{e}_i | \mathcal{T} | \mathbf{e}_j \rangle \langle \mathbf{e}_j | \mathbf{e}'_n \rangle$$

$$= R_{mi}\, T_{ij}\, R_{nj} = R_{mi}\, T_{ij}\, (R^{-1})_{jn} = [R\, T\, R^{-1}]_{mn}$$

Thus the relationship is

$$(T)' = RTR^{-1} \ . \tag{1.1.21}$$

This is in fact the (passive) transformation rule for the rank-2 tensor T, analogous to the passive transformation rule for a rank-1 tensor which is $(\mathbf{V})' = R\mathbf{V}$ (see Comment below). The actual tensor is the operator $\mathcal{T}$ while T is the matrix which represents $\mathcal{T}$ in the $\mathbf{e}_n$ basis. This tensor $\mathcal{T}$ can be expanded on the various bases in this manner

$$\mathcal{T} = T_{ij} |\mathbf{e}_i\rangle\langle\mathbf{e}_j| = (T)'_{ij} |\mathbf{e}'_i\rangle\langle\mathbf{e}'_j| = (T)''_{ij} |\mathbf{e}''_i\rangle\langle\mathbf{e}''_j| \ . \quad \text{// implied sum on i and j} \tag{1.1.22}$$

To verify that this is true, we can close for example with $\langle \mathbf{e}'_m |$ and $|\mathbf{e}'_n\rangle$ to get

$$\langle \mathbf{e}'_m | \mathcal{T} | \mathbf{e}'_n \rangle = (T)'_{ij} \langle \mathbf{e}'_m | \mathbf{e}'_i \rangle \langle \mathbf{e}'_j | \mathbf{e}'_n \rangle = (T)'_{ij}\, \delta_{mi}\delta_{jm} = (T)'_{mn}$$

Similarly,

$$\langle \mathbf{e}'_m | \mathcal{T} | \mathbf{e}'_n \rangle = T_{ij} \langle \mathbf{e}'_m | \mathbf{e}_i \rangle \langle \mathbf{e}_j | \mathbf{e}'_n \rangle = T_{ij}\, R_{mi} R_{nj} = R_{mi} T_{ij} (R^{-1})_{jn} = [RTR^{-1}]_{mn} = T'_{mn}$$

and

$$\langle \mathbf{e}_m | \mathcal{T} | \mathbf{e}_n \rangle = (T)'_{ij} \langle \mathbf{e}_m | \mathbf{e}'_i \rangle \langle \mathbf{e}'_j | \mathbf{e}_n \rangle = (T)'_{ij}\, R_{im} R_{jn} = (R^{-1})_{mi} \ \ (T)'_{ij}\, R_{jn} = [R^{-1}T'R]_{mn} = T_{mn}$$

If $\mathcal{T} = \boldsymbol{1}$, then (1.1.22) replicates the statement of completeness,

$$\boldsymbol{1} = (1)_{ij} |\mathbf{e}_i\rangle\langle\mathbf{e}_j| = \delta_{ij} |\mathbf{e}_i\rangle\langle\mathbf{e}_j| = |\mathbf{e}_i\rangle\langle\mathbf{e}_i| \ .$$

Sometimes one writes $\mathcal{T} = |\mathbf{e}_i\rangle T_{ij} \langle\mathbf{e}_j|$ so then $\boldsymbol{1} = |\mathbf{e}_i\rangle\delta_{ij}\langle\mathbf{e}_j| = |\mathbf{e}_i\rangle\langle\mathbf{e}_i|$.

Comment: The reader might be more used to the notations $T' = RTR^{-1}$ and $\mathbf{V}' = R\mathbf{V}$ to describe the transformations of rank-2 and rank-1 tensors under transformations. These two notations are appropriate if one takes the Active View of transformations where there is only one frame (Frame S) and then $\mathbf{V}'$ and $\mathbf{V}$ are different vectors in Frame S and similarly T and T' are different tensors in Frame S. Because we are dealing with two frames, Frame S and Frame S', each having different basis vectors $\mathbf{e}_n$ and $\mathbf{e}'_n$, we shall

find it more convenient (and in fact essential) that we work in the Passive View of transformations, and in this view there are no objects **V**' and T', there are only **V** and T viewed from two coordinate systems. The transformation rules are then $(T)' = RTR^{-1}$ and $(\mathbf{V})' = R\mathbf{V}$ which are shorthand statements for the component equations $(T)'_{ij} = R_{ia}T_{ab}R^{-1}_{bj}$ and $(V)'_i = R_{ij}V_j$. This subject is discussed in Section 1.3 below.

The Basis Theorem

Recall now our assumed linear combination sum (1.1.2) which states

$$\mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \qquad \text{or} \qquad |\mathbf{e}'_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \ . \tag{1.1.23}$$

One regards the matrix R_{nm} as the representation of an operator $\boldsymbol{\mathcal{R}}$ in the $\mathbf{e}_n$ basis, as in (1.1.18) for T, so

$$R_{nm} = \langle \mathbf{e}_n \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e}_m\rangle \ . \qquad \textit{(1.1.18)} \tag{1.1.24}$$

Note that

$$\delta_{nm} = \langle \mathbf{e}_n | \mathbf{e}_m\rangle = \langle \mathbf{e}_n \,|\, \boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{R}}^{-1} \,|\, \mathbf{e}_m\rangle = \langle \mathbf{e}_n \,|\, \boldsymbol{\mathcal{R}}\,|\mathbf{e}_i\rangle\langle \mathbf{e}_i|\, \boldsymbol{\mathcal{R}}^{-1} \,|\, \mathbf{e}_m\rangle = R_{ni}\,\langle \mathbf{e}_i|\, \boldsymbol{\mathcal{R}}^{-1} \,|\, \mathbf{e}_m\rangle$$

so one must conclude that

$$\langle \mathbf{e}_i|\, \boldsymbol{\mathcal{R}}^{-1} \,|\, \mathbf{e}_m\rangle = (R^{-1})_{im} \ . \tag{1.1.25}$$

Now apply $\boldsymbol{\mathcal{R}}$ to (1.1.23) to get

$$\boldsymbol{\mathcal{R}}|\mathbf{e}'_n\rangle = R_{nm}\,\boldsymbol{\mathcal{R}}|\mathbf{e}_m\rangle \ .$$

From (1.1.16) the left side is $|R\mathbf{e}'_n\rangle$ while the right side is

$$R_{nm}\,\boldsymbol{\mathcal{R}}|\mathbf{e}_m\rangle = R_{nm}\,|\mathbf{e}_i\rangle\langle\mathbf{e}_i|\boldsymbol{\mathcal{R}}|\mathbf{e}_m\rangle = R_{nm}\,|\mathbf{e}_i\rangle\,R_{im} = R_{nm}R_{im}|\mathbf{e}_i\rangle = R_{nm}R^{T}_{mi}|\mathbf{e}_i\rangle$$

$$= (RR^{T})_{ni}\,|\mathbf{e}_i\rangle = \delta_{ni}\,|\mathbf{e}_i\rangle = |\mathbf{e}_n\rangle \ .$$

Thus we have shown that

$$\mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \qquad \Rightarrow \qquad |R\mathbf{e}'_n\rangle = |\mathbf{e}_n\rangle \ \text{ or } \ \boldsymbol{\mathcal{R}}|\mathbf{e}'_n\rangle = |\mathbf{e}_n\rangle \ \text{ or } \ R\mathbf{e}'_n = \mathbf{e}_n \tag{1.1.26}$$

where on the right we show three equivalent forms of the same equation.

[In the following sequence of steps, we show Dirac notation on the left and vector notation on the right.]

Conversely to the above, suppose we know that

$$\boldsymbol{\mathcal{R}}|\mathbf{e}'_n\rangle = |\mathbf{e}_n\rangle \qquad\qquad R\mathbf{e}'_n = \mathbf{e}_n \ .$$

Inverting we get

$$|\mathbf{e'}_n\rangle = \mathcal{R}^{-1}|\mathbf{e}_n\rangle \qquad \mathbf{e'}_n = R^{-1}\mathbf{e}_n \,.$$

Since the basis $\mathbf{e}_n$ is complete, we know we can write, for some unknown coefficients A_{nm} ,

$$|\mathbf{e'}_n\rangle = A_{nm}\,|\mathbf{e}_m\rangle \qquad \mathbf{e'}_n = A_{nm}\mathbf{e}_m \,. \tag{1.1.27}$$

Comparing the last two equations one has,

$$A_{nm}\,|\mathbf{e}_m\rangle = \mathcal{R}^{-1}|\mathbf{e}_n\rangle = |R^{-1}\mathbf{e}_n\rangle \qquad A_{nm}\mathbf{e}_m = R^{-1}\mathbf{e}_n \,.$$

Now close with $\langle\mathbf{e}_i|$ on the left to get

$$A_{nm}\langle\mathbf{e}_i\,|\mathbf{e}_m\rangle = \langle\mathbf{e}_i\,|\mathcal{R}^{-1}|\mathbf{e}_n\rangle \qquad A_{nm}\,(\mathbf{e}_m)_i = [R^{-1}\mathbf{e}_n]_i = (R^{-1})_{ik}(\mathbf{e}_n)_k$$

or

$$A_{nm}\,\delta_{im} = (R^{-1})_{in} \qquad A_{nm}\,\delta_{mi} = (R^{-1})_{ik}\,\delta_{nk} = (R^{-1})_{in}$$

or

$$A_{ni} = R_{ni} \qquad A_{ni} = R_{ni} \,.$$

Therefore (1.1.27) becomes,

$$|\mathbf{e'}_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \,.$$

Thus we have shown that

$$\mathcal{R}|\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle \quad\Rightarrow\quad |\mathbf{e'}_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \tag{1.1.28}$$

or

$$R\mathbf{e'}_n = \mathbf{e}_n \quad\Rightarrow\quad \mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \,.$$

We have now proved a simple theorem which seems to have no name, so we give it a name:

The Basis Theorem:

$$\mathbf{e}_n = R\mathbf{e'}_n \quad\Leftrightarrow\quad \mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \qquad \text{// vector notation}$$

$$|R\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle \quad\Leftrightarrow\quad |\mathbf{e'}_n\rangle = R_{nm}\,|\mathbf{e}_m\rangle \qquad \text{// Dirac notation} \tag{1.1.29}$$

$$\mathcal{R}|\mathbf{e'}_n\rangle = |\mathbf{e}_n\rangle$$

The equation on the right concerns Frame S' basis vectors being linear combinations of Frame S basis vectors. The equation on the left says that the rotation operator $\mathcal{R}$ acting on $|\mathbf{e'}_n\rangle$ creates a rotated vector called $|R\mathbf{e'}_n\rangle$ (or $R\mathbf{e'}_n$) which is equal to $|\mathbf{e}_n\rangle$ (or $\mathbf{e}_n$).

We can invert both sides of this theorem to get

$$\begin{array}{lll} \mathbf{e'}_n = R^{-1}\mathbf{e}_n & \Leftrightarrow & \mathbf{e}_n = (R)^{-1}_{nm}\,\mathbf{e'}_m \quad \text{// vector notation} \\ |R^{-1}\mathbf{e}_n\rangle = |\mathbf{e'}_n\rangle & \Leftrightarrow & |\mathbf{e}_n\rangle = (R)^{-1}_{nm}\,|\mathbf{e'}_m\rangle \quad \text{// Dirac notation} \\ \boldsymbol{\mathcal{R}}^{-1}|\mathbf{e}_n\rangle = |\mathbf{e'}_n\rangle & & \end{array} \tag{1.1.30}$$

Alternate shorthand notations

$$\mathbf{e}_n = R\mathbf{e'}_n \quad \Leftrightarrow \quad (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = R\,(\mathbf{e'}_1, \mathbf{e'}_2, \mathbf{e'}_3) = \boldsymbol{\mathcal{R}}\,(\mathbf{e'}_1, \mathbf{e'}_2, \mathbf{e'}_3) \tag{1.1.31}$$

$$\mathbf{e'}_n = R_{nm}\,\mathbf{e}_m \quad \Leftrightarrow \quad \begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \\ \mathbf{e'}_3 \end{pmatrix} = R \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} . \tag{1.1.32}$$

The first alternate notation implies for example that $\mathbf{e}_1 = R\mathbf{e'}_1$. This is *not* implied by the second alternate notation which is meant to say $\mathbf{e'}_1 = R_{11}\mathbf{e}_1 + R_{12}\mathbf{e}_2 + R_{13}\mathbf{e}_3$ = linear combination of vectors. These alternate notations are useful when the basis vectors have names like $\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ or $\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}$.

The matrices R and R'

Consider now the relation $\mathbf{e}_n = R\mathbf{e'}_n$ so that R relates the Frame S and Frame S' bases, as above. In this case, taking components one gets,

$$\begin{array}{lll} (\mathbf{e}_n)_i = [R\mathbf{e'}_n]_i = R_{ij}\,(e'_n)_j & & \text{Frame S components} \\ (\mathbf{e}_n)'_i = [R\mathbf{e'}_n]'_i = (R)'_{ij}\,(e'_n)'_j & & \text{Frame S' components} \quad \text{// note prime on } (R)'_{ij} \end{array} \tag{1.1.33}$$

In Dirac notation the above two lines may be expressed as,

$$(\mathbf{e}_n)_i = \langle \mathbf{e}_i \,|\, \mathbf{e}_n \rangle = \langle \mathbf{e}_i \,|\, R\mathbf{e'}_n \rangle = \langle \mathbf{e}_i \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e'}_n \rangle = \langle \mathbf{e}_i \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e}_j \rangle\langle \mathbf{e}_j \,|\, \mathbf{e'}_n \rangle = R_{ij}(\mathbf{e'}_n)_j$$

$$(\mathbf{e}_n)'_i = \langle \mathbf{e'}_i \,|\, \mathbf{e}_n \rangle = \langle \mathbf{e'}_i \,|\, R\mathbf{e'}_n \rangle = \langle \mathbf{e'}_i \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e'}_n \rangle = \langle \mathbf{e'}_i \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e'}_j \rangle\langle \mathbf{e'}_j \,|\, \mathbf{e'}_n \rangle = (R)'_{ij}\,(\mathbf{e'}_n)'_j \ .$$

We encounter two matrices here,

$$R_{ij} = \langle \mathbf{e}_i \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e}_j \rangle$$

$$(R)'_{ij} = \langle \mathbf{e'}_i \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e'}_j \rangle \ . \tag{1.1.34}$$

Because the operator $\boldsymbol{\mathcal{R}}$ is the same operator which relates the two bases, these two matrices are the same,

$$(R)'_{ij} = \langle \mathbf{e'}_i \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e'}_j \rangle = \langle \mathbf{e'}_i \,|\, \mathbf{e}_n \rangle\langle \mathbf{e}_n \,|\, \boldsymbol{\mathcal{R}} \,|\, \mathbf{e}_m \rangle\langle \mathbf{e}_m \,|\, \mathbf{e'}_j \rangle = R_{in}\,R_{nm}\,R_{jm}$$

$$= R_{in}\, R_{nm}\, R^{T}_{mj} = R_{in}\,(RR^{T})_{nj} = R_{in}\,\delta_{nj} = R_{ij}\ . \qquad (1.1.35)$$

This fact is abundantly clear from (1.1.21) $(T)' = RTR^{-1}$ which in this case says $(R)' = RRR^{-1} = R$.

As (1.1.14) shows, R_{nm} can also be written $R_{nm} = \langle \mathbf{e'}_m | \mathbf{e}_n \rangle = \langle \mathbf{e'}_m | \mathit{1} | \mathbf{e}_n \rangle$. R_{nm} is thus the matrix element of the unity operator in a "mixed basis". R is the "basis change matrix" between the two bases. One can consider any tensor in a mixed basis, but the notation gets a bit messy. For example one could write

$$T^{(e',e)}{}_{mn} = \langle \mathbf{e'}_m | \mathcal{T} | \mathbf{e}_n \rangle$$

where instead a label prime or no prime on T, we have to supply a label which shows the two bases being used. For the unmixed bases we just say $T^{(e,e)}{}_{mn} = T_{mn}$ and $T^{(e',e')}{}_{mn} = (T)'_{mn}$.

Other Dirac Facts

We have managed so far to avoid the following Dirac notation facts, but now is a good time to get them on the table. Here we show Dirac notation on the left, and vector notation on the right :

$$|\mathbf{a}\rangle = |X\mathbf{b}\rangle = \mathcal{X}\,|\mathbf{b}\rangle \qquad \mathbf{a} = X\mathbf{b}$$

$$\langle \mathbf{a}| = \langle X\mathbf{b}| = \langle \mathbf{b}|\,\mathcal{X}^{T} \qquad \mathbf{a}^{T} = (X\mathbf{b})^{T} = \mathbf{b}^{T}X^{T}$$

$$\langle \mathbf{c}|\,\mathcal{X}\,|\mathbf{d}\rangle = \langle \mathbf{c}|\,X\mathbf{d}\rangle = \langle X\mathbf{d}\,|\mathbf{c}\rangle = \langle \mathbf{d}|\,\mathcal{X}^{T}\,|\mathbf{c}\rangle \qquad \mathbf{c}^{T}X\mathbf{d} = (\mathbf{c}^{T}X\mathbf{d})^{T} = \mathbf{d}^{T}X^{T}\mathbf{c}\ . \qquad (1.1.36)$$

Notice in $\langle \mathbf{a}| = \langle \mathbf{b}|\,\mathcal{X}^{T}$ that the operator $\mathcal{X}^{T}$ acts to the left, just as the matrix in $\mathbf{b}^{T}X^{T}$ acts to the left on the transpose vector $\mathbf{b}^{T}$. Also, $\mathbf{c}^{T}X\mathbf{d}$ is just a number, so $(\mathbf{c}^{T}X\mathbf{d}) = (\mathbf{c}^{T}X\mathbf{d})^{T}$.

Example: $R_{nm} = \langle \mathbf{e}_n |\, \mathcal{R}\, | \mathbf{e}_m \rangle$ (1.1.24) $= [\langle \mathbf{e}_n |\, \mathcal{R}\,]\, | \mathbf{e}_m \rangle$ acts to left $= \langle R^{T}\mathbf{e}_n | \mathbf{e}_m \rangle$ (1.1.36) $= \langle R^{-1}\mathbf{e}_n | \mathbf{e}_m \rangle$ real orthog $= \langle \mathbf{e'}_n | \mathbf{e}_m \rangle$ (1.1.29) $= R_{nm}$ (1.1.14)

Suppose one knows that $XX^{T} = 1$. That says $X_{ij}(X^{T})_{jk} = \delta_{ik}$ or $X_{ij}X_{kj} = \delta_{ik}$ or

$$\langle \mathbf{e}_i |\, \mathcal{X}\, | \mathbf{e}_j \rangle \langle \mathbf{e}_k |\, \mathcal{X}\, | \mathbf{e}_j \rangle = \delta_{ik}$$

or

$$\langle \mathbf{e}_i |\, \mathcal{X}\, | \mathbf{e}_j \rangle \langle \mathbf{e}_j |\, \mathcal{X}^{T}\, | \mathbf{e}_k \rangle = \delta_{ik} \qquad \text{// (1.1.36)}$$

or

$$\langle \mathbf{e}_i |\, \mathcal{X}\mathcal{X}^{T}\, | \mathbf{e}_k \rangle = \delta_{ik}\ . \qquad \text{// (1.1.20)}$$

Thus it must be that

$$XX^{T} = 1 \ \Leftrightarrow\ \mathcal{X}\mathcal{X}^{T} = \mathit{1} \qquad \text{or} \qquad XX^{-1} = 1 \ \Leftrightarrow\ \mathcal{X}\mathcal{X}^{-1} = \mathit{1} \qquad (1.1.37)$$

as one would expect.

Example:

$$(R)'_{nm} = \langle \mathbf{e}'_n | \mathcal{R} | \mathbf{e}'_m \rangle \overset{(1.1.34)}{=} \langle \mathbf{e}'_n | \mathcal{R}^T \mathcal{R} \mathcal{R} | \mathbf{e}'_m \rangle \overset{(1.1.37)}{=} \langle R\mathbf{e}'_n | \mathcal{R} | R\mathbf{e}'_m \rangle \overset{(1.1.36)}{=} \langle \mathbf{e}_n | \mathcal{R} | \mathbf{e}_m \rangle \overset{(1.1.29)}{=} R_{nm} \overset{(1.1.24)}{=} (R)'_{nm}$$

As an application of the above consider this fact, where R is our usual real orthogonal rotation matrix,

$$\mathbf{a} \bullet \mathbf{b} = [R\mathbf{a}] \bullet [R\mathbf{b}] \quad \text{or} \quad \langle \mathbf{a} | \mathbf{b} \rangle = \langle R\mathbf{a} | R\mathbf{b} \rangle \ . \tag{1.1.38}$$

Proof: In vector notation one has, using $\mathbf{e}_n$ vector components,

$$[R\mathbf{a}] \bullet [R\mathbf{b}] = [R\mathbf{a}]_k [R\mathbf{b}]_k \overset{(1.1.33)}{=} R_{ki} a_i R_{kj} b_j = (R^T)_{jk} R_{ki} a_i b_j$$

$$= (R^T R)_{ji} a_i b_j = \delta_{ji} a_i b_j = a_i b_i = \mathbf{a} \bullet \mathbf{b} \ .$$

In Dirac notation the proof reads, using (1.1.36) and (1.1.37),

$$\langle R\mathbf{a} | R\mathbf{b} \rangle = \langle \mathbf{a} | \mathcal{R}^T \mathcal{R} | \mathbf{b} \rangle = \langle \mathbf{a} | \mathbf{1} | \mathbf{b} \rangle = \langle \mathbf{a} | \mathbf{b} \rangle \ .$$

Time dependence of the R_{ij}

If Frame S is fixed and Frame S' is rotating, then we really have $\mathbf{e}_n = R(t)\mathbf{e}'_n(t)$ where the R matrix is a function of time, so we have $R_{ij}(t)$. Similarly, if Frame S is moving and Frame S' is fixed, $\mathbf{e}_n(t) = R(t)\, \mathbf{e}'_n$ and again one has $R_{ij}(t)$. Only in the case where there is no rotation between the frames are the R_{ij} independent of time. This means that $\boldsymbol{\omega} = 0$ in Fig 1. Since our document is about "rotating frames of reference" we exclude this no-rotation case from consideration.

Concatenated Two Transformations

In this and the next section we use the notation T' to mean (T)' and similarly for other tensors.

Up to this point, we have dealt with a single rotation transformation $\mathbf{e}_n = R\, \mathbf{e}'_n$ for which the following facts are true (the left side is the Basis Theorem),

$$\mathbf{e}_n = R\, \mathbf{e}'_n \overset{(1.1.29)}{\Leftrightarrow} \mathbf{e}'_n = R_{nm}\, \mathbf{e}_m \qquad \mathbf{e}'_i \bullet \mathbf{e}_j \overset{(1.1.8)}{=} R_{ij} \ , \quad T' \overset{(1.1.21)}{=} RTR^{-1} \ , \quad R \overset{(1.1.35)}{=} R' \ . \tag{1.1.39}$$

Suppose we have a second rotation transformation $\mathbf{e}'_n = S\, \mathbf{e}''_n$. Then the claim is that,

$$\mathbf{e}'_n = S\, \mathbf{e}''_n \quad \Leftrightarrow \quad \mathbf{e}''_n = S'_{nm}\, \mathbf{e}'_m \qquad \mathbf{e}''_i \bullet \mathbf{e}'_j = S'_{ij} \ , \quad T'' = S'T'S'^{-1} \ , \quad S' = S'' \ . \tag{1.1.40}$$

Again, the left side is just a statement of the Basis Theorem for this second transformation. To verify the items on the right, consider

$$\mathbf{e}''_i \bullet \mathbf{e}'_j = [S'_{im}\, \mathbf{e}'_m] \bullet \mathbf{e}'_j = S'_{im} \, [\, \mathbf{e}'_m \bullet \mathbf{e}'_j \,] = S'_{im}\, \delta_{mj} = S'_{ij} \ .$$

Then,

$$T''_{ij} = \langle \mathbf{e}''_i \,|\, \mathcal{T} \,|\, \mathbf{e}''_j \rangle = \langle \mathbf{e}''_i \,|\, e'_m \rangle\langle e'_m|\, \mathcal{T} \,|\, e'_n \rangle\langle e'_n \,|\, \mathbf{e}''_j \rangle = S'_{im}\, T'_{mn}\, S'_{jn}$$

$$= S'_{im}\, T'_{mn}\, S'^{T}_{jn} = S'_{im}\, T'_{mn}\, S'^{-1}_{jn} = (S'T'S'^{-1})_{jj}$$

which shows that

$$T'' = S'T'S'^{-1} .$$

Finally, applying this last equation to T" = S" one finds,

$$S'' = S'S'S'^{-1} = S' .$$

so all three items in the right of (1.1.40) are verified.

Notice now that

$$\mathbf{e}_n = R\,\mathbf{e}'_n = R\,(S\,\mathbf{e}''_n) = RS\,\mathbf{e}''_n . \qquad // \; |\mathbf{e}_n\rangle = \mathcal{RS}\,|\mathbf{e}''_n\rangle$$

Also,

$$\mathbf{e}''_n = S'_{nm}\,\mathbf{e}'_m = S'_{nm}\,(R_{mk}\,\mathbf{e}_k) = S'_{nm}\,R_{mk}\,\mathbf{e}_k = (S'R)_{nk}\mathbf{e}_k .$$

On the other hand, we can apply the Basis Theorem directly to Q ≡ SR to get

$$\mathbf{e}_n = RS\mathbf{e}''_n \qquad \Leftrightarrow \qquad \mathbf{e}''_n = (RS)_{nk}\,\mathbf{e}_k .$$

Comparing the right sides of the last two $\mathbf{e}''_n$ expressions one finds the seemingly contradictory result that

$$(S'R)_{nk}\mathbf{e}_k = (RS)_{nk}\,\mathbf{e}_k$$

where the matrices seem to have reverse order on the two sides. But there is no contradiction because we know that S' = RSR^{-1} and therefore S'R = RS.

Concatenated Three Transformations

Since we are going to be dealing later with Euler angles which involve *three* basis transformations, we consider finally a third rotation transformation U whose facts the reader can easily verify,

$$\mathbf{e}''_n = U\,\mathbf{e}'''_n \;\Leftrightarrow\; \mathbf{e}'''_n = U''_{nm}\,\mathbf{e}''_m \qquad \mathbf{e}'''_i \bullet \mathbf{e}''_j = U''_{ij}\,,\; T''' = U''T''U''^{-1}\,,\; U'' = U''' . \quad (1.1.41)$$

Concatenating transformations in the ways shown above one gets,

$$\mathbf{e}_n = R\mathbf{e}'_n = RS\mathbf{e}''_n = RSU\mathbf{e}'''_n \qquad // = \mathcal{RSU}\,|\mathbf{e}'''_n\rangle \qquad (1.1.42)$$

$$\mathbf{e'''}_n = U''_{nm}\,\mathbf{e''}_m = U''_{nm}S'_{mi}\,\mathbf{e'}_i = U''_{nm}S'_{mi}\,R_{ij}\,\mathbf{e}_j = (U''S'R)_{nj}\,\mathbf{e}_j\ ,$$

Now thinking RSU = Q, the Basis Theorem says,

$$\mathbf{e}_n = RSU\mathbf{e'''}_n \qquad \Leftrightarrow \qquad \mathbf{e'''}_n = (RSU)_{nk}\,\mathbf{e}_k\ . \tag{1.1.43}$$

Comparing the last two expressions for $\mathbf{e'''}_n$ we get (again with reverse ordered matrices),

$$(U''S'R)_{nj}\,\mathbf{e}_j = (RSU)_{nk}\,\mathbf{e}_k\ .$$

Can we show that in fact U''S'R = RSU ? Consider, using the tensor rules shown above,

$$U''S'R = (S'U'S'^{-1})S'R = S'U'R = (RSR^{-1})(RUR^{-1})R = RSU$$

so again there is no contradiction.

Since the matrices like R, S and U in basis e_n (Frame S) are likely to be known, whereas the others might have to be calculated, we prefer the RSU form shown in (1.1.43). In the alternate shorthand notation of (1.1.31) and (1.1.32) we shall write (1.1.42) and (1.1.43) in this manner

$$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = RSU(\mathbf{e'''}_1, \mathbf{e'''}_2, \mathbf{e'''}_3) \qquad \text{or} \quad (|\mathbf{e}_1\rangle, |\mathbf{e}_2\rangle, |\mathbf{e}_3\rangle) = \mathcal{RSU}(|\mathbf{e'''}_1\rangle, |\mathbf{e'''}_2\rangle, |\mathbf{e'''}_3\rangle) \tag{1.1.44}$$

$$\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \\ \mathbf{e}'_3 \end{pmatrix} = RSU \begin{pmatrix} \mathbf{e'''}_1 \\ \mathbf{e'''}_2 \\ \mathbf{e'''}_3 \end{pmatrix}\ . \tag{1.1.45}$$

In (1.1.45), R S and U are always matrices in the $\mathbf{e}_n$ (Frame S) basis. In (1.1.44), recall that [RSU$\mathbf{e'''}_1$] is the name of the vector obtained by applying $\mathcal{RSU}$ to the vector $\mathbf{e'''}_1$. If we take $\mathbf{e}_n$ components of this equation, *then* we may regard R, S and U as Frame S matrices, since in that case,

$$(\mathbf{e}_n)_i = (RSU)_{ij}\,(\mathbf{e'''}_n)_j = R_{in}S_{nm}U_{mj}\,(\mathbf{e'''}_n)_j\ .$$

However, if we take Frame S' (or some other frame) components, we get different matrices, for example,

$$(\mathbf{e}_n)'_i = (RSU)'_{ij}\,(\mathbf{e'''}_n)'_j = R'_{in}S'_{nm}U'_{mj}\,(\mathbf{e'''}_n)'_j\ .$$

We shall make use of (1.1.44) and (1.1.45) in our discussion of Euler Angle rotations in Appendix G.

1.2 Expansions of a vector and use of primes and parentheses

Note: We write $(a)_i$ as a component of vector $\mathbf{a}$, but $(\mathbf{e}_n)_i$ as a component of $\mathbf{e}_n$. In the first case the a in $(a)_i$ is not bolded, but since $\mathbf{e}_n$ is decorated with a label n, it gets bolded. It is just our convention.

Any vector $\mathbf{a}$ can be expanded on either set of basis vectors $\mathbf{e}_i$ (Frame S) or $\mathbf{e}'_i$ (Frame S') so that, with implied summation on i,

$$\mathbf{a} = a_i\,\mathbf{e}_i = a'_i\,\mathbf{e}'_i \quad . \qquad a_i = \mathbf{a}\bullet\mathbf{e}_i \qquad a'_i = \mathbf{a}\bullet\mathbf{e}'_i \quad . \tag{1.2.1}$$

If some other vector named $\mathbf{a}'$ is lurking in the wings, one might want to be more careful labeling components. A safe method is this (which we have already used in Section 1.1),

$$\begin{aligned} &\mathbf{a} = (a)_i\,\mathbf{e}_i = (a)'_i\,\mathbf{e}'_i && (a)_i = \mathbf{a}\bullet\mathbf{e}_i && (a)'_i = \mathbf{a}\bullet\mathbf{e}'_i \\ &\mathbf{a}' = (a')_i\,\mathbf{e}_i = (a')'_i\,\mathbf{e}'_i && (a')_i = \mathbf{a}'\bullet\mathbf{e}_i && (a')'_i = \mathbf{a}'\bullet\mathbf{e}'_i \quad . \end{aligned} \tag{1.2.2a}$$

Here, a prime *inside* a parentheses is part of the vector name, whereas a prime *outside* a parentheses denotes a vector component in Frame S' (whereas no prime outside means a component in Frame S). Unless the relationship between vectors $\mathbf{a}$ and $\mathbf{a}'$ has a certain simple form, it is very likely that $(a')_i \neq (a)'_i$. In this case the notation a'_i would be ambiguous since one doesn't know whether it refers to $(a')_i$ or $(a)'_i$. It is true that the notation a_i could be unambiguously identified with $(a)_i$, but we shall sometimes maintain the parentheses just to be uniform.

We repeat (1.2.2a) in Dirac notation

$$\begin{aligned} &|\mathbf{a}\rangle = (a)_i\,|\mathbf{e}_i\rangle = (a)'_i\,|\mathbf{e}'_i\rangle && (a)_i = \langle\mathbf{e}_i|\mathbf{a}\rangle && (a)'_i = \langle\mathbf{e}'_i|\mathbf{a}\rangle \\ &|\mathbf{a}'\rangle = (a')_i\,|\mathbf{e}_i\rangle = (a')'_i\,|\mathbf{e}'_i\rangle && (a')_i = \langle\mathbf{e}_i|\mathbf{a}'\rangle && (a')'_i = \langle\mathbf{e}'_i|\mathbf{a}'\rangle \end{aligned} \tag{1.2.2b}$$

Matrix Notation to show how components are related.

Let R be the transformation appearing in the Basis Theorem (1.1.29) such that $\mathbf{e}'_n = R_{nm}\,\mathbf{e}_m$ and $\mathbf{e}_n = R\mathbf{e}'_n$. Consider the following expansion of vector $\mathbf{a}$ on the basis vectors $\mathbf{e}'_j$,

$$\begin{aligned} \mathbf{a} &= (a)'_j\,\mathbf{e}'_j && \text{// (1.2.2)} \\ &= (a)'_j\,\{\,R_{ji}\,\mathbf{e}_i\,\} && \text{// } \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \text{ , linear combination of vectors} \\ &= (a)'_j\,\{\,(R^{-1})_{ij}\,\mathbf{e}_i\,\} && \text{// } R = (R^{-1})^{T} \text{ real orthogonal rotation} \\ &= \{\,(R^{-1})_{ij}(a)'_j\,)\mathbf{e}_i \quad . && \text{// reorder} \end{aligned} \tag{1.2.3}$$

Comparing this to $\mathbf{a} = (a)_i\,\mathbf{e}_i$ of (1.2.2) we conclude that, since $\mathbf{e}_i$ is a complete basis,

$$(a)_i = (R^{-1})_{ij}(a)'_j \qquad \text{so} \qquad (a)'_i = R_{ij}(a)_i \tag{1.2.4}$$

where R^{-1} is a 3x3 rotation matrix.

We can repeat the above discussion replacing **a** with **a'** with this result,

$$(a')_i = (R^{-1})_{ij}(a')'_j \; . \qquad \text{so} \qquad (a')'_i = R_{ij}(a')_i \tag{1.2.5}$$

These matrix equations are convenient for computing the components of a vector on the $\mathbf{e}_i$ basis if they are known in the $\mathbf{e}'_i$ basis (and vice versa) .

1.3 Active and Passive Views of rotation, and a review of dot products

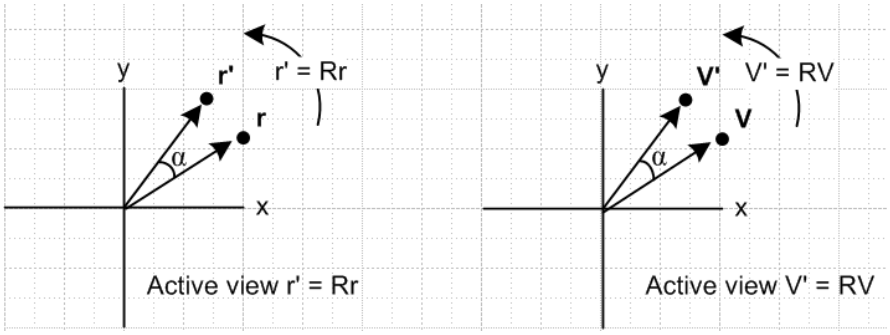
Basis Vectors, Kinematic Vectors, Active and Passive Views

We imagine an Apparatus in Frame S which "contains" (is described by, is associated with) various vectors of interest (in addition to scalars and perhaps fancier tensors). We refer to these vectors as **Kinematic Vectors**, examples being the position **r** of some particle of the apparatus, or the velocity **v** or acceleration **a** of that particle, or some electric field **E** at point **r**.

Active View

If the entire apparatus is forward-rotated (by a positive angle according to the right hand rule) by R about the Frame S origin, all these Kinematic Vectors transform in the usual manner $(V')_i = R_{ij}V_i$ or $\mathbf{V}' = R\mathbf{V}$. For example, a point that was located at position **r** in the apparatus (in Frame S) is now located at a new position $\mathbf{r}' = R\mathbf{r}$ in Frame S. The Frame S **Basis Vectors** $\mathbf{e}_n$ *do not move*, only the apparatus moves. We call this the **Active View** of rotation. The vector **V'** is a new vector in Frame S that is different from **V**. There are no vectors $\mathbf{e}'_n$ and there is no Frame S'. Graphically,

$R = R_z(\alpha) \qquad \mathbf{V}' = R\mathbf{V} = R_z(\alpha)\mathbf{V}$



(1.3.1)

In the Active View one writes,

$$(V')_i = R_{ij}V_j \qquad \Leftrightarrow \qquad \mathbf{V}' = R\mathbf{V} \qquad \text{or} \qquad (\mathbf{V}') = R\mathbf{V} \; . \tag{1.3.2}$$

Passive View

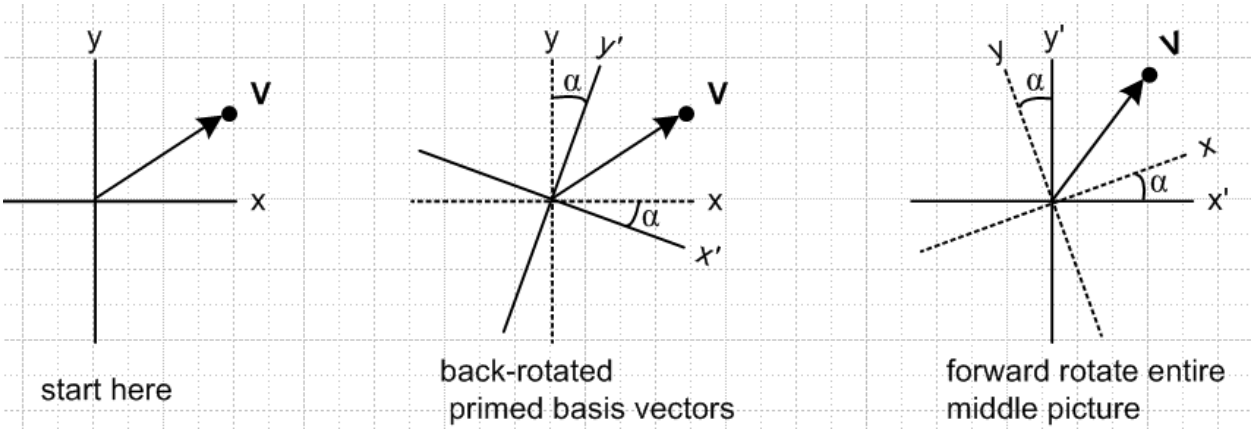
Alternatively, suppose the apparatus stays put, but the basis vectors $\mathbf{e}_n$ are back-rotated about the Frame S origin into new Frame S' basis vectors $\mathbf{e}'_n$ such that $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$. In this case, the components of vector **V** (which were V_i in Frame S) become $(V)'_i = R_{ij}V_j$ in Frame S'. There is no vector named **V**'. We call this the **Passive View** of rotation. Our convention for writing $(V)'_i = R_{ij}V_j$ in vector notation is this

$$(V)'_i = R_{ij}V_j \qquad \Leftrightarrow \qquad (\mathbf{V})' = R\mathbf{V} \qquad // \ \mathbf{e}'_n = R^{-1}\mathbf{e}_n \tag{1.3.3}$$

Graphically we illustrate the above passive view situation (also valid with **V** replaced by **r**) :

$\mathbf{e}'_n = R^{-1}\mathbf{e}_n = [R_z(\alpha)]^{-1}\,\mathbf{e}_n = R_z(-\alpha)\ \mathbf{e}_n$ $\qquad R^{-1} = R_z(-\alpha)$ = "back-rotation"

$\hat{\mathbf{x}}' = R_z(-\alpha)\hat{\mathbf{x}}$ $\qquad \hat{\mathbf{y}}' = R_z(-\alpha)\hat{\mathbf{y}}$ $\qquad$ basis vectors are "back-rotated" by right-hand-rule



Basis vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ $\qquad$ new basis vectors $\hat{\mathbf{x}}'$ and $\hat{\mathbf{y}}'$

(a) Frame S seen from Frame S $\qquad$ (b) Frame S and Frame S' seen from Frame S $\qquad$ (c) Frame S and Frame S' seen from Frame S'

(1.3.4)

Here is a Dirac interpretation of the Passive View and (1.3.3) :

$|\mathbf{e}'_n\rangle = |R^{-1}\mathbf{e}_n\rangle = \mathcal{R}^{-1}\,|\mathbf{e}_n\rangle$ $\qquad$ basis vectors $\qquad \mathbf{e}'_n = R^{-1}\mathbf{e}_n$

$(V)'_i = \langle \mathbf{e}'_i|\mathbf{V}\rangle = \langle \mathbf{e}'_i\,|\mathbf{e}_j\rangle\langle\mathbf{e}_j|\,\mathbf{V}\rangle = R_{ij}V_j$ $\qquad$ (1.1.20) $\quad$ (1.1.14) $\qquad$ kinematic vectors $\qquad (\mathbf{V})' = R\mathbf{V}$

In the passive view, one might *choose* to create a new vector **V**' according to the rule **V**' ≡ R**V**. In this case, one now has two vectors **V**' and **V** with $(V')_i = (V)'_i$. One can then replace $(\mathbf{V})' = R\mathbf{V}$ with the statement **V**' ≡ R**V** which is then exactly the same statement one sees for the Active View.

Aside: The equation **V**' ≡ R**V** really is a vector equation, and you could evaluate it in either Frame S or Frame S' as $(V')_i = R_{ij}V_j$ or $(V')'_i = [R\mathbf{V}]'_i = (R')_{ij}V'_j = R_{ij}V'_j$. The equation $(\mathbf{V})' = R\mathbf{V}$ is not a real vector equation, it is just a vector shorthand notation for $(V)'_i = R_{ij}V_j$.

However, suppose the vector **V'** already has some other meaning unrelated to the above discussion. Then we cannot use the Active View, because then **V'** = R**V** (being the actively rotated vector **V)**, will very likely not be the same as the vector **V'** from its unrelated other meaning. In this case we have an overloaded notation. In terms of components, we will have $(V')_i \neq (V)'_i$.

To restate the above, if **V'** has some other predefined meaning, we can (and will) use the Passive View, but we cannot go that extra step to create **V'** ≡ R**V** because then **V'** would be overloaded. So in this case, we must always use (**V**)' = R**V** (which does not involve the vector **V'**).

We shall directly encounter this situation in subsequent sections. For example, in Fig 1 we show the relation **r'** = **r** + **b** so the vector **r'** already has a definition, so we cannot define **r'** ≡ R**r**. Similarly, we shall write write in (6.6.c) that **v** = **v'** + **ω** x **r** + $\dot{\mathbf{b}}_{S'}$ where **v** and **v'** are the "natural" velocities of a particle in Frames S and S'. This is of course incompatible with **v'** ≡ R**v ,** but (**v**)' ≡ R**v** of the Passive View is OK. Again here we shall have $(v')_i \neq (v)'_i$.

To summarize, The Kinematic Vectors and the Basis Vectors are in two disjoint classes: no vector is in both classes. In the Active View, all the Kinematic Vectors forward-rotate while all the Basis Vectors $\mathbf{e}_n$ stay put. In the Passive View the Kinematic Vectors all stay put, while the Basis Vectors e_n are back-rotated into new basis vectors $\mathbf{e}'_n$ which define Frame S'.

This subject can be deceptive, so we roll out a simple example in an attempt to exterminate a recurring confusion.

Example: The vector $\hat{\mathbf{x}}$ is a Basis Vector so one has $\hat{\mathbf{x}}' = R^{-1}\hat{\mathbf{x}}$ in the Passive View. If the vector **r** is a point in the Apparatus, one has **r'** = R**r** in the Active View. There is no equation which says $\hat{\mathbf{x}}' = R\hat{\mathbf{x}}$ analogous to **r'** = R**r** because $\hat{\mathbf{x}}$ is not an Kinematic Vector, it is a Basis Vector.

Now suppose in Frame S some point in the apparatus *happens to be* located at **r** = $\hat{\mathbf{x}}$. This is potentially confusing since **r** is a Kinematic Vector and $\hat{\mathbf{x}}$ is a Basis Vector and in Frame S these vectors happen to be equal. Consider,

View	General **r**	Specific **r** = $\hat{\mathbf{x}}$
Active	**r'** = R**r**	**r'** = $R\hat{\mathbf{x}}$ ≠ $\hat{\mathbf{x}}'$ since $\hat{\mathbf{x}}'$ does not exist
Active + Create $\hat{\mathbf{x}}' \equiv R^{-1}\hat{\mathbf{x}}$	**r'** = R**r**	**r'** = $R\hat{\mathbf{x}}$ ≠ $\hat{\mathbf{x}}'$ (since $\hat{\mathbf{x}}' = R^{-1}\hat{\mathbf{x}}$)
Passive	(**r**)' = R**r**	(**r**)' = $R\hat{\mathbf{x}}$ ≠ $\hat{\mathbf{x}}'$ (since $\hat{\mathbf{x}}' = R^{-1}\hat{\mathbf{x}}$)
Passive + Create **r'** ≡ R**r**	**r'** = (**r**)' = R**r**	**r'** = (**r**)' = $R\hat{\mathbf{x}}$ ≠ $\hat{\mathbf{x}}'$ (since $\hat{\mathbf{x}}' = R^{-1}\hat{\mathbf{x}}$)

(1.3.5)

In all of these possible Views, although we start with **r** = $\hat{\mathbf{x}}$, we always find **r'** ≠ $\hat{\mathbf{x}}'$ or (**r**)' ≠ $\hat{\mathbf{x}}$.

The reason this can be confusing is that one says $\mathbf{p} = m\mathbf{v}$ in Frame S, so one gets $\mathbf{p}' = m\mathbf{v}'$ in the Active View, or $(\mathbf{p})' = m(\mathbf{v})'$ in Frame S' in the Passive View. Similarly, one casually thinks that with $\mathbf{r} = \hat{\mathbf{x}}$ in Frame S, one ought to get $\mathbf{r}' = \hat{\mathbf{x}}'$ or $(\mathbf{r})' = \hat{\mathbf{x}}'$ in Frame S'. In the first case $\mathbf{p}$ and $\mathbf{v}$ are both Kinematic Vectors, but in the second case $\mathbf{r}$ is a Kinematic Vector but $\hat{\mathbf{x}}$ is a Basis Vector.

Reminder: The last line of (1.3.5) is not an acceptable View if the vector $\mathbf{r}'$ already has some other definition.

Rank-2 tensors in Active and Passive View

In the Active View, the tensor $T' = RTR^{-1}$ is a new tensor in Frame S, different from T. There is no Frame S', there are no basis vectors $\mathbf{e}'_n$.

In the Passive View one has $(T)' = RTR^{-1}$ meaning $(T)'_{ij} = R_{ia}T_{ab}R^{-1}{}_{bj}$. There is no tensor named T' different from T, there is only tensor T viewed from the two frames. In Dirac notation we quote the statement made above (1.1.21),

$$(T)'_{mn} \equiv \langle \mathbf{e}'_m | \mathcal{T} | \mathbf{e}'_n \rangle = \langle \mathbf{e}'_m | \mathit{1}\mathcal{T}\mathit{1} | \mathbf{e}'_n \rangle = \langle \mathbf{e}'_m | \mathbf{e}_i \rangle\langle \mathbf{e}_i | \mathcal{T} | \mathbf{e}_j \rangle\langle \mathbf{e}_j | \mathbf{e}'_n \rangle$$

$$= R_{mi}\, T_{ij}\, R_{nj} = R_{mi}\, T_{ij}\, (R^{-1})_{jn} = [R\, T\, R^{-1}]_{mn} \qquad \Rightarrow \qquad (T)' = RTR^{-1}$$

Dot Products and Scalars

Suppose $\mathbf{a}$ and $\mathbf{b}$ are two apparatus Kinematic Vectors. Barring external definitions of $\mathbf{a}'$ or $\mathbf{b}'$, in the Active View we then have $\mathbf{a}' = R\mathbf{a}$ and $\mathbf{b}' = R\mathbf{b}$. We know using (1.1.38) that

$$\mathbf{a} \bullet \mathbf{b} = [R\mathbf{a}] \bullet [R\mathbf{b}] = \mathbf{a}' \bullet \mathbf{b}'. \tag{1.3.6}$$

As an example, one then has $\mathbf{a} \bullet \mathbf{a} = \mathbf{a}' \bullet \mathbf{a}'$ which says $|\mathbf{a}|^2 = |\mathbf{a}'|^2$. Thus a real orthogonal transformation R is one which preserves the length of a vector. Note that both regular rotations (detR = 1) and reflections (detR = -1) have this property.

Since the dot product $\mathbf{a} \bullet \mathbf{b}$ has the same value in Frame S as in Frame S', it is a "rotational scalar", as distinct from a "scalar" which sometimes just means a 1-tuple.

Instead of dealing with two Kinematic Vectors, we can look at the dot product of two Basis Vectors. We have already seen in (1.1.1) that

$$\mathbf{e}_n \bullet \mathbf{e}_m = \mathbf{e}'_n \bullet \mathbf{e}'_m = \delta_{nm}\,. \tag{1.3.7}$$

Although this dot product is the same in both Frames, it is not a scalar. It is a rank-2 tensor which is known a metric tensor.

From (1.1.3) and (1.1.8) the other possible Basis Vector dot product is

$$\mathbf{e}_n \bullet \mathbf{e}'_m = (\mathbf{e}'_m)_n = \mathbf{e}'_m \bullet \mathbf{e}_n = (\mathbf{e}_n)'_m = R_{mn} = (R)'_{mn} \quad \text{// last equality is (1.1.35)} \tag{1.3.8}$$

which is the rotation matrix. One can regard R_{mn} as a trivial rank-2 tensor since it transforms as $T' = RTR^{-1}$ for $T = R$. This dot product has "one foot in each Frame" so it makes no sense to ask if it is a scalar which has the same value in the two Frames.

What about the dot product of a Kinematic Vector with a Basis Vector? We find

$$\mathbf{a} \bullet \mathbf{e}_n = (a)_n \qquad \mathbf{a} \bullet \mathbf{e}'_n = (a)'_n \qquad \mathbf{a}' \bullet \mathbf{e}_n = (a')_n \qquad \mathbf{a}' \bullet \mathbf{e}'_n = (a')'_n \ . \tag{1.3.9}$$

These dot products are neither scalars nor tensors. They are all some kind of vector components. If **a**' has no other externally predefined meaning, we can write **a**' = R**a** and then from (1.1.38),

$$(a')_n = \mathbf{a}' \bullet \mathbf{e}_n = [R\mathbf{a}] \bullet [R\mathbf{e}'_n] = \mathbf{a} \bullet \mathbf{e}'_n = (a)'_n \equiv a'_n \tag{1.3.10}$$

In this special case we end up with $(a')_n = (a)'_n$ which we could then call a'_n without ambiguity. But as shown in (1.2.5), we are still stuck with $(a')'_n = R_{nm}(a')_m = R_{nm}a'_m = (R\mathbf{a}')_n = (R^2\mathbf{a})_n \neq a_n$.

As noted above, soon we shall be dealing with the Fig 1 equation **r'** = **r** - **b**. Since this is *not* of the form **r'** ≡ R**r** , we may *not* dispense with the parentheses, and we expect that $(r')_i$ and $(r)'_i$ will be different.

Comment: If both Frame S and Frame S' are *static* (so they are inertial frames), then both force **F** and acceleration **a** transform as kinematic variables so **F**' = R**F** and **a**' = R**a**. In this case Newton's Law **F** = m**a** in Frame S becomes **F'** = m**a'** in Frame S'. Since this equation has the same form in both Frames, it is said to be **covariant** under rotations. All valid laws of physics must be covariant under rotations, and also under the velocity boost transformations of special relativity.