Rewrite of Section 8 after 8.8.42 PhL 2.27.17

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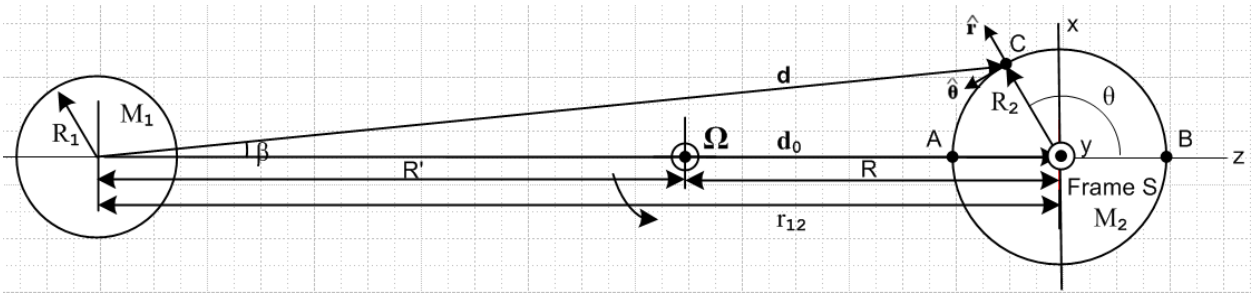
Tidal patterns for an arbitrary rotation axis of the Earth

We now turn the rotation of the Earth back on (we turned if off earlier). For the real Earth, there are many tidal complications that arise. There are land masses. Lake water has nowhere to go. There is friction between the water and the land which slows down the Earth's rotation slightly over time. There is weather and there are ocean tidal currents which do not flow infinitely fast. We shall not attempt to analyze this general situation.

Instead, we imagine an idealized Earth covered with water and the Earth turns under the water with no "friction", and an Observer just stands in the water and measures the tide height as a function of time. What does that Observer see? It of course depends on where the Earth's axis of rotation is located relative to our picture. Consider,



(8.8.43)

Frame S has basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. We now define Frame S' as having a new set of basis vectors as follows,

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_z(\varphi_1)R_x(\theta_1)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}) \qquad \text{for example} \qquad \hat{\mathbf{z}}' = R_z(\varphi_1)R_x(\theta_1)\hat{\mathbf{z}}$$

In our Passive View discussion of Section 1.3, we think of basis vectors back-rotated $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$, so we shall then define $R_1^{-1} = R_z(\varphi_1)R_x(\theta_1)$ and then we have $(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_1^{-1}(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$. The matrix of interest is shown in (E.2.2) so in Frame S components we may write,

$$\hat{\mathbf{z}}' = R_1^{-1}\,\hat{\mathbf{z}} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\varphi_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta_1\cos\varphi_1 \\ \sin\theta_1\sin\varphi_1 \\ \cos\theta_1 \end{pmatrix}. \qquad (8.8.44)$$

We now assume that the Earth rotates about this new $\hat{\mathbf{z}}'$ at rate $\varphi' = \omega t$.

As noted in (1.3.3), if a Kinematic Vector $\mathbf{V}$ has components V_i in Frame S, then it has components $(V)'_i = R_1 V_i$ in Frame S', which we write in vector notation as $(\mathbf{V})' = R_1\mathbf{V}$. Applying to $\mathbf{V} = \mathbf{r}$ we find, for the position of some point in the two frames,

$$(\mathbf{r})' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_1\,\mathbf{r} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & \cos\theta_1\sin\varphi_1 & -\sin\theta_1 \\ -\sin\varphi_1 & \cos\varphi_1 & 0 \\ \sin\theta_1\cos\varphi_1 & \sin\theta_1\sin\varphi_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} . \qquad (8.8.45a)$$

Since $R_1^{-1} = R_1^T$, the matrix appearing here is just the transpose of that appearing in (8.8.44). Inverting one gets,

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_1^{-1}\,(\mathbf{r})' = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\varphi_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} . \qquad (8.8.45b)$$

Suppose we define spherical coordinates in Frame S and Frame S' as so that (since a rotation, r' = r),

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\cos\varphi \\ r\sin\theta\sin\varphi \\ r\cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} r\sin\theta'\cos\varphi' \\ r\sin\theta'\sin\varphi' \\ r\cos\theta' \end{pmatrix} . \qquad (8.8.46)$$

Then (8.8.45b) says, cancelling the r factors,

$$\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\varphi_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \sin\theta'\cos\varphi' \\ \sin\theta'\sin\varphi' \\ \cos\theta' \end{pmatrix} . \qquad (8.8.47)$$

The last of these three equations reads

$$\cos\theta = -\sin\theta_1\sin\theta'\cos\varphi' + \cos\theta_1\cos\theta'$$

$$= \cos\theta_1\cos\theta' - \sin\theta_1\sin\theta'\cos(\omega t) \qquad (8.8.48a)$$

$$= \cos\theta_1\sin\theta'_L - \sin\theta_1\cos\theta'_L\cos(\omega t) \quad . \qquad (8.8.48b)$$

In (8.8.48b) we replace colatitude θ' by latitude $\theta'_L = \pi/2 - \theta$ which causes sin↔cos.

Recall the equation of the water surface from (8.8.33),

$$r(\theta) = R_2 + a\cos 2\theta \quad . \qquad a > 0 \qquad \textit{(8.8.33)}$$

This implies a tide height of

$$h(\theta) = a\cos 2\theta = 2\cos^2\theta - 1 \quad . \qquad (8.8.49)$$

Therefore on our idealized Earth which rotates about an axis (θ_1,φ_1) relative to Fig (8.8.43) we obtain the following tide height during the day

$$h(t) = a\cos 2\theta(t) = a\,[\,2\cos^2\theta(t) - 1\,]$$

$$= a\,[\;2\,(\cos\theta_1\cos\theta' - \sin\theta_1\sin\theta'\cos(\omega t))^2 - 1\;] \qquad (8.8.50a)$$

$$a\,[\;2\,(\cos\theta_1\sin\theta'_L - \sin\theta_1\cos\theta'_L\cos(\omega t))^2 - 1\;] \qquad (8.8.50b)$$

The tide height does not depend on the angle φ_1 because z is a symmetry axis of Fig (8.8.43) and the tides are thus symmetrical about this axis.

Example 1: Consider the case $\hat{\mathbf{z}}' = \hat{\mathbf{y}}$.

Suppose the Earth's rotation axis were in the $\hat{\mathbf{y}}$ direction in Fig (8.8.43) (pointing out of the plane of paper). In that case one has $\theta_1 = \pi/2$ and $\varphi_1 = \pi/2$, since

$$\hat{\mathbf{z}}' = R_z(\pi/2)\,R_y(\pi/2)\,\hat{\mathbf{z}} = \begin{pmatrix} \sin\theta_1\cos\varphi_1 \\ \sin\theta_1\sin\varphi_1 \\ \cos\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{\mathbf{y}} \;. \qquad (8.8.51)$$

Then from (8.8.50),

$$h(t) = a\,[\;2(\cos\theta_1\cos\theta' - \sin\theta_1\sin\theta'\cos(\omega t))^2 - 1\;]$$

$$= a\,[\;2(0\cos\theta' - 1\sin\theta'\cos(\omega t))^2 - 1\;]$$

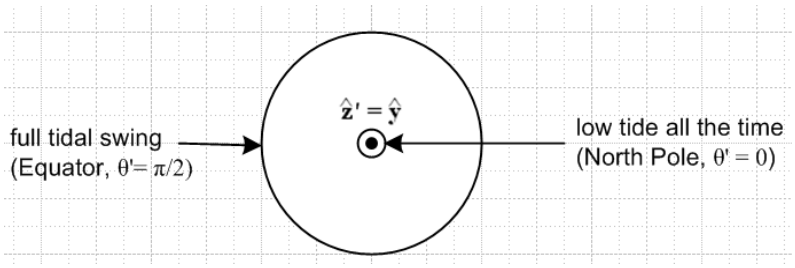
$$= a\,[\;2(-\sin\theta'\cos(\omega t))^2 - 1\;]$$

$$= a\,[\;2\sin^2\theta'\cos^2\omega t - 1\;]\,. \qquad (8.8.52)$$

If the Observer were at the Earth's equator $\theta' = \pi/2$ (which is in the plane of paper of Fig (8.8.43)), one would have

$$h(t) = a\,[\;2\cos^2\omega t - 1\;] = a\cos(2\omega t)\,.$$

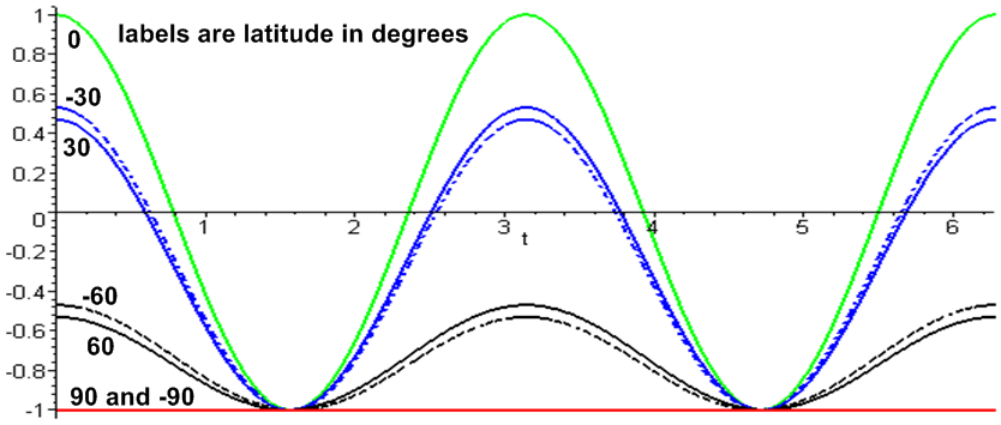
Comparing to (8.8.49), one then has $\theta = \omega t$. The Observer sees a full amplitude swing of $\pm a$ in the tide. As this Observer moves toward the pole so θ' decreases, the amplitude of the tide decreases as (8.8.52) shows. At the pole, where $\theta' = 0$, one finds $h(t) = -a$ (a constant) all the time, which seems reasonable since $\theta = \pi/2$ all the time and so $h(\theta) = a\cos 2\theta = a\cos\pi = -a$, as depicted below,



(8.8.53)

Here is a Maple rendition of this Example 1 where we set a = 1 and ω = 1 so one day lasts T = 2π. In this code we use θ'_L = lat and the first equation is (8.8.48b) :

```
costheta := cos(theta1)*sin(lat) - sin(theta1)*cos(lat)*cos(omega*t):
theta1 := evalf((Pi/180)*89):  a := 1:   omega := 1:
h := a * (2*costheta^2-1):  # tidal height
set1 := evalf((Pi/180)*[90,60,30,0]): # northern hemisphere, solid lines
p1 := plot([seq(h(t),lat = set1)],t=0..2*Pi,color
=[red,black,blue,green],thickness=2,linestyle=1):
set2 := evalf((Pi/180)*[-90,-60,-30]):# southern hemisphere, dashed lines
p2 := plot([seq(h(t),lat = set2)],t=0..2*Pi,color
=[red,black,blue],thickness=2,linestyle=4):
display(p1,p2);
```



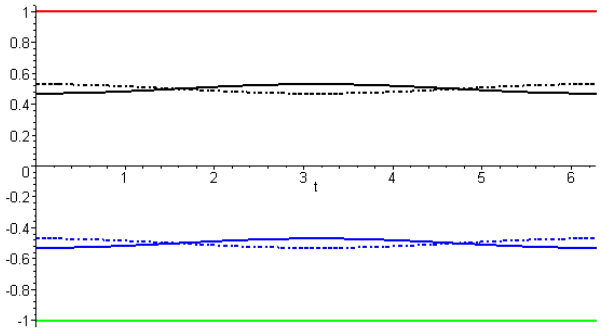
(8.8.54)

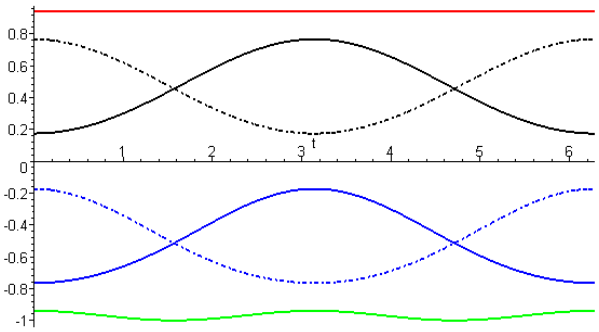
The solid lines are the equator (green) and northern hemisphere latitudes 30° (blue), 60° (black) and 90° (red); the dashed lines are southern hemisphere latitudes -30° (blue), -60°(black), and -90° (red). We set θ_1 = 89° instead of 90° to pull apart the solid and dashed lines a bit. At θ'_L = +90° (north pole) and -90° (south pole), the red solid and red dashed lines always coincide. The lines are horizontal because at each pole, there is no motion at all as the Earth rotates. The lines coincide because the tides are azimuthally symmetric about the symmetry axis as shown in (8.8.27) so both poles see the same h = -a = -1.

Example 2: Start with $\hat{\mathbf{z}}' = \hat{\mathbf{z}}$ and then slowly increase θ_1 from 0 to π/2 so $\hat{\mathbf{z}}'$ tips away from $\hat{\mathbf{z}}$.

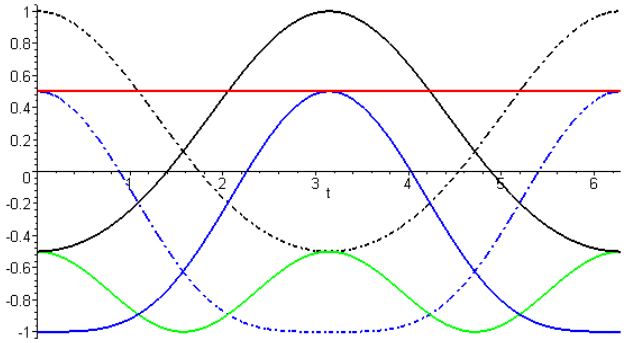
Here we show a set of plots from the above code for various values of the tilt θ_1. At θ_1 = 0 we of course expect there to be no tides at all again due to the azimuthal symmetry of the tides about the z axis.

θ_1= 1° θ_1= 10°

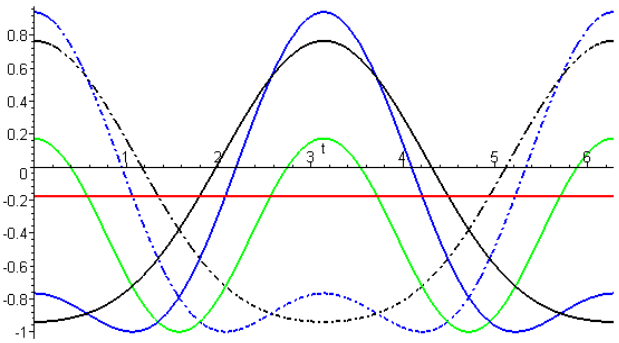




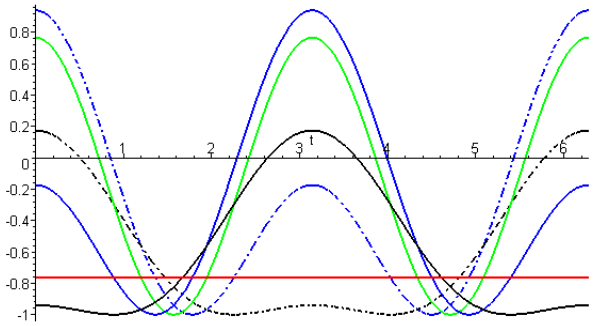
$\theta_1 = 30°$



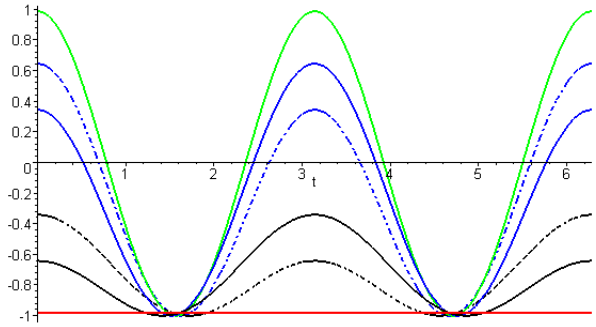
$\theta_1 = 50°$



$\theta_1 = 70°$



$\theta_1 = 85°$

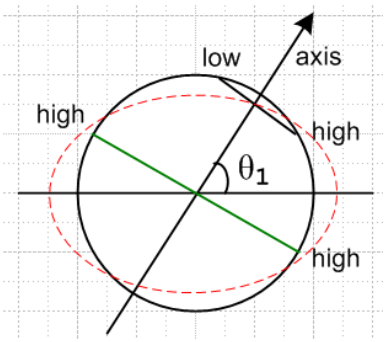


(8.8.55)

Equation (8.8.50b) shows that the curves for ± mirror latitudes have the same shape but are time-shifted half a day, and the figures bear this out.

For $\theta_1 = 90° + \Delta$ the solid and dashed curves are swapped compared to $\theta_1 = 90° - \Delta$ so we don't show plots for $\theta_1 > 90°$.

As an aid to interpreting the above curves, consider this drawing with $\theta_1 = 45°$.



(8.8.56)

• At the poles the tides are constant (red horizontal lines in plots) and decrease as θ_1 is increased.

• At the equator there will always be two equal high tides during each day, as marked by the green equatorial line. These become more pronounced as θ_1 increases.

• At high latitudes (as shown in black), we expect only one high tide per day.

• At some latitude there must be a gradual transition from two tide cycles per day at the equator to one tide cycle per day at high latitudes. For example, our $\theta_1 = 50°$ plot shows that at the low latitudes $\pm 30°$ (blue) there are two tide cycles (albeit quite unequal) while at $\pm 60°$ latitudes (black) there is only one tide cycle.

• As we shall see below, the real Earth's θ_1 lies in the rough range $60° \leq \theta_1 \leq 120°$ so the two-tide pattern is more predominant, as in the $\theta_1 = 85°$ plot.