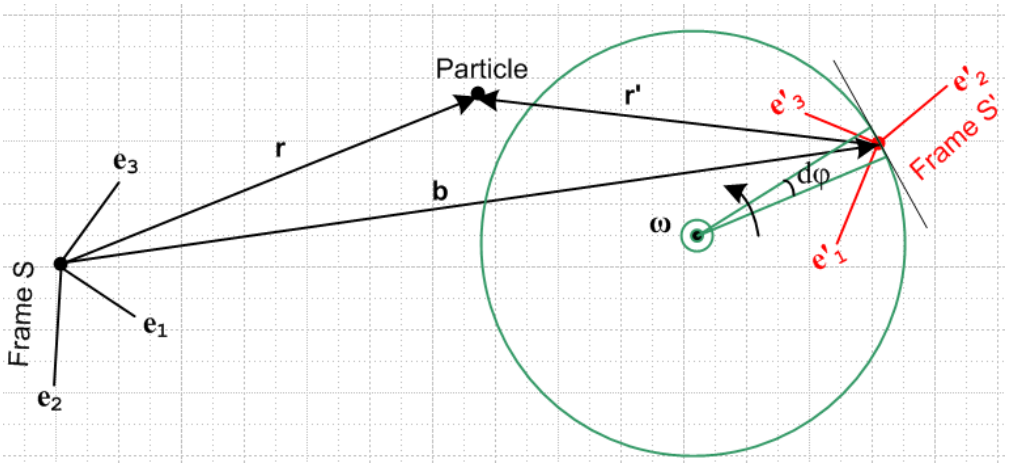
12.1 Summary of the Forward Problem equations (non-swap notation)

We now summarize the results of Sections 6, 7, 8 and 11. The first set of equations below is valid regardless of whether either of these frames is inertial (they could both be non-inertial). The second set of equations involving fictitious forces and torques assumes that Frame S is inertial (and Frame S' is not).



(12.1.1)

Definitions and Equations

r, **v**, **a**	position, natural velocity and natural acceleration in Frame S		(12.1.2)
r', **v'**, **a'**	position, natural velocity and natural acceleration in Frame S'		
ω	angular velocity of Frame S' relative to Frame S		
b	vector directed from origin of Frame S to origin of Frame S'		
$\mathbf{r} = \mathbf{b} + \mathbf{r}'$		*(6.1)*	(a)
$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S$		*(6.6a)*	(b)
$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}$		*(6.6c)*	(c)
$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S$		*(7.6a)*	(d)
S S' Euler Coriolis centripetal frame			
$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S'} + \ddot{\mathbf{b}}_{S'}$		*(7.6b)*	(e)
$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [(\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]$		*(11.2.14)*	(f)
$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} + m(\mathbf{r}'-\mathbf{c}') \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S]$ $- m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + m\,\dot{\mathbf{c}}' \times \mathbf{v}'$		*(11.2.15)*	(g)

Fictitious Forces and Torques (Section 8 and 11)

$\mathbf{F} = m\mathbf{a}$	$\mathbf{N}^{(c)} = \dot{\mathbf{L}}^{(c)}$	// true law	
$\mathbf{F'}_{\mathtt{eff}} = m\mathbf{a'}$	$\mathbf{N'}_{\mathtt{eff}}{}^{(c')} = \dot{\mathbf{L}}'^{(c')}$	// fake law	
$\mathbf{F'}_{\mathtt{eff}} = \mathbf{F} + \mathbf{F'}_{\mathtt{fict}}$	$\mathbf{N'}_{\mathtt{eff}}{}^{(c')} = \mathbf{N}^{(c)} + \mathbf{N'}_{\mathtt{fict}}{}^{(c')}$		
$\mathbf{F'}_{\mathtt{fict}} = m\mathbf{a'} - m\mathbf{a} = \dot{\mathbf{p}}' - \dot{\mathbf{p}}$	$\mathbf{N'}_{\mathtt{fict}}{}^{(c')} = \dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)}$.		(12.1.3)

For the general case, the fictitious forces can be expressed as

$$\mathbf{F'}_{\mathtt{fict}} = -m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \quad . \qquad \textit{(8.1.8)} \qquad (12.1.4)$$

frame centrifugal Coriolis Euler

For Special Case # 1 problems ($\boldsymbol{\omega}$ axis passes through Frame S origin), we have

$$\mathbf{F'}_{\mathtt{fict}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad . \qquad \text{Special Case \#1} \qquad \textit{(8.4.2)} \qquad (12.1.5)$$

centrifugal Coriolis Euler

The fictitious torques may be written,

$$\mathbf{N'}^{(c')}{}_{\mathtt{fict}} = (\mathbf{r'}-\mathbf{c'}) \times \mathbf{F'}_{\mathtt{fict}} + m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v}' \qquad \textit{(11.3.8)}$$

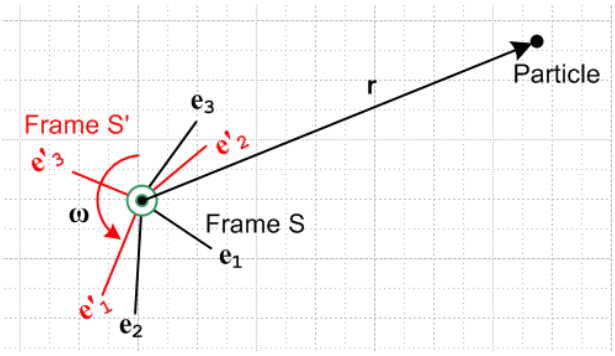
or

$$\mathbf{N'}^{(c')}{}_{\mathtt{fict}} = -(\mathbf{r'}-\mathbf{c'}) \times [\, m\ddot{\mathbf{b}}_S + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}'] \qquad (12.1.6)$$

$$+ m(\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - m\dot{\mathbf{c}}' \times \mathbf{v}' \qquad \textit{(11.3.10)}$$

Special Case #4 (Forward Problem, non-swap notation)

If the origins of Frame S and Frame S' coincide, then $\mathbf{b} = 0$. In this case one of course has $\mathbf{r} = \mathbf{r}'$ and also $\mathbf{c} = \mathbf{c}' = 0$. We then write $\mathbf{L} \equiv \mathbf{L}^{(0)}$, $\mathbf{L}' \equiv \mathbf{L}'^{(0)}$, $\mathbf{N} \equiv \mathbf{N}^{(0)}$ and $\mathbf{N}' \equiv \mathbf{N}'^{(0)}$. The new drawing is this,



(12.1.7)

We have placed the common origin in the plane of paper and are viewing things from a direction which causes the instantaneous **ω** vector to point toward the viewer. Frame S' is rotating relative to Frame S at rate **ω**. The vector **r** in general is not in the plane of paper. Here are the simplified equations obtained from those above with $\mathbf{r} = \mathbf{r}'$, $\mathbf{b} = \dot{\mathbf{b}}_S = \dot{\mathbf{b}}_{S'} = \ddot{\mathbf{b}}_S = \mathbf{c} = \mathbf{c}' = 0$:

r, **v**, **a**	position, natural velocity and natural acceleration in Frame S	(12.1.8)
r, **v'**, **a'**	position, natural velocity and natural acceleration in Frame S'	
ω	angular velocity of Frame S' relative to Frame S	
$\mathbf{r} = \mathbf{r}'$		(a)

$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'$ ALL SPECIAL CASE #4 (b) = (c)

$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$ (d) = (e)

S S' Euler Coriolis centripetal

$\mathbf{L} = \mathbf{L}' + m\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}')$ (f)

$\dot{\mathbf{L}} = \dot{\mathbf{L}}' + m\mathbf{r}' \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')\,]$ (g)

For the following items, Frame S is inertial and Frame S' is rotating

$\mathbf{F} = m\mathbf{a}$		$\mathbf{N} = \dot{\mathbf{L}}$	// true law
$\mathbf{F}'_{\mathtt{eff}} = m\mathbf{a}'$		$\mathbf{N}'_{\mathtt{eff}} = \dot{\mathbf{L}}'$	// fake law
$\mathbf{F}'_{\mathtt{eff}} = \mathbf{F} + \mathbf{F}'_{\mathtt{fict}}$		$\mathbf{N}'_{\mathtt{eff}} = \mathbf{N} + \mathbf{N}'_{\mathtt{fict}}$	
$\mathbf{F}'_{\mathtt{fict}} = m\mathbf{a}' - m\mathbf{a} = \dot{\mathbf{p}}' - \dot{\mathbf{p}}$		$\mathbf{N}'_{\mathtt{fict}} = \dot{\mathbf{L}}' - \dot{\mathbf{L}}$	(12.1.9)

$\mathbf{F}'_{\mathtt{fict}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}'$ (12.1.10)

centrifugal Coriolis Euler

$\mathbf{N}'_{\mathtt{fict}} = \mathbf{r}' \times \mathbf{F}'_{\mathtt{fict}} = -\,\mathbf{r}' \times [\,m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}']$ (12.1.11)