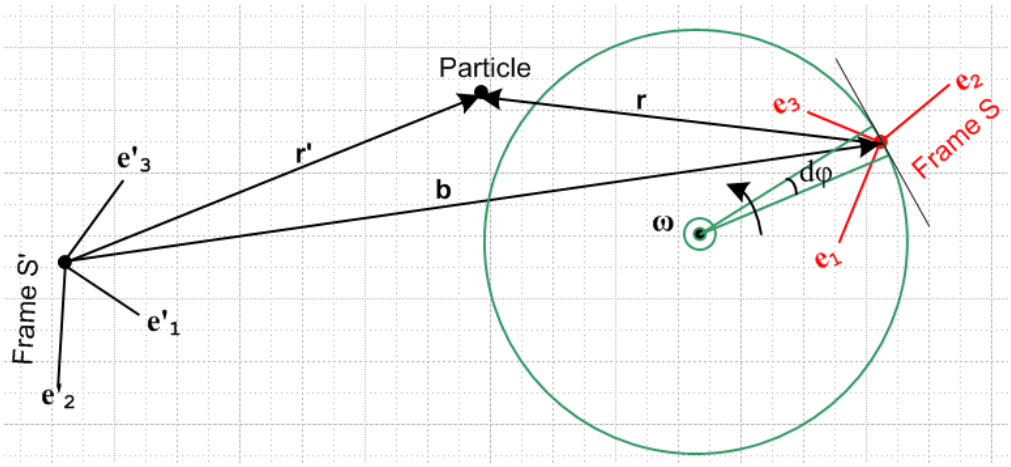
12.2 Summary of the Forward Problem equations (swap notation)

We are now going to translate everything in Section 12.1 into swap notation using rules (13.2.2).
Fig (12.2.1) is the same as Fig (12.1.1) but the primes are swapped with no-primes.

The first set of equations below is valid regardless of whether either of these frames is inertial (they could both be non-inertial). The second set of equations involving fictitious forces assumes that Frame S' is inertial (and Frame S is not).



(12.2.1)

Definitions and Equations

r', **v'**, **a'**	position, natural velocity and natural acceleration in Frame S'		(12.2.2)
r, **v**, **a**	position, natural velocity and natural acceleration in Frame S		
ω	angular velocity of Frame S relative to Frame S'		
b	vector directed from origin of Frame S' to origin of Frame S		

$$\mathbf{r'} = \mathbf{b} + \mathbf{r} \qquad (6.1)_S \qquad \text{(a)}$$

$$\mathbf{v'} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'} \qquad (6.6a)_S \qquad \text{(b)}$$

$$\mathbf{v'} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_{S} \qquad (6.6c)_S \qquad \text{(c)}$$

$$\mathbf{a'} = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \ddot{\mathbf{b}}_{S'} \qquad (7.6a)_S \qquad \text{(d)}$$

S' S Euler Coriolis centripetal frame

$$\mathbf{a'} = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + 2\boldsymbol{\omega} \times \dot{\mathbf{b}}_{S} + \ddot{\mathbf{b}}_{S} \qquad (7.6b)_S \qquad \text{(e)}$$

$$\mathbf{L'}^{(c')} = \mathbf{L}^{(c)} + m(\mathbf{r}-\mathbf{c}) \times [\,(\boldsymbol{\omega} \times \mathbf{r}) + \dot{\mathbf{b}}_{S'}\,] \qquad (11.2.14)_S \qquad \text{(f)}$$

$$\dot{\mathbf{L}}'^{(c')} = \dot{\mathbf{L}}^{(c)} + m(\mathbf{r}-\mathbf{c}) \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \ddot{\mathbf{b}}_{S'}\,]$$
$$- m(\dot{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c} + \dot{\mathbf{b}}_{S'}) \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}) + m\,\dot{\mathbf{c}} \times \mathbf{v} \qquad (11.2.15)_S \qquad \text{(g)}$$

Fictitious Forces and Torques (Section 8 and 11)

$\mathbf{F}' = m\mathbf{a}'$	$\mathbf{N}'^{(c')} = \dot{\mathbf{L}}'^{(c')}$	// true law
$\mathbf{F}_{\mathtt{eff}} = m\mathbf{a}$	$\mathbf{N}_{\mathtt{eff}}^{(c)} = \dot{\mathbf{L}}^{(c)}$	// fake law
$\mathbf{F}_{\mathtt{eff}} = \mathbf{F}' + \mathbf{F}_{\mathtt{fict}}$	$\mathbf{N}_{\mathtt{eff}}^{(c)} = \mathbf{N}'^{(c')} + \mathbf{N}_{\mathtt{fict}}^{(c)}$	
$\mathbf{F}_{\mathtt{fict}} = m\mathbf{a} - m\mathbf{a}' = \dot{\mathbf{p}} - \dot{\mathbf{p}}'$	$\mathbf{N}_{\mathtt{fict}}^{(c)} = \dot{\mathbf{L}}^{(c)} - \dot{\mathbf{L}}'^{(c')}$.	(12.2.3)

For the general case, the fictitious forces can be expressed as

$$\mathbf{F}_{\mathtt{fict}} = -m\ddot{\mathbf{b}}_{S'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad (8.1.8)_{\mathtt{s}} \qquad (12.2.4)$$

frame centrifugal Coriolis Euler

For Special Case # 1 problems ($\boldsymbol{\omega}$ axis passes through Frame S' origin), we have

$$\mathbf{F}_{\mathtt{fict}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad \text{Special Case \#1} \qquad (8.4.2)_{\mathtt{s}} \qquad (12.2.5)$$

centrifugal Coriolis Euler

The fictitious torques may be written,

$$\mathbf{N}^{(c)}{}_{\mathtt{fict}} = (\mathbf{r}-\mathbf{c}) \times \mathbf{F}_{\mathtt{fict}} + m(\dot{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c} + \dot{\mathbf{b}}_{S'}) \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}) - m\dot{\mathbf{c}} \times \mathbf{v} \qquad (11.3.8)_{\mathtt{s}}$$

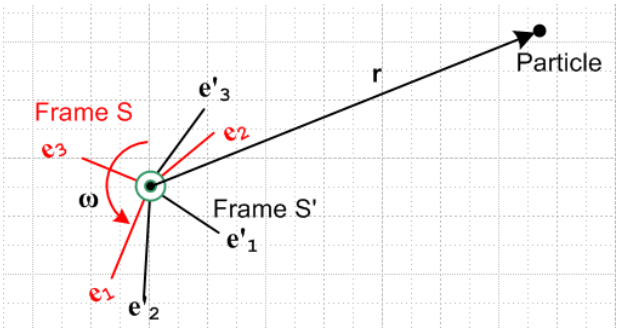
or

$$\mathbf{N}^{(c)}{}_{\mathtt{fict}} = -(\mathbf{r}-\mathbf{c}) \times [\, m\ddot{\mathbf{b}}_{S'} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r}] \qquad (12.2.6)$$

$$+ m(\dot{\mathbf{c}} + \boldsymbol{\omega} \times \mathbf{c} + \dot{\mathbf{b}}_{S'}) \times (\mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_{S'}) - m\dot{\mathbf{c}} \times \mathbf{v} \qquad (11.3.10)_{\mathtt{s}}$$

Special Case #4 (Forward Problem, swap notation)

If the origins of Frame S and Frame S' coincide, then $\mathbf{b} = 0$. In this case one of course has $\mathbf{r} = \mathbf{r}'$ and also $\mathbf{c} = \mathbf{c}'$ for the torque and angular momentum reference points. It is usual in this case to select $\mathbf{c} = \mathbf{c}' = 0$. We then write $\mathbf{L} \equiv \mathbf{L}^{(0)}$, $\mathbf{L}' \equiv \mathbf{L}'^{(0)}$, $\mathbf{N} \equiv \mathbf{N}^{(0)}$ and $\mathbf{N}' \equiv \mathbf{N}'^{(0)}$. The new drawing is this,



(12.2.7)

We have placed the common origin in the plane of paper and are viewing things from a direction which causes the instantaneous **ω** vector to point toward the viewer. Frame S is rotating relative to Frame S' at rate **ω**. The vector **r** in general is not in the plane of paper. Here are the simplified equations obtained from those above with $\mathbf{r} = \mathbf{r}'$, $\mathbf{b} = \dot{\mathbf{b}}_S = \dot{\mathbf{b}}_{S'} = \ddot{\mathbf{b}}_S = \mathbf{c} = \mathbf{c}' = 0$:

r, **v'**, **a'**	position, natural velocity and natural acceleration in Frame S'	(12.2.8)
r, **v**, **a**	position, natural velocity and natural acceleration in Frame S	
ω	angular velocity of Frame S relative to Frame S'	
$\mathbf{r}' = \mathbf{r}$		(a)

$\mathbf{v}' = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r}$ ALL SPECIAL CASE #4 (b) = (c)

$\mathbf{a}' = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ (d) = (e)

S' S Euler Coriolis centripetal

$\mathbf{L}' = \mathbf{L} + m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r})$ (f)

$\dot{\mathbf{L}}' = \dot{\mathbf{L}} + m\mathbf{r} \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\,]$ (g)

For the following items, Frame S' is inertial and Frame S is rotating

$\mathbf{F}' = m\mathbf{a}'$	$\mathbf{N}' = \dot{\mathbf{L}}'$	// true law	
$\mathbf{F}_{eff} = m\mathbf{a}$	$\mathbf{N}_{eff} = \dot{\mathbf{L}}$	// fake law	
$\mathbf{F}_{eff} = \mathbf{F}' + \mathbf{F}_{fict}$	$\mathbf{N}_{eff} = \mathbf{N}' + \mathbf{N}_{fict}$		
$\mathbf{F}_{fict} = m\mathbf{a} - m\mathbf{a}' = \dot{\mathbf{p}} - \dot{\mathbf{p}}'$	$\mathbf{N}_{fict} = \dot{\mathbf{L}} - \dot{\mathbf{L}}'$		(12.2.9)

$\mathbf{F}_{fict} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ (12.2.10)

centrifugal Coriolis Euler

$\mathbf{N}_{fict} = \mathbf{r} \times \mathbf{F}_{fict} = -\,\mathbf{r} \times [\,m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r}]$ (12.2.11)