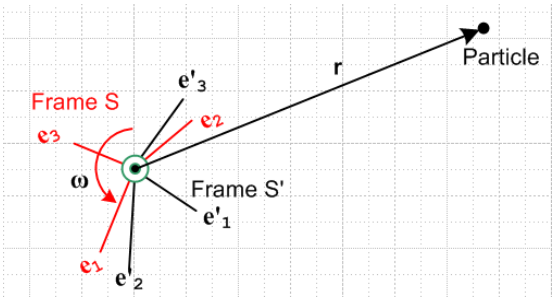
****

If the origins of Frame S' and Frame S are the same point, then **b** = 0. In this case one of course has **r'** = **r** and also **c'** = **c** for the torque and angular momentum reference points. It is usual in this case to simply select **c'** = **c** = 0. We then write $\mathbf{X'}^{(0)}$ and $\mathbf{X}^{(0)}$ as **X'** and **X**, where **X** = **L** or **N**. So we define our Special Case #4 to have **b** = **c'** = **c** = 0, which causes a simplification of many equations of Section 12.1. Here is the picture corresponding to Fig (12.1.1),



(12.4.1)

We have placed the common origin in the plane of paper and are viewing things from a direction which causes the instantaneous **ω** vector to point to the viewer, as in Fig (12.2.1). Frame S is rotating relative to Frame S' at rate **ω,** again just as in Fig (12.2.1). The vector **r** in general is not in the plane of paper. Here are the simplified equations obtained from Section 12.2 with **r** = **r'**, $\mathbf{b} = \dot{\mathbf{b}}_{S'} = \dot{\mathbf{b}}_{S} = \ddot{\mathbf{b}}_{S'} = \mathbf{c'} = \mathbf{c} = 0$:

r, **v**, **a**	position, natural velocity and natural acceleration in Frame S	(12.4.2)
r, **v'**, **a'**	position, natural velocity and natural acceleration in Frame S'	
ω	angular velocity of Frame S relative to Frame S'	

$$\mathbf{r'} = \mathbf{r} \qquad \text{(a)}$$

$$\mathbf{v'} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} \qquad \text{(b) = (c)}$$

$$\mathbf{a'} = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad \text{(d) = (e)}$$

S' S Euler Coriolis centripetal

$$\mathbf{L'} = \mathbf{L} + m\mathbf{r} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad \text{(f)}$$

$$\dot{\mathbf{L}}' = \dot{\mathbf{L}} + m\mathbf{r} \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \underset{\text{(g)}}{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}\,]$$

For the following items, Frame S' is inertial and Frame S is rotating

$\mathbf{F}_{\mathbf{eff}} = m\mathbf{a}$ // fake Newton's Law in rotating Frame S

$$\mathbf{F}_{\mathbf{eff}} = \mathbf{F'} + \mathbf{F}_{\mathbf{fict}} \; . \qquad (12.4.3)$$

$$\mathbf{F}_{\mathbf{fict}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \; . \qquad (12.4.4)$$

centrifugal Coriolis Euler

$$\mathbf{N}_{\mathbf{fict}} = \mathbf{r} \times \mathbf{F}_{\mathbf{fict}} = -\,\mathbf{r} \times [\,m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r}] \qquad (12.4.6)$$

****

As a reminder, we quote from above:

Swap Notation. The meaning of "swap notation" is that, in Fig 1, the vectors $\boldsymbol{\omega}$ and $\mathbf{b}$ stay put, but all other vectors undergo $\mathbf{V} \leftrightarrow \mathbf{V}'$. This latter group includes basis vectors $\mathbf{e}_i \leftrightarrow \mathbf{e}'_i$, $\mathbf{r} \leftrightarrow \mathbf{r}'$, $\mathbf{v} \leftrightarrow \mathbf{v}'$, $\mathbf{a} \leftrightarrow \mathbf{a}'$ and of course Frame S $\leftrightarrow$ Frame S'. This is nothing more than a change of the way things are labeled. If a non-swap notation equation has the number (x.x.x), then the corresponding equation in swap notation will be given the number $(x.x.x)_s$. *(8.5.10)*

****

Suppose the two frame origins line up so $\mathbf{b} \equiv 0$. And suppose then that both torques are taken relative to this common point. Then we also have $\mathbf{c} = 0$ and $\mathbf{c}' = 0$. The two equations above then become,

$$\mathbf{L}^{(0)} = \mathbf{L}'^{(0)} + m(\mathbf{r}') \times [(\boldsymbol{\omega} \times \mathbf{r}')] \qquad (11.2.14)$$

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} + m(\mathbf{r}') \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')]$$

$$(d\mathbf{L}/dt)_S = (d\mathbf{L}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{L} \quad ??$$

These last two lines do not seem consistent to me.

What is the meaning of the point r' here? Suppose it is a point in the rigid object. Then $\mathbf{v}' = 0$ since all points of a rigid object are at rest in frame S'. Then we have

$$\mathbf{L}^{(0)} = \mathbf{L}'^{(0)} + m\mathbf{r}' \times [(\boldsymbol{\omega} \times \mathbf{r}')]$$

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} + m\mathbf{r}' \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')]$$

Now I suppose if our point is fixed in Frame S', it has no $\mathbf{L}'^{(0)}$. Then we have

$$\mathbf{L}^{(0)} = m\mathbf{r}' \times [(\boldsymbol{\omega} \times \mathbf{r}')]$$

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} + m\mathbf{r}' \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')]$$

In this case I can write

$$\boldsymbol{\omega} \times \mathbf{L} = \boldsymbol{\omega} \times \mathbf{L}^{(0)} = \boldsymbol{\omega} \times [m\mathbf{r}' \times [(\boldsymbol{\omega} \times \mathbf{r}')] = m\,\boldsymbol{\omega} \times (\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}'))$$

This has a similarity to the non-Euler term

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} + m\mathbf{r}' \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'))$$

Can I show this to be true:

$$\mathbf{A} \times (\mathbf{B} \times (\mathbf{A} \times \mathbf{B})) = \mathbf{B} \times (\mathbf{A} \times (\mathbf{A} \times \mathbf{B})) \quad ?$$

I do know this

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\mathbf{b} - (\mathbf{a} \bullet \mathbf{b})\mathbf{c}$$

$$\mathbf{B} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \bullet \mathbf{B})\mathbf{A} - (\mathbf{B} \bullet \mathbf{A})\mathbf{B}$$

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \bullet \mathbf{B})\mathbf{A} - (\mathbf{A} \bullet \mathbf{A})\mathbf{B}$$

So I would have to show that

$$\mathbf{A} \times ((\mathbf{B} \bullet \mathbf{B})\mathbf{A} - (\mathbf{B} \bullet \mathbf{A})\mathbf{B}) = \mathbf{B} \times ((\mathbf{A} \bullet \mathbf{B})\mathbf{A} - (\mathbf{A} \bullet \mathbf{A})\mathbf{B}) \quad ?$$

$$\mathbf{A} \times (0 - (\mathbf{B} \bullet \mathbf{A})\mathbf{B}) = \mathbf{B} \times ((\mathbf{A} \bullet \mathbf{B})\mathbf{A}) \quad ?$$

$$-(\mathbf{B} \bullet \mathbf{A})\,\mathbf{A} \times \mathbf{B} = (\mathbf{A} \bullet \mathbf{B})\;\; \mathbf{B} \times \mathbf{A}$$

So it is true. I have just shown that

$$\boldsymbol{\omega} \times (\mathbf{r}' \times (\boldsymbol{\omega} \times \mathbf{r}')) = \mathbf{r}' \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'))$$

and therefore I have shown

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} + m\mathbf{r}' \times (\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}'))$$

$$\underset{\text{space}}{\dot{\mathbf{L}}^{(0)}} = \underset{\text{body}}{\dot{\mathbf{L}}'^{(0)}} + \boldsymbol{\omega} \times \mathbf{L}^{(0)}$$

$$(d\mathbf{L}/dt)_S = (d\mathbf{L}/dt)_{S'} + \boldsymbol{\omega} \times \mathbf{L}$$

So with all these simplifications, my result at least does replicate their result.
Their result however is true under ALL conditions!

Goldstein's approach seems so much simpler. My general result with origin reference points is,

$$\mathbf{L}^{(0)} = \mathbf{L}'^{(0)} + m(\mathbf{r}') \times [\,(\boldsymbol{\omega} \times \mathbf{r}')] \qquad (11.2.14)$$

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} + m(\mathbf{r}') \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')]$$

How is the second term here ever going to be ω x L since **L** does not include $\dot{\boldsymbol{\omega}}$???

and also

$$\mathbf{L}^{(0)} = \mathbf{r} \times m\mathbf{v} \qquad (1.9.4)$$

$$\dot{\mathbf{L}}^{(0)} = \mathbf{r} \times m\mathbf{a} \qquad (1.9.5)$$

$$\mathbf{L}'^{(0)} = \mathbf{r}' \times m\mathbf{v}' \qquad (1.9.6)$$

$$\dot{\mathbf{L}}'^{(0)} = \mathbf{r}' \times m\mathbf{a}' \; . \qquad (1.9.7)$$

Now we should still have

$$\dot{\mathbf{L}}^{(0)} = \dot{\mathbf{L}}'^{(0)} + \boldsymbol{\omega} \times \mathbf{L}^{(0)}$$

or

$$\mathbf{r} \times m\mathbf{a} = \mathbf{r}' \times m\mathbf{a}' + \boldsymbol{\omega} \times (\mathbf{r} \times m\mathbf{v})$$

or

$$\mathbf{r} \times \mathbf{a} = \mathbf{r}' \times \mathbf{a}' + \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v})$$

OK, now go back to summary results

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\, \boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}'$$

And let's see if we are OK

$$\mathbf{r} \times \mathbf{a} = \mathbf{r}' \times \mathbf{a}' + \boldsymbol{\omega} \times (\mathbf{r} \times \mathbf{v}) \qquad ??$$

$$\mathbf{r} \times [\, \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\, \boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')] = \mathbf{r}' \times \mathbf{a}' + \boldsymbol{\omega} \times (\mathbf{r} \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}']) \qquad ??$$

$$\mathbf{r}' \times [\, \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\, \boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')] = \boldsymbol{\omega} \times (\mathbf{r}' \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}']) \qquad ??$$

The Euler term is never going to work.