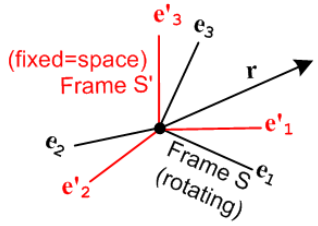
10. Notation comparison with Goldstein (1950) and Goldstein, Poole and Safko (2001)

In this Section we compare our notation for rotating-frame kinematics and non-inertial-frame physics to that of Goldstein (1950) and Goldstein, Poole and Safko (GFS 2001).

These books don't say much about the locations of the origins of the reference frames they use. The discussion of rotating frames and the G Rule appears in Goldstein Sections 4.8, 4.9 (GPS 4.9, 4.10). In Goldstein (p135) we are told that **r** is a "vector from the origin of the terrestrial system" and that "terrestrial measurements are usually made with respect to a coordinate system fixed *in* the earth, which therefore rotates with a constant angular velocity **ω** relative to *the inertial system*". Having studied their rotational equations, it is our conclusion that they are using our Special Case #1 of Section 4.4 where the rotation axis passes through the Frame S origin.

I am just trying to see if I can make if fly with Special Case #1 :

that their two frames of reference must have their origins co-sited at the center of the Earth,



(10.1)

More generally, Goldstein and GPS assume that **b** = 0 which means that the rotation axis passes through the origin of *both* frames. This then is the combination of our Special Case #1 and #2 which we shall call Special Case #4. When **b** = 0, we have **r** = **r'** so there is only one position vector to worry about. In applications the common origin is placed at a point in a rotating rigid object which is fixed in space.

Here are some of our equations simplified to Special Case #1 :

$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}$ *(6.11)*

$\mathbf{v}_S = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$. // Goldstein p 135 (4-104)

so assume $\mathbf{v} = \mathbf{v}_S$ and $\mathbf{v}' = \mathbf{v}_r$ and $\mathbf{r} = \mathbf{r}$ and insert into new table below

Next we have

$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ me *(7.6a)*

S S' Euler Coriolis centripetal

STOP. What (7.6a) says is this (and this is no-swap notation)

$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S$. (7.6a)

S S' Euler Coriolis centripetal frame

and I know in Special Case #1 that

$$\dot{\mathbf{b}}_S \equiv \boldsymbol{\omega} \times \mathbf{b} \qquad \Rightarrow \qquad \ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \dot{\boldsymbol{\omega}} \times \dot{\mathbf{b}}_S$$

Inserting this into the above gives

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\boldsymbol{\omega}} \times \mathbf{b} + \dot{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{b}) \quad .$$

$$= \mathbf{a}' + \dot{\boldsymbol{\omega}} \times (\mathbf{r}'+\mathbf{b}) + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{b})$$

$$= \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\boldsymbol{\omega}} \times (\boldsymbol{\omega} \times \mathbf{b})$$

Now Goldstein says this

$$\mathbf{a}_s = \mathbf{a}_r + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad \text{// Goldstein p 135 (4-105)}$$

We are in agreement as long as $\dot{\boldsymbol{\omega}} = 0$ where we associate $\mathbf{a}_r = \mathbf{a}'$ and $\mathbf{v}' = \mathbf{v}_r$ and $\mathbf{r}' = \mathbf{r}$. So my little table would have to say

our non-swap notation	Goldstein authors	(10.3)

velocity comparison

$\mathbf{v} = \mathbf{v}_s$ and $\mathbf{v}' = \mathbf{v}_r$ and $\mathbf{r} = \mathbf{r}$

$\mathbf{v}$	$\mathbf{v}_s$
$\mathbf{v}'$	$\mathbf{v}_r$
$\mathbf{r}$	$\mathbf{r}$

accel comparison

$\mathbf{a}$	$\mathbf{a}_s$
$\mathbf{v}'$	$\mathbf{v}_r$
$\mathbf{r}'$	$\mathbf{r}$

But we already have a contradiction. His r associates with but my r and my r', but that can only be the case for Special Case #4. So Special Case #1 does not fly with Goldstein, leave section as was!

$$m\mathbf{a}' = \mathbf{F}'_{\mathtt{eff}} = \mathbf{F} + \mathbf{F}'_{\mathtt{fict}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \textit{(8.1.4, 7, 8)}$$

We start our comparison with equation *(6.6c)* :

$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}$ *(6.6c)*

$\mathbf{v}_s = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r}$. // Goldstein p 135 (4-104)

// GPS p 175 (4.88)

The next comparison is (we add $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ to their equations),

$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ *(7.6a)*

$\mathbf{a}_s = \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$. // Goldstein p 135 (4-105)

// GPS p 175 (4.89)

And finally,

$m\mathbf{a}' = \mathbf{F}'_{\mathbf{eff}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ *(8.1.4, 7, 8)*

$m\mathbf{a}_r = \mathbf{F}_{\mathbf{eff}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$. // Goldstein p 135 (4-106,7)

// GPS p 175 (4.90,1)

Based on these comparisons, we construct the following translation table where Goldstein's rotating frame is associated with our Frame S' in our non-swap notation:

(10.3)

our non-swap notation	Goldstein authors	$\mathbf{b} = 0$ so $\mathbf{r} = \mathbf{r}'$
S'	r	name of the rotating frame (r = rotating or body)
S	s	name of the fixed frame (s = space)
$(d/dt)_{S'}$	$(d/dt)_r$	time derivative in the rotating frame
$(d/dt)_S$	$(d/dt)_s$	time derivative in the fixed frame
$\mathbf{r}'$	$\mathbf{r}$	position in rotating frame ($\mathbf{r}' = \mathbf{r}$)
$\mathbf{v}'$	$\mathbf{v}_r$	velocity in rotating frame
$\mathbf{a}'$	$\mathbf{a}_r$	acceleration in rotating frame
$\mathbf{r}$	$\mathbf{r}$	position in fixed frame
$\mathbf{v}$	$\mathbf{v}_s$	velocity in fixed frame (s = space)
$\mathbf{a}$	$\mathbf{a}_s$	acceleration in fixed frame
$\mathbf{F}$	$\mathbf{F}$	force in fixed frame = true force in rotating frame
$\mathbf{F}'_{\mathbf{eff}}$	$\mathbf{F}_{\mathbf{eff}}$	total effective force in the rotating frame
$\mathbf{F}_{\mathbf{fict}}$		total fictitious force in the rotating frame
$\mathbf{b}$	0	location of the rotating frame origin (measured in the fixed frame)
$\dot{\mathbf{b}}_{S'}$	0	velocity of the rotating frame origin (measured in the fixed frame)
$\ddot{\mathbf{b}}_{S'}$	0	acceleration of the rotating frame origin (measured in the fixed frame)

Since **b** = 0 is assumed, the Goldstein and GPS texts only treat a special case of the "rotating frames of reference" scenario we depict in Fig 1 in which **b**(t) is a general dynamic vector.

In the Goldstein and GPS discussion of the Euler angles (Section 4-4) their figure shows that the "space" frame has unprimed axes (our Frame S) and the "body" frame has primed axes (our Frame S'). The body frame is a frame that is fixed within the body of a rotating object like a top, while the space frame is inertial. For a top, the origins of both Frame S and Frame S' are co-sited at the non-moving tip of the rotating top.

However, in the Goldstein and GPS discussion of rigid body motion on Chapter 5, the authors switch to our swap notation so that now the rotating body-frame variables are Frame S (no primes). The same footnote (nearly) appears in both books which we quote, regarding the rotating body frame:

> *In Chapter 4, such a system was denoted by primes. As components along spatial axes are rarely used here, this convention will be dropped from now on to simplify the notation. Unless otherwise specified, all coordinates used for the rest of the chapter refer to systems fixed in the rigid body.

The phrase "spatial axes" means the axes of the "space" frame of reference.

For example, the rotating-frame **ω** components which were called $\omega_{x'}$, $\omega_{y'}$, $\omega_{z'}$ on page 134 4-103 (GPS p 174 4.87) are referred to in Chapter 5 as ω_x, ω_y, ω_z. As an exercise, we derive these components two ways in Appendix H.