Rotation and Tides PhL 1.19.17

Since the red ellipse of Fig ** and its equation (8.8.31) is based on the angle θ of Fig (8.8.17), the red ellipse is really an ellipsoid in a three-dimensional version of (8.8.32). If it happened that the axis of the Earth's rotation pointed toward the viewer in (8.8.18), one could calculate the intersection of the red ellipsoid with a line of latitude,

Plan B. The tide height is given by $r(\theta) = R_2 + a\cos 2\theta$. Think of θ as an angle of spherical coordinates where the z axis points to the right in Fig ***.

Imagine a different set of spherical angles θ', φ' based on a different axis $\hat{z}'$. How are these angles related to θ,φ? I think I can answer that question.

Let's assume that $\hat{\mathbf{z}}' = R\,\hat{\mathbf{z}}$ where R then defines your new $\hat{\mathbf{z}}'$ in Fig **. I think we then know that

$\mathbf{r}' = R^{-1}\mathbf{r}$ in Cartesian coordinates.

Start over. We have two frames

Frame S $\mathbf{r} = x,y,z$

Frame S' $\mathbf{r}' = x',y',z'$

For an active rotation R we would say $\mathbf{r}' = R\mathbf{r}$ and the axes stay fixed.

This is equivalent to a passive rotation of the axes "backward" by R^{-1}.

Start over. Start with north pole vector $\hat{\mathbf{z}}$. I want to rotate that vector in a simple manner so it lines up with some new vector $\hat{\mathbf{z}}'$. Here is an easy way to do that

$\hat{\mathbf{z}}' = R_z(\varphi_1)\, R_y(\theta_1)\, \hat{\mathbf{z}}$ as in (E.2.2)

Then

$$\hat{\mathbf{z}}' = \begin{pmatrix} \sin\theta_1\cos\varphi_1 \\ \sin\theta_1\sin\varphi_1 \\ \cos\theta_1 \end{pmatrix}$$

Now I want to define a set of coordinates with respect to this new $\hat{\mathbf{z}}'$axis. But I know from (E.2.2) that

$$\hat{\boldsymbol{\varphi}}' = \begin{pmatrix} -\sin\varphi_1 \\ \cos\varphi_1 \\ 0 \end{pmatrix} \qquad \hat{\boldsymbol{\theta}}' = \begin{pmatrix} \cos\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 \\ -\sin\theta_1 \end{pmatrix}$$

So I now know the spherical coordinates unit vectors in this new Frame S'. As we do this rotation, ANY point in Frame S gets rotated in this manner, so here is what happens to a general vector **r**,

$$\mathbf{r}' = R_{\mathbf{z}}(\varphi_1)\, R_{\mathbf{y}}(\theta_1)\, \mathbf{r}$$

which says from App E

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\varphi & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{with 1 subscripts}$$

$$\begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta & \sin\theta\cos\varphi \\ \cos\theta\sin\varphi & \cos\varphi & \sin\theta\sin\varphi \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Now I have it hand entered so I can then write

$$\mathbf{r}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_1\, \mathbf{r}$$

So this says have the Cartesian coordinates move, but how to the Spherical coordinates move??? If I cancel the radius R_2 I can write the above as

$$\begin{pmatrix} \sin\theta'\cos\varphi' \\ \sin\theta'\sin\varphi' \\ \cos\theta' \end{pmatrix} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix}$$

I am interested in a latitude line with θ' = constant and how this affects cosθ which in turn determines the tides. I would like to know the inverse of this matrix.

I guess I will just create a whole new Maple file to compute all this stuff. It is done. I can say then that

$$\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\varphi & \cos\theta\sin\varphi & -\sin\varphi \\ -\sin\varphi & \cos\varphi & 0 \\ \sin\theta\cos\varphi & \sin\theta\sin\varphi & \cos\theta \end{pmatrix} \begin{pmatrix} \sin\theta'\cos\varphi' \\ \sin\theta'\sin\varphi' \\ \cos\theta' \end{pmatrix} \qquad \text{with 1's added, so}$$

$$\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & \cos\theta_1\sin\varphi_1 & -\sin\theta_1 \\ -\sin\varphi_1 & \cos\varphi_1 & 0 \\ \sin\theta_1\cos\varphi_1 & \sin\theta_1\sin\varphi_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \sin\theta'\cos\varphi' \\ \sin\theta'\sin\varphi' \\ \cos\theta' \end{pmatrix} \qquad \text{with 1's added, so}$$

I then know that

$$\cos\theta = \sin\theta_1\cos\varphi_1\sin\theta'\cos\varphi' + \sin\theta_1\sin\varphi_1\sin\theta'\sin\varphi' + \cos\theta_1\cos\theta'$$

Now suppose we are at a fixed latitude and $\varphi' = \omega t$ for earth rotation. Then

$$\cos\theta(t) = \sin\theta_1\cos\varphi_1\sin\theta'\cos\omega t + \sin\theta_1\sin\varphi_1\sin\theta'\sin\omega t + \cos\theta_1\cos\theta'$$

and this tells how $\theta(t)$ varies during the day for latitude line θ' on the true Earth

Meanwhile

$$\cos 2\theta = 2\cos^2\theta - 1$$

So

$$r(\theta) = R_2 + a\cos 2\theta$$

$$= R_2 + a\,(\,2\cos^2\theta - 1\,)$$

$$= R_2 + a\,(\,2[\sin\theta_1\cos\varphi_1\sin\theta'\cos\omega t + \sin\theta_1\sin\varphi_1\sin\theta'\sin\omega t + \cos\theta_1\cos\theta']^2 - 1\,)$$

More interesting to just do the height of the tide effect

$$h(t) = a\cos 2\theta = a\,(\,2\cos^2\theta - 1\,)$$

$$= a\,(\,2[\sin\theta_1\cos\varphi_1\sin\theta'\cos\omega t + \sin\theta_1\sin\varphi_1\sin\theta'\sin\omega t + \cos\theta_1\cos\theta']^2 - 1\,)$$

Example: Suppose $\hat{\mathbf{z}}'$ points out of the plane of paper in Fig *** so

$$\hat{\mathbf{z}}' = R_z(\varphi_1)\,R_y(\theta_1)\,\hat{\mathbf{z}} = R_z(0)\,R_y(\pi/2)\,\hat{\mathbf{z}}$$

Then

$$\cos\theta(t) = \sin\theta_1\cos\varphi_1\sin\theta'\cos\omega t + \sin\theta_1\sin\varphi_1\sin\theta'\sin\omega t + \cos\theta_1\cos\theta'$$

$$= \sin\theta'\cos\omega t$$

Then we get

$$h(t) = a\,(\,2\cos^2\theta - 1\,) = a\,(\,2\sin^2\theta'\cos^2\omega t - 1\,)$$

If $\theta' = \pi/2$ so we are at the equator , then get $\cos\theta = \cos\omega t$ so simply $\theta = \omega t$ and

$$h(t) = a\cos 2\theta = a\cos(2\omega t)$$

and we get maximum tides.

As we move to the pole, $\theta' \rightarrow 0$, we get

$h(t) = -a$

and we just sit at low tide all the time.