Tide potential Calculation PhL 1.18.17

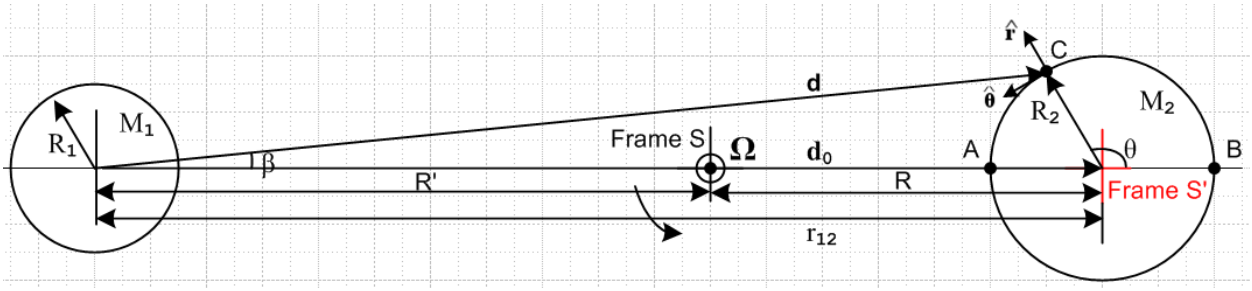
Start with

$$\mathbf{F}_{tid} = mM_1G\left(\hat{\mathbf{d}}_0/d_0^2 - \hat{\mathbf{d}}/d^2\right)$$

Omit the leading constant since it won't matter

$$\mathbf{Q} = \left(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\right)$$

and here is our picture



In Cartesian coordinates we can write

$$d_x = d_0 + R_2\cos\theta \qquad d_y = R_2\sin\theta$$

$$\mathbf{d} = (d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}$$

$$\mathbf{d}_0 = d_0\hat{\mathbf{x}}$$

Here I spend a lot of time trying to get the potential by doing line integration of the force.

The potential is

$$-V = \int_0^\theta \mathbf{Q}\bullet d\mathbf{s} \qquad d\mathbf{s} = R_2 d\theta\,\hat{\boldsymbol{\theta}}$$

then

$$-V = \int_0^\theta \left(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\right) R_2 d\theta\,\hat{\boldsymbol{\theta}}$$

$$= \int_0^\theta \left(d_0\hat{\mathbf{x}}/d_0^3 - [(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}]/d^3\right)\bullet R_2 d\theta\,\hat{\boldsymbol{\theta}}$$

$$= \int_0^{\theta} \{[\, 1/d_0^2 - (d_0 + R_2 \cos\theta)/d^3 \,]\hat{\mathbf{x}} + [R_2 \sin\theta/d^3]\,\hat{\mathbf{y}}\} \bullet R_2 d\theta\, \hat{\boldsymbol{\theta}}$$

$$= R_2 \int_0^{\theta} \{[\, 1/d_0^2 - (d_0 + R_2 \cos\theta)/d^3 \,]\hat{\mathbf{x}} \bullet \hat{\boldsymbol{\theta}} + [R_2 \sin\theta/d^3]\,\hat{\mathbf{y}} \bullet \hat{\boldsymbol{\theta}}\} d\theta$$

$$= R_2 \int_0^{\theta} \{[\, 1/d_0^2 - (d_0 + R_2 \cos\theta)/d^3 \,](-\sin\theta) + [R_2 \sin\theta/d^3] \cos\theta\} d\theta \qquad \text{// (E.5.5)}$$

$$= R_2 \int_0^{\theta} \{ -\sin\theta/d_0^2 + \sin\theta d_0/d^3 + R_2 \sin\theta\cos\theta/d^3 + R_2 \sin\theta\cos\theta/d^3 \}\, d\theta$$

$$= R_2 \int_0^{\theta} \{ -\sin\theta/d_0^2 + \sin\theta d_0/d^3 + 2R_2 \sin\theta\cos\theta/d^3 \}\, d\theta$$

$$= R_2 \int_0^{\theta} \{ -\sin\theta/d_0^2 + \sin\theta d_0/d^3 + R_2 \sin 2\theta/d^3 \}\, d\theta$$

Now write

$$d^2 = d_0^2 + R_2^2 - 2d_0R_2\cos(\pi-\theta) = d_0^2 + R_2^2 + 2d_0R_2\cos\theta = d_0^2[\, 1 + (R_2/d_0)^2 + 2\,(R_2/d_0)\cos\theta \,]$$

$$\approx d_0^2[\, 1 + 2\,(R_2/d_0)\cos\theta \,]$$

$$d \approx d_0 [\, 1 + 2(R_2/d_0)\cos\theta]^{1/2}$$

$$d^{-2} \approx d_0^{-2} [\, 1 + 2(R_2/d_0)\cos\theta]^{-1} \approx d_0^{-2} [\, 1 - 2(R_2/d_0)\cos\theta \,]$$

$$d^{-3} \approx d_0^{-3} [\, 1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3} [\, 1 - 3(R_2/d_0)\cos\theta \,]$$

Then have

$$= R_2 \int_0^{\theta} \{ -\sin\theta/d_0^2 + \sin\theta d_0/d^3 + R_2 \sin 2\theta/d^3 \}\, d\theta$$

$$= R_2 \int_0^{\theta} \{-\sin\theta/d_0^2 + \sin\theta d_0 * d_0^{-3} [\, 1 - 3(R_2/d_0)\cos\theta \,] + R_2 \sin 2\theta * d_0^{-3} [\, 1 - 3(R_2/d_0)\cos\theta \,]\} d\theta$$

$$= (R_2/d_0^3) \int_0^{\theta} \{-\sin\theta d_0 + \sin\theta d_0 [\, 1 - 3(R_2/d_0)\cos\theta \,] + R_2 \sin 2\theta [\, 1 - 3(R_2/d_0)\cos\theta \,]\} d\theta$$

$$= (R_2/d_0^3) \int_0^{\theta} \{ \sin\theta d_0 [- 3(R_2/d_0)\cos\theta \,] + R_2 \sin 2\theta [\, 1 - 3(R_2/d_0)\cos\theta \,]\} d\theta$$

$$= (R_2/d_0^3) \int_0^\theta \{ -3R_2\sin\theta\cos\theta + R_2\sin2\theta - 3R_2(R_2/d_0)\sin2\theta\cos\theta \} d\theta$$

$$= (R_2^2/d_0^3) \int_0^\theta \{ -3\sin\theta\cos\theta + \sin2\theta - 3(R_2/d_0)\sin2\theta\cos\theta \} d\theta$$

$$= (R_2^2/d_0^3) \int_0^\theta \{ -(3/2)\sin2\theta + \sin2\theta - 3(R_2/d_0)\sin2\theta\cos\theta \} d\theta$$

$$= (R_2^2/d_0^3) \int_0^\theta \{ -(1/2)\sin2\theta - 3(R_2/d_0)\sin2\theta\cos\theta \} d\theta$$

$$\approx (R_2^2/d_0^3) \int_0^\theta \{ -(1/2)\sin2\theta \} d\theta$$

$$= (-1/2)(R_2^2/d_0^3) \int_0^\theta \sin2\theta d\theta$$

$$= (-1/2)(R_2^2/d_0^3)(1/2)[1 - \cos2\theta]$$

Putting back the constant factor, here is my result for the potential function which goes with the tidal force

$$V(\theta) = +(1/4)\, mM_1G\,(R_2^2/d_0^3)[1 - \cos2\theta]$$

Butakov gives

$$R(\theta) = R_2 + a\cos(2\theta)\,, \qquad (8.7.27)$$

$$a = (3/4)R_2\,(M_1/M_2)(R_2/r_{12})^3 \quad . \qquad (8.7.28)$$

Let's go back to the start,

$$\mathbf{F}_{tid} = mM_1G\,(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3)$$

$$\mathbf{d} = (d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}$$
$$\mathbf{d}_0 = d_0\hat{\mathbf{x}}$$

$$\mathbf{F}_{tid} = mM_1G\,(d_0\hat{\mathbf{x}}/d_0^3 - [(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}]/d^3)$$

now use

$$d^{-3} \approx d_0^{-3}[1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3}[1 - 3(R_2/d_0)\cos\theta]$$

so then

$$\mathbf{F}_{tid} = mM_1G\left(d_0\hat{\mathbf{x}}/d_0^3 - \left[(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}} \right] \right) d_0^{-3}\left[1 - 3(R_2/d_0)\cos\theta \right]$$

$$= (mM_1G/d_0^3)\left(d_0\hat{\mathbf{x}} - \left[(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}} \right] \left[1 - 3(R_2/d_0)\cos\theta \right] \right)$$

$$= (mM_1G/d_0^2)\left(\hat{\mathbf{x}} - \left[(1 + (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} + (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} \right] \left[1 - 3(R_2/d_0)\cos\theta \right] \right)$$

$$= (mM_1G/d_0^2)\left(\hat{\mathbf{x}} - \hat{\mathbf{x}} - (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} + 3(R_2/d_0)\cos\theta\,\hat{\mathbf{x}} \right)$$

$$= (mM_1G/d_0^2)\left(-(R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} + 3(R_2/d_0)\cos\theta\,\hat{\mathbf{x}} \right)$$

$$= (mM_1G/d_0^2)(R_2/d_0)\left(-\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}} + 3\cos\theta\,\hat{\mathbf{x}} \right)$$

$$= (mM_1G/d_0^2)(R_2/d_0)\left(-\sin\theta\,\hat{\mathbf{y}} + 2\cos\theta\,\hat{\mathbf{x}} \right)$$

Now check this at the compass points

$\mathbf{F}_{tid}(0) = (mM_1G/d_0^2)(R_2/d_0)\,2\hat{\mathbf{x}}$	point B	to the right
$\mathbf{F}_{tid}(\pi) = -(mM_1G/d_0^2)(R_2/d_0)\,2\hat{\mathbf{x}}$	point A	to the left
$\mathbf{F}_{tid}(\pi/2) = -(mM_1G/d_0^2)(R_2/d_0)\,\hat{\mathbf{y}}$	down at top	
$\mathbf{F}_{tid}(-\pi/2) = +(mM_1G/d_0^2)(R_2/d_0)\,\hat{\mathbf{y}}$	up at bottom	

where I toss terms of order $(R_2/d_0)^2$. Compass agrees. I like the general formula:

$$\mathbf{F}_{tid}(\theta) \approx (mM_1G/d_0^2)(R_2/d_0)\left(-\sin\theta\,\hat{\mathbf{y}} + 2\cos\theta\,\hat{\mathbf{x}} \right)$$

Well, I guess this is my first statement of F_{tid} in polar coordinates.

Now let's try again doing the integral

$$-V = \int_0^\theta \mathbf{F}_{tid}(\theta) \bullet d\mathbf{s} \qquad d\mathbf{s} = R_2 d\theta\,\hat{\boldsymbol{\theta}}$$

So then

$$-V = (mM_1G/d_0^2)(R_2/d_0)\int_0^\theta \left(-\sin\theta\,\hat{\mathbf{y}} + 2\cos\theta\,\hat{\mathbf{x}} \right) \bullet R_2 d\theta\,\hat{\boldsymbol{\theta}}$$

$$= R_2\,(mM_1G/d_0^2)(R_2/d_0)\int_0^\theta \left(-\sin\theta\,\hat{\mathbf{y}} \bullet \hat{\boldsymbol{\theta}} + 2\cos\theta\,\hat{\mathbf{x}} \bullet \hat{\boldsymbol{\theta}} \right) d\theta$$

$$= R_2\,(mM_1G/d_0^2)(R_2/d_0)\int_0^{\theta}(-\sin\theta\cos\theta + 2\cos\theta[-\sin\theta])\,d\theta$$

$$= R_2\,(mM_1G/d_0^2)(R_2/d_0)\int_0^{\theta}(-3\sin\theta\cos\theta)\,d\theta$$

$$= R_2\,(mM_1G/d_0^2)(R_2/d_0)\int_0^{\theta}(-(3/2)\,2\sin\theta\cos\theta)\,d\theta$$

$$= R_2\,(mM_1G/d_0^2)(R_2/d_0)\int_0^{\theta}(-(3/2)\sin2\theta)\,d\theta$$

$$= -(3/2)\,R_2\,(mM_1G/d_0^2)(R_2/d_0)\int_0^{\theta}\sin2\theta\,d\theta$$

$$= -(3/2)R_2\,(mM_1G/d_0^2)(R_2/d_0)(1/2)(1-\cos2\theta)$$

$$= -(3/2)R_2^2\,(mM_1G/d_0^3)(1/2)(1-\cos2\theta)$$

$$= -(3/4)R_2^2\,(mM_1G/d_0^3)(1-\cos2\theta)$$

And therefore

$$V = (3/4)r^2\,(mM_1G/d_0^3)(1-\cos2\theta) \qquad r = R_2$$

$$\mathbf{F}_{tid}(\theta) = -\nabla V = -\hat{\boldsymbol{\theta}}\,(1/R_2)\partial_\theta V - \hat{\mathbf{r}}\partial_r V$$

$$= -\hat{\boldsymbol{\theta}}\,(1/R_2)\,(3/4)r^2\,(mM_1G/d_0^3)2\sin2\theta - \hat{\mathbf{r}}\,(3/4)2r\,(mM_1G/d_0^3)(1-\cos2\theta)$$

$$= (3/4)(mM_1G/d_0^3)\,[\,-\hat{\boldsymbol{\theta}}\,R_2 2\sin2\theta \;-\hat{\mathbf{r}}2R_2(1-\cos2\theta)\,]$$

$$= (3/4)(mM_1G/d_0^3)\,[\,-(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}})\,R_2 2\sin2\theta \;-(\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}})2R_2(1-\cos2\theta)\,]$$

$$= -(3/4)(mM_1G/d_0^3)\,[\,(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}})\,R_2 2\sin2\theta + (\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}})2R_2(1-\cos2\theta)\,]$$

$$= -(3/4)(mM_1G/d_0^3)\,R_2 2\,[\,(-\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}})\sin2\theta + (\cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}})(1-\cos2\theta)\,]$$

$$= -(3/4)(mM_1G/d_0^3)\,R_2 2\,[\,\hat{\mathbf{x}}(-\sin\theta\sin2\theta + \cos\theta(1-\cos2\theta) + \hat{\mathbf{y}}\,(\cos\theta\sin2\theta + \sin\theta\,(1-\cos2\theta)\,]$$

Now:

$$-\sin\theta\sin2\theta + \cos\theta(1-\cos2\theta) = -\sin\theta\,2\sin\theta\cos\theta + \cos\theta\,2\sin^2\theta$$

$= -2\sin^2\theta\cos\theta + 2\cos\theta \sin^2\theta = 0$

$\cos\theta \sin 2\theta + \sin\theta (1-\cos 2\theta) = \cos\theta 2\sin\theta\cos\theta + \sin\theta\, 2\sin^2\theta$

$= 2\sin\theta\cos^2\theta + 2\sin\theta \sin^2\theta = 2\sin\theta$

so the result is then

$$\mathbf{F}_{tid}(\theta) = -(3/4)(mM_1G/d_0^3)\, R_2 2\, [\, \hat{\mathbf{x}}\, (0) + \hat{\mathbf{y}}\, (2\sin\theta)\,]$$

which is the wrong answer. The correct answer is

$$\mathbf{F}_{tid} = (mM_1G/d_0^2)(R_2/d_0)\, (-\sin\theta\, \hat{\mathbf{y}} + 2\cos\theta\, \hat{\mathbf{x}})$$

So things are just hugely messed up in this entire doc, I guess I need to start over.

I later realized that in hybrid notation I could find the potential by inspection! I got this hint from reading how Taylor did it.