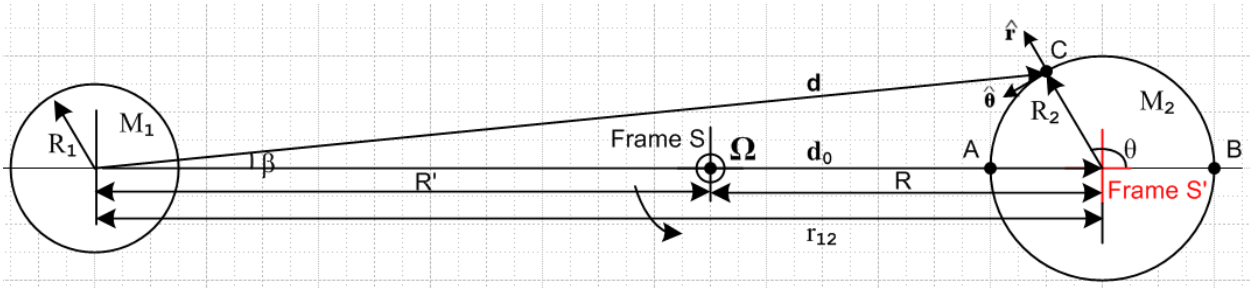
Tide potential Calculation PhL 1.18.17

Start with

$$\mathbf{F}_{tid} = mM_1G\left(\hat{\mathbf{d}}_0/d_0^2 - \hat{\mathbf{d}}/d^2\right)$$

$$= mM_1G\left(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\right)$$

and here is our picture



In Cartesian coordinates we can write

$$d_x = d_0 + R_2\cos\theta \qquad d_y = R_2\sin\theta$$

$$\mathbf{d} = (d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}$$

$$\mathbf{d}_0 = d_0\hat{\mathbf{x}}$$

Then,

$$\mathbf{F}_{tid} = mM_1G\left(d_0\hat{\mathbf{x}}/d_0^3 - \left[(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}\right]/d^3\right)$$

now use

$$d^2 = R_2^2 + d_0^2 - 2d_0R_0\cos(\pi-\theta) = R_2^2 + d_0^2 + 2d_0R_0\cos\theta = d_0^2[\,1 + (R_2/d_0)^2 + 2\,(R_2/d_0)\cos\theta\,]$$

$$\approx d_0^2[\,1 + 2\,(R_2/d_0)\cos\theta\,]$$

to get

$$d^{-3} \approx d_0^{-3}\,[\,1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3}\,[\,1 - 3(R_2/d_0)\cos\theta\,]\,.$$

Then,

$$\mathbf{F}_{tid} = mM_1G\left(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\right)$$

$\approx$ $mM_1G\,(\ d_0\hat{\mathbf{x}}/d_0^3 - \mathbf{[}\,(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}\mathbf{]}\,)\,d_0^{-3}\,[\ 1 - 3(R_2/d_0)\cos\theta\ ]\ )$

$= (mM_1G/d_0^2)\,(\ \hat{\mathbf{x}} - \mathbf{[}\,(1 + (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} + (R_2/d_0)\sin\theta\,\hat{\mathbf{y}}\mathbf{]}\,[\ 1 - 3(R_2/d_0)\cos\theta\ ]\,)$

$= (mM_1G/d_0^2)\,(\ \hat{\mathbf{x}} - \hat{\mathbf{x}} - (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} + 3(R_2/d_0)\cos\theta\ \hat{\mathbf{x}}\ ) + O((R_2/d_0)^2)$

$\approx (mM_1G/d_0^2)\,(\ -(R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} + 3(R_2/d_0)\cos\theta\,\hat{\mathbf{x}}\ )$

$= (mM_1G/d_0^2)(R_2/d_0)\,(\ -\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}} + 3\cos\theta\,\hat{\mathbf{x}}\ )$

$= (mM_1G/d_0^2)(R_2/d_0)\,(\ 2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}})$

Here I have again derived the hybrid Cartesian/polar coordinates version of Ft_{tid}

Now check this at the compass points

$\mathbf{F}_{tid}(0) = (mM_1G/d_0^2)(R_2/d_0)\,2\hat{\mathbf{x}}$	point B	to the right
$\mathbf{F}_{tid}(\pi) = -(mM_1G/d_0^2)(R_2/d_0)\,2\hat{\mathbf{x}}$	point A	to the left
$\mathbf{F}_{tid}(\pi/2) = -(mM_1G/d_0^2)(R_2/d_0)\,\hat{\mathbf{y}}$	down at top	
$\mathbf{F}_{tid}(-\pi/2) = +(mM_1G/d_0^2)(R_2/d_0)\,\hat{\mathbf{y}}$	up at bottom	

Compass agrees. I like the general formula:

$\mathbf{F}_{tid}(\theta) \approx (mM_1G/d_0^2)(R_2/d_0)\,(\ 2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}})$

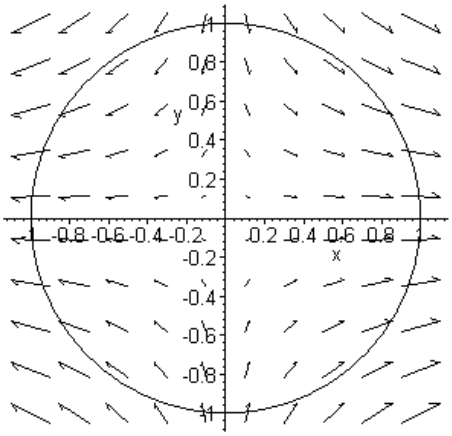
How might I plot this?

$x = r\cos\theta$
$y = r\sin\theta$

$\mathbf{F}_{tid}(r,\theta) \approx (mM_1G/d_0^2)(1/d_0)\,(\ 2r\cos\theta\,\hat{\mathbf{x}} - r\sin\theta\,\hat{\mathbf{y}})$

$= (mM_1G/d_0^3)\,(\ 2x\hat{\mathbf{x}} - y\hat{\mathbf{y}}\ )$ // and now Cartesian!

```
with(plots):with(plottools):
s1 := fieldplot([2*x,-y],x=-1..1,y=-1..1,grid=[10,10]):
s2 := circle([0,0],1):
display(s1,s2,scaling=constrained);
```



Very good. You want to think of $R_2 = r$ as a variable.

What is it in polar coordinates? ($r = R_2$)

$$\mathbf{F}_{tid}(\theta) \approx (mM_1G/d_0^2)(r/d_0)\,(\,2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}})$$

$$= (mM_1G/d_0^2)(r/d_0)(\,2\cos\theta\,[\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] - \sin\theta\,[\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}])$$

$$= (mM_1G/d_0^2)(r/d_0)[\,(2\cos^2\theta-\sin^2\theta)\hat{\mathbf{r}} + (-2\sin\theta\cos\theta - \sin\theta\cos\theta)\hat{\boldsymbol{\theta}}]$$

$$= (mM_1G/d_0^2)(r/d_0)[\,(3\cos^2\theta-1)\hat{\mathbf{r}} + (-3\sin\theta\cos\theta)\hat{\boldsymbol{\theta}}\,]$$

$$= (mM_1G/d_0^2)(r/d_0)[\,(3\{1/2)(1+\cos2\theta)\}-1)\,\hat{\mathbf{r}} + (-(3/2)\sin2\theta)\hat{\boldsymbol{\theta}}\,]$$

$$= (1/2)(mM_1G/d_0^2)(r/d_0)[\,(3\{(1+\cos2\theta)\}-2)\,\hat{\mathbf{r}} + (-3\sin2\theta)\hat{\boldsymbol{\theta}}\,]$$

$$= (1/2)(mM_1G/d_0^2)(r/d_0)[\,(1 + 3\cos2\theta)\,\hat{\mathbf{r}} - 3\sin2\theta\hat{\boldsymbol{\theta}}\,]$$

$$= (3/2)(mM_1G/d_0^2)(r/d_0)[\,([1/3] + \cos2\theta)\,\hat{\mathbf{r}} - \sin2\theta\hat{\boldsymbol{\theta}}\,]$$

This says

$\hat{\boldsymbol{\theta}}$ piece $= -(3/2)(mM_1G/d_0^2)(r/d_0)\sin2\theta$	horizontal	(5) Butikov
$\hat{\mathbf{r}}$ piece $= (3/2)(mM_1G/d_0^2)(r/d_0)\,(\cos2\theta + 1/3)$	vertical	(6) Butikov

So far so good! (8.8.31)

Can this force be represented by a potential? In electrostatics we have

$\mathbf{E} = -\nabla\varphi \quad \Rightarrow \quad \text{div}\,\mathbf{E} = -\nabla^2\varphi = 0$ region of no charge

$\text{curl}\mathbf{E} = -\nabla \times (\nabla\varphi) = 0$

So **E** must have zero divergence and zero curl in order to be rep by a potential. So consider

$\text{div}\,\mathbf{F}_{tid} = \text{div}(\, 2x\hat{\mathbf{x}} - y\hat{\mathbf{y}}\,) = \partial^2_x(2x) + \partial^2_y(-y) = 0 + 0 = 0$

$\text{curl}\,\mathbf{F}_{tid} = [\partial_x F_y - \partial_y F_x]\,\hat{z} + \text{cyclic}$

$[\,0 - 0] + 0 + 0 = 0$

Therefore you NOT represent $\mathbf{F}_{tid}$ by a potential.

WRONG!!! First of all, div $\mathbf{E} = -\nabla^2\varphi$ = "ρ" and ≠ 0.
Second of all, conservativity only involves the curl!

So the tidal force is a conservative force. yes

Here then is the force for which we seek a potential

$\mathbf{F}_{g2} = -[(GM_2m)/r^3]\ \mathbf{r}$ Earth gravity ok

$\mathbf{F}_{tid} = (mM_1G/d_0^3)\,(\,2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}\,)$ $\dim(F) = M\,L/T^2$ ok

$\dim(GMm/r^2) = M\,L/T^2$ $\dim(GMm) = M\,L^3/T^2$

Now consider $\dim(V) = M\,L^2/T^2$

$\nabla V(x,y) = -[(GM_2m)/r^3]\ \mathbf{r} + (mM_1G/d_0^3)\,(\,2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}\,)$ ok

$= -[(GM_2m)/r^3](\,x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}\,) + (mM_1G/d_0^3)\,(\,2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}\,)$

$= [\,-(GM_2m)/r^3) + (mM_1G/d_0^3)2\,]\ x\,\hat{\mathbf{x}} + [\,-(GM_2m)/r^3) - (mM_1G/d_0^3)\,]\ y\,\hat{\mathbf{y}}$

$= Gm\,\{\,[\,-M_2/r^3 + 2(M_1/d_0^3)\,]\ x\,\hat{\mathbf{x}} + [\,-M_2/r^3 - (M_1/d_0^3)\,]\ y\,\hat{\mathbf{y}}\}$ ok

Now suppose you try as ansatz,

$V(x,y) = k[A^2/x^2 + B^2/y^2 - 1]$ ok // an arbitrary ellipse centered at Earth center

$\nabla V(x,y) = -2kA^2/x^3\ \hat{x} - 2kB^2/y^3\ \hat{y}$ ok // dim(k) = $M L^2/T^2$

Comment: Here I am assuming that V(x,y) is a multiple of the desired ellipse equation so I guess the surface V(x,y) = 0 would be an ellipse. Should have said $V(x,y) = k[A^2/x^2 + B^2/y^2]$ and then any surface V(x,y) = constant would be an ellipse. What I did not know is that the actual surface is a quartic and not an ellipse, so this was all hopeless. (Hence Paradox below)

This does not seem to work, why is that? You would need now to have

$$-2kA^2/x^3\ \hat{x} = Gm\ [\ -M_2/r^3 + 2(M_1/d_0^3)\ ]\ x\ \hat{x}$$
$$-2kB^2/y^3\ \hat{y} = Gm\ [\ -M_2/r^3 - (M_1/d_0^3)\ ]\ y\ \hat{y}$$

or

$$-2kA^2 = Gm\ [\ -M_2/r^3 + 2\ (M_1/d_0^3)\ ]\ x^4$$
$$-2kB^2 = Gm\ [\ -M_2/r^3 - (M_1/d_0^3)\ ]\ y^4 \qquad r^2 = x^2+y^2$$

Just check dimensions please.

$\dim(kA^2) = M L^4/T^2$ $\dim(GmMx^4/r^3) = M L^3/T^2 * L = M L^4/T^2$

so dimensions are OK. I do know that

$r << d_0$ $x << d_0$ $y << d_0$

but this does not seem to help at all. I am stumped, this will take another day or two of burned time to figure out. It is so simple, what am I doing wrong here?

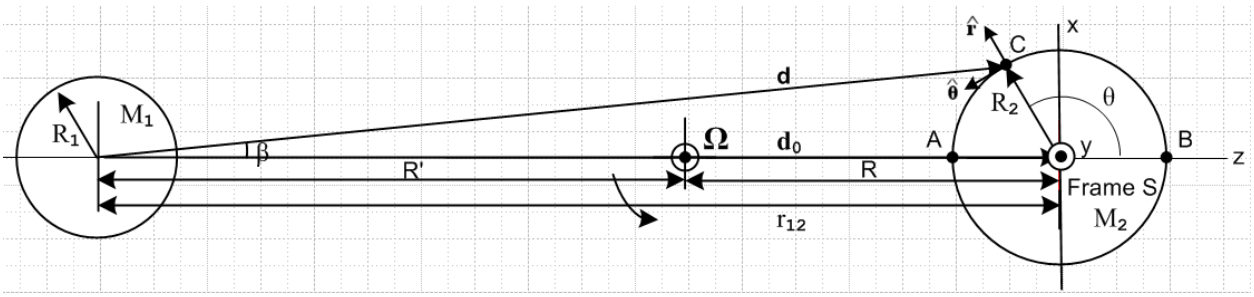
Paradox #1. Suppose I let $M_1 \rightarrow 0$ so the moon goes away slowly. The last two equations then become

$$-2kA^2 = Gm\ [\ -M_2/r^3\ ]\ x^4$$
$$-2kB^2 = Gm\ [\ -M_2/r^3\ ]\ y^4 \qquad r^2 = x^2+y^2$$

I expect the potential surface to be a circle rather than an ellipse since there are now no tidal forces, and that means I expect to have A = B. But instead I have $A \neq B$. This should give $A = B = R_2$ somehow.

Maybe I have to do this in 3D space and not fake 2D space?

Start Over! Use spherical coordinates r,θ,φ centered on the Earth with z to the right.



$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$\mathbf{d}_0 = d_0\hat{\mathbf{z}}$

$\mathbf{C} = \mathbf{r} = r\sin\theta\cos\varphi\ \hat{\mathbf{x}} + r\sin\theta\sin\varphi\ \hat{\mathbf{y}} + r\cos\theta\ \hat{\mathbf{z}} = \mathbf{x}$

Moon center = $\mathbf{c}_m = -d_0\hat{\mathbf{z}}$

$\mathbf{d} = \mathbf{C} - \mathbf{c}_m = r\sin\theta\cos\varphi\ \hat{\mathbf{x}} + r\sin\theta\sin\varphi\ \hat{\mathbf{y}} + r\cos\theta\ \hat{\mathbf{z}} + d_0\hat{\mathbf{z}}$

$= r\sin\theta\cos\varphi\ \hat{\mathbf{x}} + r\sin\theta\sin\varphi\ \hat{\mathbf{y}} + (r\cos\theta+d_0)\hat{\mathbf{z}}$

$d^2 = (r\sin\theta\cos\varphi)^2 + (r\sin\theta\sin\varphi)^2 + (r\cos\theta+d_0)^2$

$= r^2\sin^2\theta + (d_0 + r\cos\theta)^2$

$= r^2\sin^2\theta + d_0^2(1 + [r/d_0]\cos\theta)^2$

$\approx r^2\sin^2\theta + d_0^2(1 + 2\,[r/d_0]\cos\theta)$

$= d_0^2 [\ [r/d_0]^2\sin^2\theta + d_0^2(1 + 2\,[r/d_0]\cos\theta)$

$\approx d_0^2(1 + 2\,[r/d_0]\cos\theta)$ // ignore quadratic term

This is same as in Section 8.8, no surprise. So still get

$$d^{-3} \approx d_0^{-3} [\ 1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3} [\ 1 - 3(R_2/d_0)\cos\theta\ ] . \qquad (8.8.22)$$

Now:

$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$= mM_1G \{ [d_0\hat{\mathbf{z}}]/d_0^3 - \mathbf{[}\ r\sin\theta\cos\varphi\ \hat{\mathbf{x}} + r\sin\theta\sin\varphi\ \hat{\mathbf{y}} + (r\cos\theta+d_0)\hat{\mathbf{z}}\mathbf{]}/d^3 \}$

$= mM_1G \{ \hat{\mathbf{z}}\ d_0^2 - \mathbf{[}\ r\sin\theta\cos\varphi\ d^{-3}\hat{\mathbf{x}} + r\sin\theta\sin\varphi d^{-3}\ \hat{\mathbf{y}} + (r\cos\theta+d_0)d^{-3}\hat{\mathbf{z}}\mathbf{]}\}$

$= mM_1G \{ - [r\sin\theta\cos\varphi\, d^{-3}\hat{\mathbf{x}} + r\sin\theta\sin\varphi d^{-3}\hat{\mathbf{y}} + \{d_0^2+(r\cos\theta+d_0)d^{-3}\}\hat{\mathbf{z}}]\}$

STOP. Why not do this all in Cartesians.

Start over

$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$\mathbf{d}_0 = d_0\hat{\mathbf{z}}$

$\mathbf{C} = \mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

Moon center = $\mathbf{c}_m = -d_0\hat{\mathbf{z}}$

$\mathbf{d} = \mathbf{C} - \mathbf{c}_m = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} + d_0\hat{\mathbf{z}}$

$= x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (d_0+z)\hat{\mathbf{z}}$

$d^2 = x^2 + y^2 + (d_0+z)^2$

$= d_0^2\{ [1 + (z/d_0)]^2 + (x/d_0)^2 + (y/d_0)^2 \}$

$\approx d_0^2\{ [1 +2 (z/d_0) + (z/d_0)^2] + (x/d_0)^2 + (y/d_0)^2 \}$

$\approx d_0^2[1 +2 (z/d_0)]$ dropping all quadratic terms

$d^{-3} \approx d_0^{-3}[1 +2 (z/d_0)]^{-3/2} \approx d_0^{-3}(1- 3(z/d_0))$

Now have

$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$= mM_1G \{ d_0\hat{\mathbf{z}}/d_0^3 - [x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (d_0+z)\hat{\mathbf{z}}]/d^3 \}$

$= mM_1G \{ \hat{\mathbf{z}}/d_0^2 - [(x/d^3)\hat{\mathbf{x}} + (y/d^3)\hat{\mathbf{y}} + ((d_0+z)/d^3)\hat{\mathbf{z}}] \}$

$= mM_1G \{ - (x/d^3)\hat{\mathbf{x}} - (y/d^3)\hat{\mathbf{y}} - ((d_0+z)/d^3 - 1/d_0^2)\hat{\mathbf{z}}] \}$

$= - mM_1G \{ (x/d^3)\hat{\mathbf{x}} + (y/d^3)\hat{\mathbf{y}} + ((d_0+z)/d^3 - 1/d_0^2)\hat{\mathbf{z}}] \}$

Now look at each component separately

$(x/d^3) \approx d_0^{-3}(1- 3(z/d_0)) x \approx x/d_0^3$

$(y/d^3) \approx d_0^{-3}(\ 1-3(z/d_0)\ )\ y \approx y/d_0^3$

$(d_0+z)/d^3 - 1/d_0^2 = [(d_0+z)/d_0^3]\ (\ 1-3(z/d_0)\ ) - 1/d_0^2$

$= [(1+(z/d_0))/d_0^2]\ (\ 1-3(z/d_0)\ ) - 1/d_0^2$

$= \{\ [(1+(z/d_0))]\ (\ 1-3(z/d_0)\ ) - 1\ \}/d_0^2$

$= \{\ 1 + (z/d_0) - 3(z/d_0) - 3(z/d_0)^2 - 1\}/d_0^2$

$\approx \{\ -2(z/d_0)\ \}/d_0^2 = -2z/d_0^3$

We then end up with

$\mathbf{F}_{tid} = -mM_1G\{\ (x/d_0^3)\hat{\mathbf{x}} + (y/d_0^3)\hat{\mathbf{y}} + (-2z/d_0^3)\hat{\mathbf{z}}\ \}$

$= -(mM_1G/d_0^3)\{\ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\ \}$

and this then is my 3D generalization of previous result where I had y = 0. Could make 3D field plot.

I also have

$\mathbf{F}_{g2} = -[(GM_2m)/r^3]\ \mathbf{r}$

The total force acting on a particle m of water is then

$\mathbf{F} = \mathbf{F}_{tid} + \mathbf{F}_{g2} = -(mM_1G/d_0^3)\{\ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\ \} - [(GM_2m)/r^3]\ \mathbf{r}$

$= mG\ \{\ (M_1/d_0^3 - M_2/r^3)x\ \hat{\mathbf{x}} + (M_1/d_0^3 - M_2/r^3)y\hat{\mathbf{y}} + (-2M_1/d_0^3 - M_2/r^3)z\ \hat{\mathbf{z}}\ \}$

Now suppose I drop the $1/d_0^3$ terms since $d_0 >> r$ and this is OK for $M_1 \rightarrow 0$ as well. then

$\mathbf{F} = \mathbf{F}_{tid} + \mathbf{F}_{g2} = -(mM_1G/d_0^3)\{\ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\ \} - [(GM_2m)/r^3]\ \mathbf{r}$

$= mG\ \{\ -M_2/r^3)x\ \hat{\mathbf{x}} + (-M_2/r^3)y\hat{\mathbf{y}} + (-M_2/r^3)z\ \hat{\mathbf{z}}\ \}$

and only gravity is left.

NOW: Here is an ansatz ellipsoid for the potential surface :

$V(x,y,z) = k[A^2/x^2 + A^2/y^2 + C^2/z^2\ ]$ // repeating the above hoping for ellipsoid

$\nabla V = -2k[\ (A^2/x^3)\hat{\mathbf{x}} + (A^2/y^3)\hat{\mathbf{y}} + (C^2/z^3)\hat{\mathbf{z}}\ ]$

Now I have the same problem as before I think. I have to identify: $\nabla V = \mathbf{F}_{tid}$ and then the following must be true:

$$2k[(A^2/x^3)\hat{\mathbf{x}} + (A^2/y^3)\hat{\mathbf{y}} + (C^2/z^3)\hat{\mathbf{z}}] = (mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \} - [(GM_2m)/r^3]\ \mathbf{r}$$

$$= (mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \} - [(GM_2m)/r^3] \{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}\ \}$$

$$= mG\ \{ (M_1/d_0^3 - M_2/r^3)x\,\hat{\mathbf{x}} + (M_1/d_0^3 - M_2/r^3)y\hat{\mathbf{y}} + (-2M_1/d_0^3 - M_2/r^3)\,\hat{\mathbf{z}}\ \}$$

which then requires that

$$2k(A^2/x^3) = mG\ (M_1/d_0^3 - M_2/r^3)x$$

$$2k(A^2/y^3) = mG\ (M_1/d_0^3 - M_2/r^3)y$$

$$2k(C^2/z^3) = mG\ (-2M_1/d_0^3 - M_2/r^3)$$

So converting to 3D coordinates has done NOTHING to explain my problem.

Paradox 1 Reappears: Suppose I turn off the moon. Then I get

$$2k(A^2/x^3) = mG\ (- M_2/r^3)x$$

$$2k(A^2/y^3) = mG\ (- M_2/r^3)y$$

$$2k(C^2/z^3) = mG\ (- M_2/r^3)$$

and this is supposed to give a sphere!!!

So I just wasted 2 hours maybe seeing if going from 2D to 3D would fix this paradox. A bad hunch.

Comment: Sometimes to fix a problem you take a wrong tack in hopes of a solution. Going from 2D to 3D was such a wrong tack and it did not resolve the paradox!

Let's try a simpler problem! There is no moon, only earth gravity so

$$\mathbf{F} = mG\ \{ - M_2/r^3)x\,\hat{\mathbf{x}} + (- M_2/r^3)y\hat{\mathbf{y}} + (- M_2/r^3)z\,\hat{\mathbf{z}}\ \}$$

This is supposed to be related to a sphere.

$$x^2 + y^2 + z^2 = A^2$$

How is this sphere connected? It is not hard to see in sphericals

$$\mathbf{F} = -(mG/r^2)\hat{\mathbf{r}}$$

$$V(r) = (mG/r^2)$$

So V is constant on a sphere of radius r. But how does that work in Cartesians?

$$V = (mG)\frac{1}{x^2+y^2+z^2}$$

The sphere is "on the bottom". So you don't say

$$V(r) = k\,(x^2+y^2+z^2)$$

But still V = constant would give a sphere either way.