Tide potential Calculation PhL 1.18.17

Just before starting this v3 doc I must have realized from reading Taylor that you can obtain the potential directly by looking at $\mathbf{F}_{tid}$ if you use hybrid polar/Cartesian coordinates.

We seek a potential V(x,y) which solves this equation (no minus sign)

$$\nabla V(x,y) = \mathbf{F}_{tid} = -[(GM_2m)/r^2]\,\hat{\mathbf{r}} + (mM_1G/d_0^3)(2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}) \qquad // \; r = \sqrt{x^2+y^2} \qquad (8.8.63)$$

The solution is the following (by inspection)

$$V(x,y) = (GM_2m/r) + (GM_1m/d_0^3)(x^2 - y^2/2)\,. \qquad // \; \nabla(1/r) = -(1/r^2)\hat{\mathbf{r}}, \quad r > 0 \qquad (8.8.64)$$

Of interest are surfaces on which V is a constant, so

$$(M_2/r) + (M_1/d_0^3)(x^2 - y^2/2) = k \qquad (8.8.65)$$

$$M_2^2 = r^2[\,k - (M_1/d_0^3)(x^2 - y^2/2)\,]^2 = k^2r^2\,[1 - (M_1/kd_0^3)(x^2 - y^2/2)]^2$$

$$\approx k^2r^2\,[1 - (M_1/kd_0^3)(2x^2 - y^2)] \qquad \Rightarrow$$

$$(M_2/k)^2 = r^2\,[1 - \varepsilon(2x^2 - y^2)] \qquad \varepsilon \equiv (M_1/kd_0^3) \qquad (8.8.66)$$

$$= (x^2+y^2)\,(1 - 2\varepsilon x^2 + \varepsilon y^2) \qquad \dim(M_1/k) = L \qquad (8.8.66)$$

$$= r^2\,(1 - 2\varepsilon r^2\cos^2\theta + \varepsilon r^2\sin^2\theta) \qquad \dim(\varepsilon) = L^{-2} \qquad (8.8.66)$$

$$= r^2\,[1 - 2\varepsilon r^2(1/2)(1+\cos2\theta) + \varepsilon r^2(1/2)(1-\cos2\theta)\,]$$

$$= r^2\,[1 - 2\varepsilon r^2(1/2) + \varepsilon r^2(1/2) - \cos2\theta\{\,2\varepsilon r^2(1/2) + \varepsilon r^2(1/2)\,)]$$

$$= r^2\,[1 - \varepsilon r^2(1/2) - \cos2\theta\{\,3\varepsilon r^2(1/2\,)]$$

$$= r^2\,[\{1 - \varepsilon r^2(1/2)\} - \{3\varepsilon r^2(1/2\,)\cos2\theta\,]$$

Consider these two closed curves in the x-y plane

$$r^2\,(1 - 2\varepsilon r^2\cos^2\theta + \varepsilon r^2\sin^2\theta) = (M_2/k)^2 \qquad // \text{ a quartic}$$

$$r^2 = R^2 \qquad // \text{ a circle}$$

Back up again. Write

$$(M_2/k)^2 = r^2 (1 - 2\varepsilon r^2\cos^2\theta + \varepsilon r^2\sin^2\theta) = r^2 + \varepsilon r^4(\sin^2\theta - 2\cos^2\theta)$$

or

$$\varepsilon(\sin^2\theta - 2\cos^2\theta)r^4 + r^2 - (M_2/k)^2 = 0$$

or

$$\varepsilon(\sin^2\theta - 2\cos^2\theta)r^4 + r^2 - c^2 = 0 \qquad c \equiv (M_2/k) \quad \dim(c) = L \quad \dim(\varepsilon) = L^{-2}$$

or

This is a quadratic equation in r^2 whose solution is

$$r^2 = \{ -1 - [1 - 4\,\varepsilon(\sin^2\theta - 2\cos^2\theta)c^2\,]^{1/2} \}/\{ 2\varepsilon(\sin^2\theta - 2\cos^2\theta)\}$$

$$\approx \{ -1 - [1 - 2\,\varepsilon(\sin^2\theta - 2\cos^2\theta)c^2 + (1/2)\varepsilon^2(\sin^2\theta - 2\cos^2\theta)^2c^4] \}/\{ 2\varepsilon(\sin^2\theta - 2\cos^2\theta)\}$$

$$\approx \{ -1 - 1 + 2\,\varepsilon(\sin^2\theta - 2\cos^2\theta)c^2 - (1/2)\varepsilon^2(\sin^2\theta - 2\cos^2\theta)^2c^4] \}/\{ 2\varepsilon(\sin^2\theta - 2\cos^2\theta)\}$$

$$\approx \{2\,\varepsilon(\sin^2\theta - 2\cos^2\theta)c^2 - (1/2)\varepsilon^2(\sin^2\theta - 2\cos^2\theta)^2c^4] \}/\{ 2\varepsilon(\sin^2\theta - 2\cos^2\theta)\}$$

$$\approx \{2\,\varepsilon c^2 - (1/2)\varepsilon^2(\sin^2\theta - 2\cos^2\theta)c^4] \}/\{ 2\varepsilon\}$$

$$\approx \{2\,c^2 - (1/2)\varepsilon(\sin^2\theta - 2\cos^2\theta)c^4] \}/\{ 2\}$$

$$\approx c^2 - (1/4)\varepsilon(\sin^2\theta - 2\cos^2\theta)c^4]$$

$$= c^2\,[\,1 - (1/4)\varepsilon(\sin^2\theta - 2\cos^2\theta)c^2\,] \qquad (8.8.68)$$

Then

$$r = c[\,1 - (1/4)\varepsilon(\sin^2\theta - 2\cos^2\theta)c^2\,]^{1/2}$$

$$\approx c[\,1 - (1/8)\varepsilon(\sin^2\theta - 2\cos^2\theta)c^2\,] \qquad (8.8.68)$$

Now finally I can compute the average over θ,

$$\langle r(\theta)\rangle = c[\,1 - (1/8)\varepsilon\{(1/2) - 2(1/2)\}c^2\,]$$

$$= c[\,1 - (1/16)c^2\,\varepsilon\,\{1-2\}\,]$$

$$= c[\,1 + (1/16)c^2\varepsilon\,] \qquad (8.8.69)$$

We now set this to R_2 !!!! so then

$$R_2 = c[\,1 + (1/16)c^2\varepsilon\,]$$

$$= c\,[\,1 + \alpha\varepsilon\,c^2\,] \qquad \alpha = (1/16)$$

$= c + \alpha\varepsilon c^3$ $\qquad\qquad$ $\dim(\alpha\varepsilon) = L^{-2}$

How about just saying

$$R_2 = c + \alpha\varepsilon c^3 = c\,[\,1 + \alpha\varepsilon c^2] \approx c\,[\,1 + \alpha\varepsilon R_2^2] = c + \alpha\varepsilon R_2^3$$

Then

$$c = R_2 - \alpha\varepsilon R_2^3 = R_2(1-\alpha\varepsilon R_2^2)$$

So I can now replace

$$(M_2/k) = R_2(1-\alpha\varepsilon R_2^2) \qquad\qquad (M_2/k) \approx R_2 \qquad\qquad (8.8.70)$$

and then the locus of the cubic is this

$$R_2^2(1-\alpha\varepsilon R_2^2)^2 = (x^2+y^2)\,[1 -\varepsilon(2x^2 - y^2)] \qquad (8.8.71)$$

$$\varepsilon = (M_1/kd_0^3) \approx (M_1/[M_2/R_2]d_0^3) = (M_1/M_2)R_2/d_0^3$$

What happens now at the right edge of the quadric where y = 0:

$$R_2^2(1-\alpha\varepsilon R_2^2)^2 = (x^2)\,[1 -\varepsilon(2x^2)] = x^2(1 - 2\varepsilon x^2) \approx x^2(1 - 2\varepsilon R_2^2)$$

Then it seems that

$$x = R_2(1-\alpha\varepsilon R_2^2) * (1 - 2\varepsilon R_2^2)^{-1/2}$$

$$\approx R_2(1-\alpha\varepsilon R_2^2) * (1+ \varepsilon R_2^2)$$

$$\approx R_2\,[\,1 + \varepsilon R_2^2(1-\alpha)]$$

$$\approx R_2\,[\,1 + (M_1/M_2)R_2/d_0^3 R_2^2(15/16)]$$

$$\approx R_2\,[\,1 + (M_1/M_2)(R_2/d_0)^3(15/16)] \qquad (8.8.72)$$

So the high tide is then

$$h_{high} = (M_1/M_2)(R_2/d_0)^3 R_2(15/16) \qquad (8.8.72)$$

Now what happens at the top edge of the quadric where x = 0

$$R_2^2(1-\alpha\varepsilon R_2^2)^2 = (x^2)\,[1 -\varepsilon(2x^2)] = y^2(1+\varepsilon y^2) \approx y^2(1 + \varepsilon R_2^2)$$

It seems then that

$$y = R_2(1-\alpha\varepsilon R_2^2) * (1 + \varepsilon R_2^2)^{-1/2}$$

$$\approx R_2(1-\alpha\varepsilon R_2^2) * (1-[1/2]\varepsilon R_2^2)$$

$$\approx R_2(1 -\alpha\varepsilon R_2^2 -[1/2]\varepsilon R_2^2)$$

$$\approx R_2(1 -\varepsilon R_2^2(1/16 + 8/16))$$

$$\approx R_2(1 -\varepsilon R_2^2(9/16))$$

$$\approx R_2(1 -(M_1/M_2)R_2/d_0^3 R_2^2(9/16))$$

$$\approx R_2(1 -(M_1/M_2)(R_2/d_0)^3(9/16)) \qquad (8.8.72)$$

So the low tide is then (abs val)

$$h_{low} = (M_1/M_2)(R_2/d_0)^3 R_2(9/16)$$

We then add these to get

$$\text{"2a"} = (M_1/M_2)(R_2/d_0)^3 R_2 [(15/16)+ (9/16)]$$

$$= (M_1/M_2)(R_2/d_0)^3 R_2 [(24/16)]$$

$$= (M_1/M_2)(R_2/d_0)^3 R_2 [(3/2)] \qquad (8.8.72)$$

Then

$$\text{"a"} = (3/4)\ (M_1/M_2)(R_2/d_0)^3 R_2 \qquad \text{// it works!!!!} \qquad (8.8.73)$$

This locus is close to the locus of a circle of some radius R

$$R^2 = (x^2+y^2)$$

or

$$R^2 = (x^2+y^2)(1 - 2\varepsilon x^2+\varepsilon y^2) \qquad \varepsilon \equiv (M_1/kd_0^3) \qquad R = M_2/k = \text{constant}$$

Since d_0 is large, ε is small, and $R^2 \approx (x^2+y^2)$ so $1/k \approx R/M_2$. Then

$$(M_2/k)^2 \approx (x^2+y^2)(1 - 2\varepsilon x^2+\varepsilon y^2) \qquad \varepsilon = (M_1/M_2)/d_0^3$$

This is a quartic curve which resembles an ellipse. For example, for $\varepsilon = .05$ and $M_2/k = 1$:

```
f := (x^2+y^2)*(1-2*e*x^2+e*y^2);
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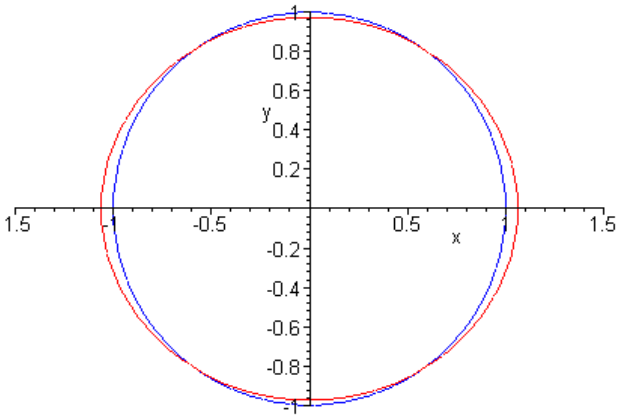
$$f := (x^2+y^2)(1-2ex^2+ey^2)$$

```
e := 0.05:
s1 := implicitplot(f=1,x=-1.5..1.5,y=-2..2, scaling = constrained,
view=[-1.5..1.5,-1..1],color=red):
s2 := implicitplot(x^2+y^2=1,x=-1.5..1.5,y=-2..2,color=blue):
display(s1,s2);
```



where the blue curve is a circle of radius 1.

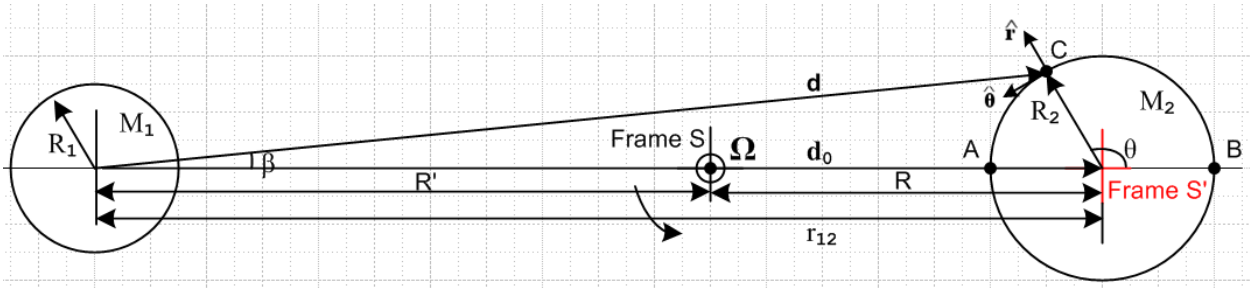
$Gm(M_1/d_0^3)(x^2/2) - Gm(M_1/d_0^3)(y^2/2)\ \}$

Start with

$$\mathbf{F}_{tid} = mM_1G\left(\hat{\mathbf{d}}_0/d_0^2 - \hat{\mathbf{d}}/d^2\right)$$

$$= mM_1G\left(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\right)$$

and here is our picture



In Cartesian coordinates we can write

$$d_x = d_0 + R_2\cos\theta \qquad d_y = R_2\sin\theta$$

$$\mathbf{d} = (d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}$$

$$\mathbf{d}_0 = d_0\hat{\mathbf{x}}$$

Then,

$$\mathbf{F}_{tid} = mM_1G\left(d_0\hat{\mathbf{x}}/d_0^3 - \left[(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}\right]/d^3\right)$$

now use

$$d^2 = R_2^2 + d_0^2 - 2d_0R_0\cos(\pi-\theta) = R_2^2 + d_0^2 + 2d_0R_0\cos\theta = d_0^2[\,1 + (R_2/d_0)^2 + 2\,(R_2/d_0)\cos\theta\,]$$

$$\approx d_0^2[\,1 + 2\,(R_2/d_0)\cos\theta\,]$$

to get

$$d^{-3} \approx d_0^{-3}\,[\,1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3}\,[\,1 - 3(R_2/d_0)\cos\theta\,]\,.$$

Then,

$$\mathbf{F}_{tid} = mM_1G\left(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\right)$$

$$\approx mM_1G\left(d_0\hat{\mathbf{x}}/d_0^3 - \left[(d_0 + R_2\cos\theta)\,\hat{\mathbf{x}} + R_2\sin\theta\,\hat{\mathbf{y}}\right] \right) d_0^{-3}\,[\,1 - 3(R_2/d_0)\cos\theta\,]\,)$$

$$= (mM_1G/d_0^2)\left(\hat{\mathbf{x}} - \left[(1 + (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} + (R_2/d_0)\sin\theta\,\hat{\mathbf{y}}\right]\,[\,1 - 3(R_2/d_0)\cos\theta\,]\,\right)$$

$$= (mM_1G/d_0^2)\left(\hat{\mathbf{x}} - \hat{\mathbf{x}} - (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} + 3(R_2/d_0)\cos\theta\,\hat{\mathbf{x}}\right) + O((R_2/d_0)^2)$$

$$\approx (mM_1G/d_0^2)\left(- (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} + 3(R_2/d_0)\cos\theta\,\hat{\mathbf{x}}\right)$$

$$= (mM_1G/d_0^2)(R_2/d_0)\left(- \cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}} + 3\cos\theta\,\hat{\mathbf{x}}\right)$$

$$= (mM_1G/d_0^2)(R_2/d_0)\left(2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}}\right)$$

Now check this at the compass points

$\mathbf{F}_{tid}(0) = (mM_1G/d_0^2)(R_2/d_0)\,2\hat{\mathbf{x}}$	point B	to the right
$\mathbf{F}_{tid}(\pi) = -(mM_1G/d_0^2)(R_2/d_0)\,2\hat{\mathbf{x}}$	point A	to the left
$\mathbf{F}_{tid}(\pi/2) = -(mM_1G/d_0^2)(R_2/d_0)\,\hat{\mathbf{y}}$	down at top	
$\mathbf{F}_{tid}(-\pi/2) = +(mM_1G/d_0^2)(R_2/d_0)\,\hat{\mathbf{y}}$	up at bottom	

Compass agrees. I like the general formula:

$$\mathbf{F}_{tid}(\theta) \approx (mM_1G/d_0^2)(R_2/d_0)\,(2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}})$$

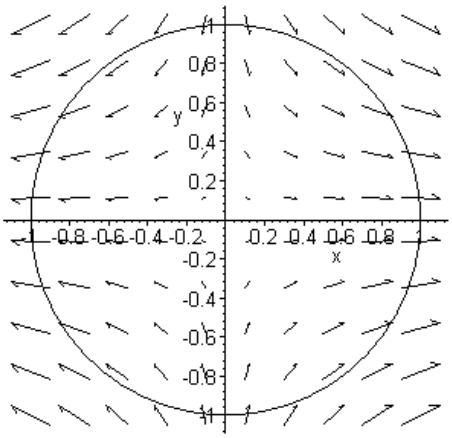
How might I plot this?

$x = r\cos\theta$
$y = r\sin\theta$

$$\mathbf{F}_{tid}(r,\theta) \approx (mM_1G/d_0^2)(1/d_0)\,(2r\cos\theta\,\hat{\mathbf{x}} - r\sin\theta\,\hat{\mathbf{y}})$$

$$= (mM_1G/d_0^3)\,(2x\hat{\mathbf{x}} - y\hat{\mathbf{y}})$$

```
with(plots):with(plottools):
s1 := fieldplot([2*x,-y],x=-1..1,y=-1..1,grid=[10,10]):
s2 := circle([0,0],1):
display(s1,s2,scaling=constrained);
```



Very good. You want to think of $R_2 = r$ as a variable.

What is it in polar coordinates? ($r = R_2$)

$$\mathbf{F}_{tid}(\theta) \approx (mM_1G/d_0^2)(r/d_0)\,(2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}})$$

$$= (mM_1G/d_0^2)(r/d_0)(\,2\cos\theta\,[\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] - \sin\theta\,[\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}])$$

$$= (mM_1G/d_0^2)(r/d_0)[\,(2\cos^2\theta - \sin^2\theta)\hat{\mathbf{r}} + (-2\sin\theta\cos\theta - \sin\theta\cos\theta)\hat{\boldsymbol{\theta}}]$$

$$= (mM_1G/d_0^2)(r/d_0)[\,(3\cos^2\theta - 1)\hat{\mathbf{r}} + (-3\sin\theta\cos\theta)\hat{\boldsymbol{\theta}}\,]$$

$$= (mM_1G/d_0^2)(r/d_0)[\,(3\{1/2)(1+\cos2\theta)\} - 1)\,\hat{\mathbf{r}} + (-(3/2)\sin2\theta)\hat{\boldsymbol{\theta}}\,]$$

$$= (1/2)(mM_1G/d_0^2)(r/d_0)[\,(3\{(1+\cos2\theta)\} - 2)\,\hat{\mathbf{r}} + (-3\sin2\theta)\hat{\boldsymbol{\theta}}\,]$$

$$= (1/2)(mM_1G/d_0^2)(r/d_0)[\ (1 + 3\cos2\theta)\ \hat{\mathbf{r}}\ - 3\sin2\theta\hat{\boldsymbol{\theta}}\ ]$$
$$= (3/2)(mM_1G/d_0^2)(r/d_0)[\ ([1/3] + \cos2\theta)\ \hat{\mathbf{r}}\ - \sin2\theta\hat{\boldsymbol{\theta}}\ ]$$

This says

$\hat{\boldsymbol{\theta}}$ piece $= -(3/2)(mM_1G/d_0^2)(r/d_0)\sin2\theta$	horizontal	(5) Butikov
$\hat{\mathbf{r}}$ piece $= (3/2)(mM_1G/d_0^2)(r/d_0)(\cos2\theta + 1/3)$	vertical	(6) Butikov

So far so good!

Can this force be represented by a potential? In electrostatics we have

$$\mathbf{E} = -\nabla\varphi \quad \Rightarrow \quad \text{div}\ \mathbf{E} = -\nabla^2\varphi = 0 \quad \text{region of no charge}$$
$$\text{curl}\mathbf{E} = -\nabla \times (\nabla\varphi) = 0$$

So **E** must have zero divergence and zero curl in order to be rep by a potential. So consider

$$\text{div}\ \mathbf{F}_{tid} = \text{div}(\ 2x\hat{\mathbf{x}} - y\hat{\mathbf{y}}\ ) = \partial^2_x(2x) + \partial^2_y(-y) = 0 + 0 = 0$$

$$\text{curl}\ \mathbf{F}_{tid} = [\partial_x F_y - \partial_y F_x]\ \hat{z} + \text{cyclic}$$

$$[\ 0 - 0] + 0 + 0 = 0$$

Therefore you NOT represent $\mathbf{F}_{tid}$ by a potential.

WRONG!!! First of all, div $\mathbf{E} = -\nabla^2\varphi$ = "ρ" and ≠ 0.
Second of all, conservativity only involves the curl!

So the tidal force is a conservative force.

Here then is the force for which we seek a potential

$\mathbf{F}_{g2} = -[(GM_2m)/r^3]\ \mathbf{r}$	Earth gravity	ok
$\mathbf{F}_{tid} = (mM_1G/d_0^3)(\ 2x\ \hat{\mathbf{x}} - y\ \hat{\mathbf{y}}\ )$	$\dim(F) = M\ L/T^2$	ok

$\dim(GMm/r^2) = M\ L/T^2$ $\qquad$ $\dim(GMm) = M\ L^3/T^2$

Now consider $\qquad$ $\dim(V) = M\ L^2/T^2$

$$\nabla V(x,y) = -[(GM_2m)/r^3]\ \mathbf{r} + (mM_1G/d_0^3)\,(2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}) \qquad \text{ok}$$

$$= -[(GM_2m)/r^3](x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}) + (mM_1G/d_0^3)\,(2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}})$$

$$= [-(GM_2m)/r^3 + (mM_1G/d_0^3)2]\ x\,\hat{\mathbf{x}} + [-(GM_2m)/r^3 - (mM_1G/d_0^3)]\ y\,\hat{\mathbf{y}}$$

$$= Gm\{[-M_2/r^3 + 2(M_1/d_0^3)]\ x\,\hat{\mathbf{x}} + [-M_2/r^3 - (M_1/d_0^3)]\ y\,\hat{\mathbf{y}}\} \qquad \text{ok}$$

Now suppose you try as ansatz, **WRONG!!!!**

$V(x,y) = k[A^2/x^2 + B^2/y^2 - 1]$ ok // an arbitrary ellipse centered at Earth center

$\nabla V(x,y) = -2kA^2/x^3\,\hat{\mathbf{x}} - 2kB^2/y^3\,\hat{\mathbf{y}}$ ok // dim(k) = $M\,L^2/T^2$

This does not seem to work, why is that? You would need now to have

$$-2kA^2/x^3\,\hat{\mathbf{x}} = Gm[-M_2/r^3 + 2(M_1/d_0^3)]\ x\,\hat{\mathbf{x}}$$
$$-2kB^2/y^3\,\hat{\mathbf{y}} = Gm[-M_2/r^3 - (M_1/d_0^3)]\ y\,\hat{\mathbf{y}}$$

or

$$-2kA^2 = Gm[-M_2/r^3 + 2(M_1/d_0^3)]\ x^4$$
$$-2kB^2 = Gm[-M_2/r^3 - (M_1/d_0^3)]\ y^4 \qquad r^2 = x^2+y^2$$

Just check dimensions please.

$\dim(kA^2) = M\,L^4/T^2$ $\dim(GmMx^4/r^3) = M\,L^3/T^2 * L = M\,L^4/T^2$

so dimensions are OK. I do know that

$r \ll d_0$ $x \ll d_0$ $y \ll d_0$

but this does not seem to help at all. I am stumped, this will take another day or two of burned time to figure out. It is so simple, what am I doing wrong here?

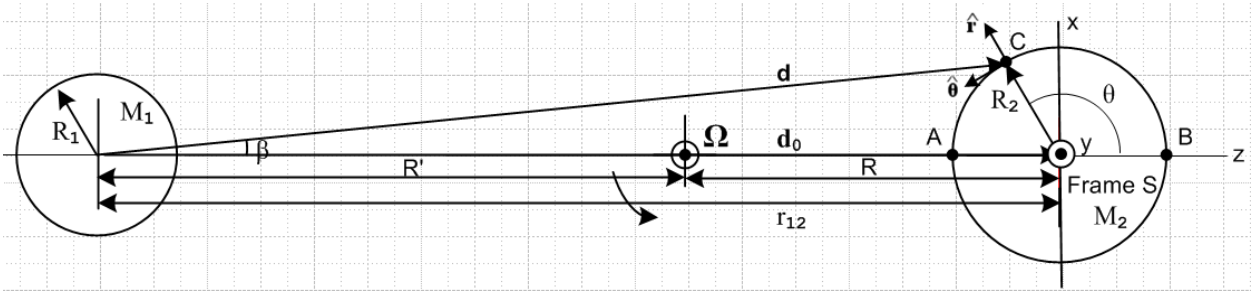
Paradox #1. Suppose I let $M_1 \rightarrow 0$ so the moon goes away slowly. The last two equations then become

$$-2kA^2 = Gm[-M_2/r^3]\ x^4$$
$$-2kB^2 = Gm[-M_2/r^3]\ y^4 \qquad r^2 = x^2+y^2$$

I expect the potential surface to be a circle rather than an ellipse since there are now no tidal forces, and that means I expect to have A = B. But instead I have $A \neq B$. This should give $A = B = R_2$ somehow.

Maybe I have to do this in 3D space and not fake 2D space?

Start Over! Use spherical coordinates r,θ,φ centered on the Earth with z to the right.



$\mathbf{F}_{tid} = mM_1G\{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$\mathbf{d}_0 = d_0\hat{\mathbf{z}}$

$\mathbf{C} = \mathbf{r} = r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}} = \mathbf{x}$

Moon center = $\mathbf{c}_m = -d_0\hat{\mathbf{z}}$

$\mathbf{d} = \mathbf{C} - \mathbf{c}_m = r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + r\cos\theta\,\hat{\mathbf{z}} + d_0\hat{\mathbf{z}}$

$= r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + (r\cos\theta+d_0)\hat{\mathbf{z}}$

$d^2 = (r\sin\theta\cos\varphi)^2 + (r\sin\theta\sin\varphi)^2 + (r\cos\theta+d_0)^2$

$= r^2\sin^2\theta + (d_0 + r\cos\theta)^2$

$= r^2\sin^2\theta + d_0^2(1 + [r/d_0]\cos\theta)^2$

$\approx r^2\sin^2\theta + d_0^2(1 + 2[r/d_0]\cos\theta)$

$= d_0^2[\,[r/d_0]^2\sin^2\theta + d_0^2(1 + 2[r/d_0]\cos\theta)$

$\approx d_0^2(1 + 2[r/d_0]\cos\theta)$ // ignore quadratic term

This is same as in Section 8.8, no surprise. So still get

$$d^{-3} \approx d_0^{-3}[\,1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3}[\,1 - 3(R_2/d_0)\cos\theta\,]\,. \qquad (8.8.22)$$

Now:

$\mathbf{F}_{tid} = mM_1G\{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$= mM_1G\{ [d_0\hat{\mathbf{z}}]/d_0^3 - \mathbf{[}\, r\sin\theta\cos\varphi\,\hat{\mathbf{x}} + r\sin\theta\sin\varphi\,\hat{\mathbf{y}} + (r\cos\theta+d_0)\hat{\mathbf{z}}\mathbf{]}/d^3 \}$

$= mM_1G \{ \hat{\mathbf{z}}\, d_0^2 - [\, r\sin\theta\cos\varphi\, d^{-3}\hat{\mathbf{x}} + r\sin\theta\sin\varphi d^{-3}\, \hat{\mathbf{y}} + (r\cos\theta+d_0)d^{-3}\hat{\mathbf{z}}]\}$

$= mM_1G \{ - [\, r\sin\theta\cos\varphi\, d^{-3}\hat{\mathbf{x}} + r\sin\theta\sin\varphi d^{-3}\, \hat{\mathbf{y}} + \{d_0^2+(r\cos\theta+d_0)d^{-3}\}\ \hat{\mathbf{z}}]\}$

STOP. Why not do this all in Cartesians.

Start over

$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$\mathbf{d}_0 = d_0\hat{\mathbf{z}}$
$\mathbf{C} = \mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$

Moon center = $\mathbf{c}_m = -d_0\hat{\mathbf{z}}$

$\mathbf{d} = \mathbf{C} - \mathbf{c}_m = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}} + d_0\hat{\mathbf{z}}$

$= x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + (d_0+z)\,\hat{\mathbf{z}}$

$d^2 = x^2 + y^2 + (d_0+z)^2$

$= d_0^2\{ [\, 1 + (z/d_0)\,]^2 + (x/d_0)^2 + (y/d_0)^2 \}$

$\approx d_0^2\{ [\, 1 + 2\,(z/d_0) + (z/d_0)^2\,] + (x/d_0)^2 + (y/d_0)^2 \}$

$\approx d_0^2[\, 1 + 2\,(z/d_0)]$ dropping all quadratic terms

$d^{-3} \approx d_0^{-3}[\, 1 + 2\,(z/d_0)]^{-3/2} \approx d_0^{-3}(\, 1 - 3(z/d_0)\,)$

Now have

$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$= mM_1G \{ d_0\hat{\mathbf{z}}/d_0^3 - [\, x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + (d_0+z)\,\hat{\mathbf{z}}]/d^3 \}$

$= mM_1G \{ \hat{\mathbf{z}}/d_0^2 - [\, (x/d^3)\,\hat{\mathbf{x}} + (y/d^3)\,\hat{\mathbf{y}} + ((d_0+z)/d^3)\,\hat{\mathbf{z}}] \}$

$= mM_1G \{ - (x/d^3)\,\hat{\mathbf{x}} - (y/d^3)\,\hat{\mathbf{y}} - ((d_0+z)/d^3 - 1/d_0^2)\,\hat{\mathbf{z}}] \}$

$= -mM_1G \{ (x/d^3)\,\hat{\mathbf{x}} + (y/d^3)\,\hat{\mathbf{y}} + ((d_0+z)/d^3 - 1/d_0^2)\,\hat{\mathbf{z}}] \}$

Now look at each component separately

$$(x/d^3) \approx d_0^{-3}(1 - 3(z/d_0))\, x \approx x/d_0^3$$

$$(y/d^3) \approx d_0^{-3}(1 - 3(z/d_0))\, y \approx y/d_0^3$$

$$(d_0+z)/d^3 - 1/d_0^2 = [(d_0+z)/d_0^3](1 - 3(z/d_0)) - 1/d_0^2$$

$$= [(1+(z/d_0))/d_0^2](1 - 3(z/d_0)) - 1/d_0^2$$

$$= \{ [(1+(z/d_0))](1 - 3(z/d_0)) - 1 \}/d_0^2$$

$$= \{ 1 + (z/d_0) - 3(z/d_0) - 3(z/d_0)^2 - 1 \}/d_0^2$$

$$\approx \{ -2(z/d_0) \}/d_0^2 = -2z/d_0^3$$

We then end up with

$$\mathbf{F}_{tid} = -mM_1G\{ (x/d_0^3)\hat{\mathbf{x}} + (y/d_0^3)\hat{\mathbf{y}} + (-2z/d_0^3)\hat{\mathbf{z}} \}$$

$$= -(mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \}$$

and this then is my 3D generalization of previous result where I had y = 0. Could make 3D field plot.

I also have

$$\mathbf{F}_{g2} = -[(GM_2m)/r^3]\ \mathbf{r}$$

The total force acting on a particle m of water is then

$$\mathbf{F} = \mathbf{F}_{tid} + \mathbf{F}_{g2} = -(mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \} - [(GM_2m)/r^3]\ \mathbf{r}$$

$$= mG\ \{ (M_1/d_0^3 - M_2/r^3)x\,\hat{\mathbf{x}} + (M_1/d_0^3 - M_2/r^3)y\hat{\mathbf{y}} + (-2M_1/d_0^3 - M_2/r^3)z\,\hat{\mathbf{z}} \}$$

Now suppose I drop the $1/d_0^3$ terms since $d_0 >> r$ and this is OK for $M_1 \rightarrow 0$ as well. then

$$\mathbf{F} = \mathbf{F}_{tid} + \mathbf{F}_{g2} = -(mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \} - [(GM_2m)/r^3]\ \mathbf{r}$$

$$= mG\ \{ -M_2/r^3)x\,\hat{\mathbf{x}} + (-M_2/r^3)y\hat{\mathbf{y}} + (-M_2/r^3)z\,\hat{\mathbf{z}} \}$$

and only gravity is left.

NOW: Here is an ansatz ellipsoid for the potential surface :

$$V(x,y,z) = k[A^2/x^2 + A^2/y^2 + C^2/z^2]$$

$$\nabla V = -2k[\ (A^2/x^3)\hat{\mathbf{x}} + (A^2/y^3)\hat{\mathbf{y}} + (C^2/z^3)\hat{\mathbf{z}}\ ]$$

Now I have the same problem as before I think. I have to identify: $\nabla V = \mathbf{F}_{tid}$ and then the following must be true:

$$2k[\ (A^2/x^3)\hat{\mathbf{x}} + (A^2/y^3)\hat{\mathbf{y}} + (C^2/z^3)\hat{\mathbf{z}}\ ] = (mM_1G/d_0^3)\{\ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\ \} - [(GM_2m)/r^3]\ \mathbf{r}$$

$$= (mM_1G/d_0^3)\{\ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}}\ \} - [(GM_2m)/r^3]\ \{x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}}\ \}$$

$$= mG\ \{\ (M_1/d_0^3 - M_2/r^3)x\ \hat{\mathbf{x}} + (M_1/d_0^3 - M_2/r^3)y\hat{\mathbf{y}} + (-2M_1/d_0^3 - M_2/r^3)\ \hat{\mathbf{z}}\ \}$$

which then requires that

$$2k(A^2/x^3) = mG\ (M_1/d_0^3 - M_2/r^3)x$$

$$2k(A^2/y^3) = mG\ (M_1/d_0^3 - M_2/r^3)y$$

$$2k(C^2/z^3) = mG\ (-2M_1/d_0^3 - M_2/r^3)$$

So converting to 3D coordinates has done NOTHING to explain my problem.

Paradox 1 Reappears: Suppose I turn off the moon. Then I get

$$2k(A^2/x^3) = mG\ (-M_2/r^3)x$$

$$2k(A^2/y^3) = mG\ (-M_2/r^3)y$$

$$2k(C^2/z^3) = mG\ (-M_2/r^3)$$

and this is supposed to give a sphere!!!

So I just wasted 2 hours maybe seeing if going from 2D to 3D would fix this paradox. A bad hunch.

Let's try a simpler problem! There is no moon, only earth gravity so

$$\mathbf{F} = mG\ \{\ -M_2/r^3)x\ \hat{\mathbf{x}} + (-M_2/r^3)y\hat{\mathbf{y}} + (-M_2/r^3)z\ \hat{\mathbf{z}}\ \}$$

This is supposed to be related to a sphere.

$$x^2 + y^2 + z^2 = A^2$$

How is this sphere connected? It is not hard to see in sphericals

$$\mathbf{F} = -(mG/r^2)\hat{\mathbf{r}}$$

$$V(r) = (mG/r^2)$$

So V is constant on a sphere of radius r. But how does that work in Cartesians?

$$V = (mG)\frac{1}{x^2+y^2+z^2}$$

The sphere is "on the bottom". So you don't say

$$V(r) = k\,(x^2+y^2+z^2)$$

You just need to have "V be constant on a sphere".

In this case I can integrate the force to get a potential:

$$\mathbf{F} = -(mG/r^2)\hat{\mathbf{r}}$$

$$V(r) = \int_{\mathbf{r}}^{\infty} \mathbf{F} \bullet \mathbf{d}r = -\int_{\mathbf{r}}^{\infty} (mG/r^2) = mG/r$$

If I try to do that in Cartesians, how does it go?

$$\mathbf{F} = mG\ \{\ (-M_2/r^3)x\,\hat{\mathbf{x}} + (-M_2/r^3)y\hat{\mathbf{y}} + (-M_2/r^3)z\,\hat{\mathbf{z}}\ \}$$

$$V(r) = \int_{\mathtt{path}} \mathbf{F} \bullet \mathbf{d}r = \int_{\mathtt{path}} \{\ \{\ (-M_2/r^3)xdx + (-M_2/r^3)ydy + (-M_2/r^3)zdz\ \}$$

You can pick any path you want to make the integral be simple. Suppose I try a path from the divergent origin and do it in Cartesian steps. Then

$$V(r) = \int_{0}^{x} (-M_2/r^3)xdx + \int_{0}^{y} (-M_2/r^3)ydy + \int_{0}^{z} (-M_2/r^3)zdz$$

I will need:

$$\int_{\infty}^{x} \frac{xdx}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2+z^2}} \qquad \text{// says Maple}$$

Somehow it integral is path independent it seems that you get

$$\frac{1}{\sqrt{x^2}} + \left\{ \frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2}} \right\} + \left\{ \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{1}{\sqrt{x^2+y^2}} \right\} = 1/r$$

and there is your hard-won result.

Int (∞,0,0) to (x,0,0)] x integrand
+ Int (x,0,0) to (x,y,0)] y integrand
+ Int (x,y,0) to (x,y,z)] z integrand giving what I quote just above.

So in the sphere case, you just want to show that V = V(r) so it is constant on a sphere.

Another way:

$$V(r) = \int_{\infty 00}^{x00} (-M_2/r^3)xdx + \int_{x00}^{xy0} (-M_2/r^3)ydy + \int_{xy0}^{xyz} (-M_2/r^3)zdz$$

The three indefinite integrals are all the same!! So could write

$$V(r) = I_1(x,0,0) - I_1(\infty,0,0) + I_1(x,y,0) - I_1(x,0,0) + I_1(x,y,z) - I_1(x,y,0)$$

$$= I_1(x,y,z) - I_1(\infty,0,0) = I_1(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

What do you do for an ellipse? What is constant on an ellipse? The string distance.

I think I have stated the problem correctly, going back to 2D, $\nabla V = + \mathbf{F}$

$$\nabla V(x,y) = Gm \{ [-M_2/r^3 + 2(M_1/d_0^3)] x \hat{\mathbf{x}} + [-M_2/r^3 - (M_1/d_0^3)] y \hat{\mathbf{y}} \}$$

Compute V(x,y) and show it is constant on the desired ellipse.

How about just *guessing* the V(x,y) ! For the two $1/r^3$ terms I know that

$$V_{13}(x,y) = GmM_2/r \qquad r = \sqrt{x^2+y^2}$$

For term #2 my guess is

$$V_2(x,y) = Gm\ 2(M_1/d_0^3)\ (x^2/2)$$

For term #4 my guess is

$$V_2(x,y) = -Gm\ (M_1/d_0^3)\ (y^2/2)$$

Then my candidate potential is this

$$V(x,y) = GmM_2/r + Gm\ 2(M_1/d_0^3)\ (x^2/2) - Gm\ (M_1/d_0^3)\ (y^2/2) \}$$

$$= Gm \{ M_2/r + 2(M_1/d_0^3)\ (x^2/2) - (M_1/d_0^3)\ (y^2/2) \}$$

So this is constant not on a circle but on a distorted circle. We then have this equation

$M_2/r + 2(M_1/d_0^3)(x^2/2) - (M_1/d_0^3)(y^2/2) = k$ // some constant

$M_2/r = k - 2(M_1/d_0^3)(x^2/2) + (M_1/d_0^3)(y^2/2)$

$M_2^2 = (x^2+y^2)[\, k - 2(M_1/d_0^3)(x^2/2) + (M_1/d_0^3)(y^2/2)]^2$

$= (x^2+y^2)k^2\,[1 - 2(M_1/kd_0^3)(x^2/2) + (M_1/kd_0^3)(y^2/2)]^2$

$\approx k^2(x^2+y^2)\,[\, 1 + 2*\{-2(M_1/kd_0^3)(x^2/2) + (M_1/kd_0^3)(y^2/2)\}\,]$

$\approx k^2(x^2+y^2)\,[\, 1 - 4(M_1/kd_0^3)(x^2/2) + 2(M_1/kd_0^3)(y^2/2)\,]$

$\approx k^2(x^2+y^2)\,[\, 1 - 2(M_1/kd_0^3)x^2 + (M_1/kd_0^3)y^2\,]$

$M_2^2 \approx k^2(x^2+y^2)\,[\, 1 - 2\varepsilon x^2 + \varepsilon y^2\,]$ $\varepsilon = (M_1/kd_0^3)$

$(M_2/k)^2 \approx (x^2+y^2)\,[\, 1 - 2\varepsilon x^2 + \varepsilon y^2\,]$

Since $\varepsilon << 1$. we see that $R_2 \approx M_2/k$ = radius of the basic circle (R_2 = radius of Earth). My quartic is now

$R_2^2 \approx (x^2+y^2)\,[\, 1 - 2\varepsilon x^2 + \varepsilon y^2\,]$ $k = (M_2/R_2)$ $\varepsilon = (M_1/kd_0^3) = (M_1/M_2)\,R_2/d_0^3$

Imagine plotting this thing.

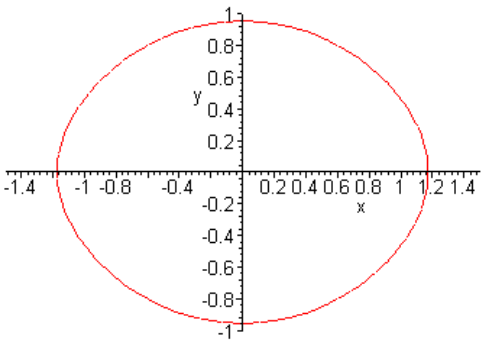
```
restart;with(plots):
e := 0.1;
```

$e := .1$

```
f := (x^2+y^2)*(1-2*e*x^2+e*y^2);
```

$f := (x^2+y^2)(1-.2x^2+.1y^2)$

```
implicitplot(f=1,x=-1.5..1.5,y=-2..2, scaling = constrained, view=[-1.5..1.5,-1..1]);
```



So there is your ellipse-like curve. How far to the right of 1 does it go? At that point we have

$$R_2^2 \approx x^2 \ [\ 1 - 2\varepsilon x^2\ ]$$

which you have to solve for x. Write as

$$a \approx z \ [\ 1 - 2\varepsilon z\ ]$$

```
restart;
f := a-z+2*e*z^2;
```

$$f := a - z + 2 e z^2$$

```
s := solve(f=0,z);
```

$$s := \frac{1}{4}\frac{1+\sqrt{1-8ea}}{e}, \frac{1}{4}\frac{1-\sqrt{1-8ea}}{e}$$

```
g := s[2];
```

$$g := \frac{1}{4}\frac{1-\sqrt{1-8ea}}{e}$$

```
series(g,e);
```

$$a + 2a^2 e + 8a^3 e^2 + 40a^4 e^3 + 224a^5 e^4 + O(e^5)$$

So the solution is that $z = a + 2ea^2$ which I translate to

$$x^2 = R_2^2 + 2\varepsilon R_2^4 = R_2^2(1 + 2\varepsilon R_2^2)$$

$$x = R_2\sqrt{1 + 2\varepsilon R_2^2} \approx R_2(1 + \varepsilon R_2^2) = R_2 + \varepsilon R_2^3$$

Then the little tide height is

$$h = \varepsilon R_2^3 = (M_1/M_2)\, R_2/d_0^3 * R_2^3 = R_2\ (M_1/M_2)\ (R_2/d_0)^3$$

My answer to this question from Section 8,8 is this:

$$a = (3/4)\, R_2\ (M_1/M_2)\, (R_2/d_0)^3$$

which is pretty close. So why don't I have a 3/4 sitting there?

I think I could redo this doing a better job.

You see the distortion, but this is a quartic which I have to approximate with a quadratic somehow. General problem:

$$f(x,y) = x^2+y^2 + \varepsilon_1(x^2+y^2)x^2 + \varepsilon_2(x^2+y^2)y^2$$

$$= x^2+y^2 + \varepsilon_1 x^4 + (\varepsilon_1+\varepsilon_2)x^2y^2 + \varepsilon_2 y^4$$

The equation of an ellipse is this

$$A^2y^2+B^2x^2 = x^2y^2$$

Could do a least squares fit to estimate A and B. Go back to

$$M_1^2 \approx k^2(x^2+y^2)\ [\ 1 + 2\varepsilon x^2 - \varepsilon y^2\ ] \qquad ` \qquad \varepsilon = 2(M_1/kd_0^3)$$

Accept this as a quartic, not an ellipse. Have

$$(x^2+y^2)\ [\ 1 + 2\varepsilon x^2 - \varepsilon y^2\ ] = (M_1/k)^2$$

At the left edge how far is it displaced from the circle :

$$x_{circle} = (M_1/k)$$

$$x_q^2\ [\ 1 + 2\varepsilon x_q^2\ ] = (M_1/k)^2$$

$$2\varepsilon x_q^4 + x_q^2 - (M_1/k)^2 = 0$$

$$x_q^2 = [\ -1 \pm\sqrt{1+4(2\varepsilon)(M_1/k)^2}\ ]/(4\varepsilon)$$

$$\approx [\ -1 \pm(1 + 4\varepsilon(M_1/k)^2 -(1/8)\ 8^2\varepsilon^2(M_1/k)^4\ )\ ]\ /(4\varepsilon)$$

$$\approx [\ -1 +(1 + 4\varepsilon(M_1/k)^2 -(1/8)\ 8^2\varepsilon^2(M_1/k)^4\ )\ ]\ /(4\varepsilon)$$

$$\approx [\ 4\varepsilon(M_1/k)^2 -8\varepsilon^2(M_1/k)^4\ )\ ]\ /(4\varepsilon)$$

$$= [\ (M_1/k)^2 -2\varepsilon(M_1/k)^4$$

$$= (M_1/k)^2 -2\varepsilon(M_1/k)^4$$

$$= (M_1/k)^2\ (\ 1 -2\varepsilon(M_1/k)^2)$$

$$= x_{circle}^2\ (\ 1 -2\varepsilon(M_1/k)^2)$$

$$x_q = x_{circle}\sqrt{1 - 2\varepsilon(M_1/k)^2}$$

Sign is wrong, but there is a plan here where you evaluate the thing "a".

I am not sure why this method is so horrible! It gives a quartic to which you have to fit an ellipse, and then you can get A and B for the ellipse, incredibly ugly!

Taylor never mentions "ellipse" but just does what I am doing above to get the tide height!

Go back to the surface which I think is exact

$$M_1^2 \approx k^2(x^2+y^2)\,[\,1 - 2\varepsilon x^2 - \varepsilon y^2\,] \qquad \varepsilon = 2(M_1/kd_0^3)$$