Tide potential Calculation PhL 1.18.17

We seek a potential V(x,y) which solves this equation (no minus sign)

$$\nabla V(x,y) = \mathbf{F}_{tid} = -[(GM_2m/r^2)]\ \hat{\mathbf{r}} + (mM_1G/d_0^3)(2x\ \hat{\mathbf{x}} - y\ \hat{\mathbf{y}}) \qquad // \ r = \sqrt{x^2+y^2} \qquad (8.8.63)$$

The solution is the following (by inspection)

$$V(x,y) = (GM_2m/r) + (GM_1m/d_0^3)(x^2 - y^2/2)\ . \qquad // \ \nabla(1/r) = -(1/r^2)\hat{\mathbf{r}},\ \ r > 0 \qquad (8.8.64)$$

Of interest are surfaces on which V is a constant, so

$$(M_2/r) + (M_1/d_0^3)(x^2 - y^2/2) = k \qquad (8.8.65)$$

$$M_2^2 = r^2[k - (M_1/d_0^3)(x^2 - y^2/2)]^2 = k^2r^2[1 - (M_1/kd_0^3)(x^2 - y^2/2)]^2$$

$$\approx k^2r^2[1 - (M_1/kd_0^3)(2x^2 - y^2)] \qquad \Rightarrow$$

$$(M_2/k)^2 = r^2[1 - \varepsilon(2x^2 - y^2)] \qquad \varepsilon \equiv (M_1/kd_0^3)$$

$$= (x^2+y^2)(1 - 2\varepsilon x^2 + \varepsilon y^2) \qquad \dim(M_1/k) = L$$

$$= r^2(1 - 2\varepsilon r^2\cos^2\theta + \varepsilon r^2\sin^2\theta) \qquad \dim(\varepsilon) = L^{-2} \qquad (8.8.66)$$

Maple file "tides by potential method" has code and graphic: (8.8.67)

This is a quadratic equation in r^2 which has this solution (after some work), to lowest order in ε

$$r(\theta) \approx (M_2/k)[1 - (1/8)\varepsilon(\sin^2\theta - 2\cos^2\theta)(M_2/k)^2] \quad // \text{ see v3 of current doc} \qquad (8.8.68)$$

As was done in **, we require that the average value of $r(\theta)$ over θ be R_2 . We find

$$\langle r(\theta)\rangle = c[1 - (1/8)\varepsilon\{(1/2) - 2(1/2)\}c^2]$$

$$= c[1 - (1/16)c^2\varepsilon\{1-2\}]$$

$$= (M_2/k)[1 + (1/16)(M_2/k)^2\varepsilon] = R_2 \qquad (8.8.69)$$

Solving for (M_2/k) gives

$$(M_2/k) = R_2(1 - \alpha\varepsilon R_2^2) \qquad (M_2/k) \approx R_2 \qquad (8.8.70)$$

so the locus of the cubic is this

$$R_2^2(1-\alpha\varepsilon R_2^2)^2 = (x^2+y^2)\,[1-\varepsilon(2x^2-y^2)] \qquad (8.8.71)$$

$$\varepsilon = (M_1/kd_0^3) \approx (M_1/[M_2/R_2]d_0^3) = (M_1/M_2)R_2/d_0^3 \qquad (8.8.71)$$

At the right edge of the quartic we set y = 0 and find that

$$x \approx R_2\,[\,1 + (15/16)(M_1/M_2)(R_2/d_0)^3] \qquad (8.8.72)$$

which says that the high tide is

$$|\Delta|_{high} = (15/16)(M_1/M_2)R_2^4/d_0^3$$

At the top edge of the quartic we set x = 0 and find that

$$|\Delta|_{low} = (9/16)(M_1/M_2)R_2^4/d_0^3 \qquad (8.8.72)$$

The height difference is then

$$H = |\Delta|_{high} + |\Delta|_{high} = (M_1/M_2)R_2^4/d_0^3\,\{\,(15/16)+(9/16)\,\}$$

$$= (3/2)(M_1/M_2)R_2^4/d_0^3 \qquad (8.8.72)$$

In (8.8.33) we have H = 2a so the result is then

$$a = (3/4)(M_1/M_2)R_2^4/d_0^3 \qquad (8.8.73)$$

in agreement with (8.8.41) .

What happens now at the right edge of the quadric where y = 0:

$$R_2^2(1-\alpha\varepsilon R_2^2)^2 = (x^2)\,[1-\varepsilon(2x^2)] = x^2(1-2\varepsilon x^2) \approx x^2(1-2\varepsilon R_2^2)$$

Then it seems that

$$x = R_2(1-\alpha\varepsilon R_2^2) * (1-2\varepsilon R_2^2)^{-1/2}$$

$$\approx R_2(1-\alpha\varepsilon R_2^2) * (1+\varepsilon R_2^2)$$

$$\approx R_2\,[\,1+\varepsilon R_2^2(1-\alpha)]$$

$$\approx R_2\,[\,1+(M_1/M_2)R_2/d_0^3R_2^2(15/16)]$$

$$\approx R_2\,[\,1+(M_1/M_2)(R_2/d_0)^3(15/16)] \qquad (8.8.72)$$

So the high tide is then

$$h_{high} = (M_1/M_2)(R_2/d_0)^3 R_2(15/16)$$

Now what happens at the top edge of the quadric where x = 0

$$R_2^2(1-\alpha\varepsilon R_2^2)^2 = (x^2)\,[1 -\varepsilon(2x^2)] = y^2(1+\varepsilon y^2) \approx y^2(1 + \varepsilon R_2^2)$$

It seems then that

$$y = R_2(1-\alpha\varepsilon R_2^2) * (1 + \varepsilon R_2^2)^{-1/2}$$

$$\approx R_2(1-\alpha\varepsilon R_2^2) * (1-[1/2]\varepsilon R_2^2)$$

$$\approx R_2(1 -\alpha\varepsilon R_2^2 -[1/2]\varepsilon R_2^2)$$

$$\approx R_2(1 -\varepsilon R_2^2(1/16 + 8/16))$$

$$\approx R_2(1 -\varepsilon R_2^2(9/16))$$

$$\approx R_2(1 -(M_1/M_2)R_2/d_0^3 R_2^2(9/16))$$

$$\approx R_2(1 -(M_1/M_2)(R_2/d_0)^3(9/16)) \qquad (8.8.72)$$

So the low tide is then (abs val)

$$h_{low} = (M_1/M_2)(R_2/d_0)^3 R_2(9/16)$$

We then add these to get

$$\text{"2a"} = (M_1/M_2)(R_2/d_0)^3 R_2\,[\,(15/16)+ (9/16)\,]$$

$$= (M_1/M_2)(R_2/d_0)^3 R_2\,[\,(24/16)\,]$$

$$= (M_1/M_2)(R_2/d_0)^3 R_2\,[\,(3/2)\,] \qquad (8.8.72)$$

Then

$$\text{"a"} = (3/4)\ (M_1/M_2)(R_2/d_0)^3 R_2 \qquad \text{// it works!!!!} \qquad (8.8.73)$$

This locus is close to the locus of a circle of some radius R

$$R^2 = (x^2+y^2)$$

or

$R^2 = (x^2+y^2)(1 - 2\varepsilon x^2+\varepsilon y^2)$ $\quad \varepsilon \equiv (M_1/kd_0^3)$ $\quad R = M_2/k = \text{constant}$

Since d_0 is large, ε is small, and $R^2 \approx (x^2+y^2)$ so $1/k \approx R/M_2$. Then

$(M_2/k)^2 \approx (x^2+y^2)(1 - 2\varepsilon x^2+\varepsilon y^2)$ $\quad \varepsilon = (M_1/M_2)/d_0^3$

This is a quartic curve which resembles an ellipse. For example, for $\varepsilon = .05$ and $M_2/k = 1$:

```
f := (x^2+y^2)*(1-2*e*x^2+e*y^2);
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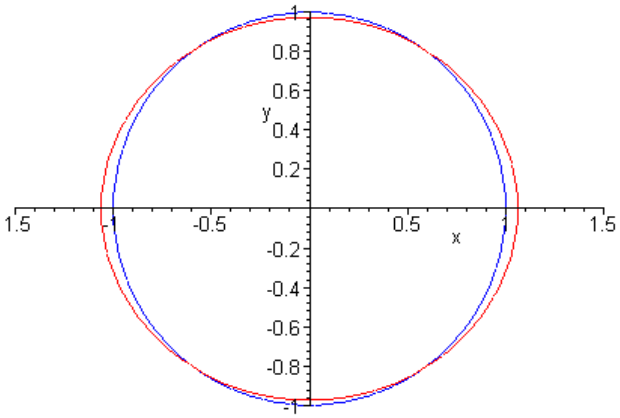
$$f := (x^2+y^2)(1-2ex^2+ey^2)$$

```
e := 0.05:
s1 := implicitplot(f=1,x=-1.5..1.5,y=-2..2, scaling = constrained,
view=[-1.5..1.5,-1..1],color=red):
s2 := implicitplot(x^2+y^2=1,x=-1.5..1.5,y=-2..2,color=blue):
display(s1,s2);
```



where the blue curve is a circle of radius 1.

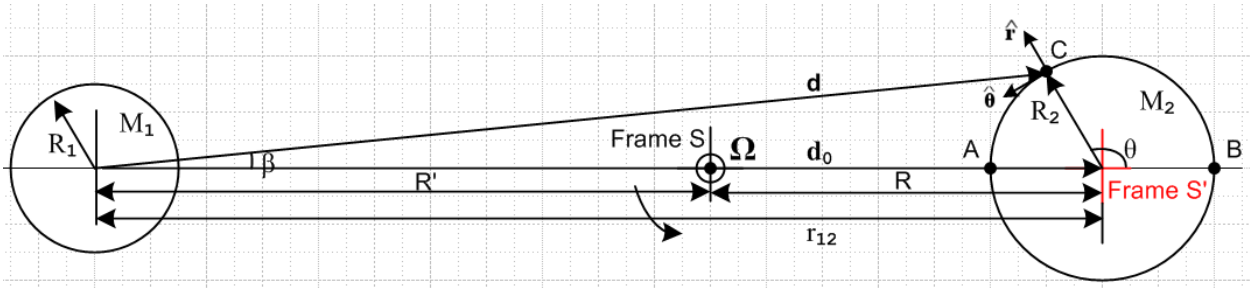
$Gm(M_1/d_0^3)(x^2/2) - Gm(M_1/d_0^3)(y^2/2)\ \}$

Start with

$$\mathbf{F}_{tid} = mM_1G\left(\hat{\mathbf{d}}_0/d_0^2 - \hat{\mathbf{d}}/d^2\right)$$

$$= mM_1G\left(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\right)$$

and here is our picture



In Cartesian coordinates we can write

$d_x = d_0 + R_2 \cos\theta$ $\qquad\qquad$ $d_y = R_2 \sin\theta$

$\mathbf{d} = (d_0 + R_2 \cos\theta)\,\hat{\mathbf{x}} + R_2 \sin\theta\,\hat{\mathbf{y}}$

$\mathbf{d}_0 = d_0 \hat{\mathbf{x}}$

Then,

$$\mathbf{F}_{tid} = mM_1 G \left(d_0\hat{\mathbf{x}}/d_0^3 - \left[(d_0 + R_2 \cos\theta)\,\hat{\mathbf{x}} + R_2 \sin\theta\,\hat{\mathbf{y}} \right]/d^3 \right)$$

now use

$$d^2 = R_2^2 + d_0^2 - 2d_0R_0\cos(\pi-\theta) = R_2^2 + d_0^2 + 2d_0R_0\cos\theta = d_0^2[\,1 + (R_2/d_0)^2 + 2\,(R_2/d_0)\cos\theta\,]$$

$$\approx d_0^2[\,1 + 2\,(R_2/d_0)\cos\theta\,]$$

to get

$$d^{-3} \approx d_0^{-3}\,[\,1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3}\,[\,1 - 3(R_2/d_0)\cos\theta\,] .$$

Then,

$$\mathbf{F}_{tid} = mM_1 G \left(\mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \right)$$

$$\approx mM_1 G \left(d_0\hat{\mathbf{x}}/d_0^3 - \left[(d_0 + R_2 \cos\theta)\,\hat{\mathbf{x}} + R_2 \sin\theta\,\hat{\mathbf{y}} \right] \right) d_0^{-3}\,[\,1 - 3(R_2/d_0)\cos\theta\,]$$

$$= (mM_1G/d_0^2) \left(\hat{\mathbf{x}} - \left[(1 + (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} + (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} \right] [\,1 - 3(R_2/d_0)\cos\theta\,] \right)$$

$$= (mM_1G/d_0^2) \left(\hat{\mathbf{x}} - \hat{\mathbf{x}} - (R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} + 3(R_2/d_0)\cos\theta\,\hat{\mathbf{x}} \right) + O((R_2/d_0)^2)$$

$$\approx (mM_1G/d_0^2) \left(-(R_2/d_0)\cos\theta)\,\hat{\mathbf{x}} - (R_2/d_0)\sin\theta\,\hat{\mathbf{y}} + 3(R_2/d_0)\cos\theta\,\hat{\mathbf{x}} \right)$$

$$= (mM_1G/d_0^2)(R_2/d_0) \left(-\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}} + 3\cos\theta\,\hat{\mathbf{x}} \right)$$

$$= (mM_1G/d_0^2)(R_2/d_0) \left(2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}} \right)$$

Now check this at the compass points

$\mathbf{F}_{tid}(0) = (mM_1G/d_0^2)(R_2/d_0)\,2\hat{\mathbf{x}}$ $\qquad$ point B $\qquad$ to the right

$\mathbf{F}_{tid}(\pi) = -(mM_1G/d_0^2)(R_2/d_0)\,2\hat{\mathbf{x}}$ $\qquad$ point A $\qquad$ to the left

$\mathbf{F}_{tid}(\pi/2) = -(mM_1G/d_0^2)(R_2/d_0)\,\hat{\mathbf{y}}$ down at top

$\mathbf{F}_{tid}(-\pi/2) = +(mM_1G/d_0^2)(R_2/d_0)\,\hat{\mathbf{y}}$ up at bottom

Compass agrees. I like the general formula:

$$\mathbf{F}_{tid}(\theta) \approx (mM_1G/d_0^2)(R_2/d_0)\,(2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}})$$

How might I plot this?

$x = r\cos\theta$
$y = r\sin\theta$

$$\mathbf{F}_{tid}(r,\theta) \approx (mM_1G/d_0^2)(1/d_0)\,(2r\cos\theta\,\hat{\mathbf{x}} - r\sin\theta\,\hat{\mathbf{y}})$$

$$= (mM_1G/d_0^3)\,(2x\hat{\mathbf{x}} - y\hat{\mathbf{y}})$$

```
with(plots):with(plottools):
s1 := fieldplot([2*x,-y],x=-1..1,y=-1..1,grid=[10,10]):
s2 := circle([0,0],1):
display(s1,s2,scaling=constrained);
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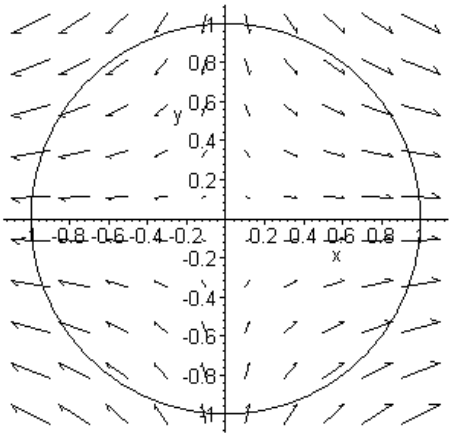


Very good. You want to think of $R_2 = r$ as a variable.

What is it in polar coordinates? ($r = R_2$)

$$\mathbf{F}_{tid}(\theta) \approx (mM_1G/d_0^2)(r/d_0)\,(2\cos\theta\,\hat{\mathbf{x}} - \sin\theta\,\hat{\mathbf{y}})$$

$$= (mM_1G/d_0^2)(r/d_0)(2\cos\theta\,[\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}] - \sin\theta\,[\sin\theta\,\hat{\mathbf{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}])$$

$$= (mM_1G/d_0^2)(r/d_0)[(2\cos^2\theta - \sin^2\theta)\hat{\mathbf{r}} + (-2\sin\theta\cos\theta - \sin\theta\cos\theta)\hat{\boldsymbol{\theta}}]$$

$$= (mM_1G/d_0^2)(r/d_0)[\ (3\cos^2\theta-1)\hat{\mathbf{r}} + (-3\sin\theta\cos\theta)\hat{\boldsymbol{\theta}}\ ]$$

$$= (mM_1G/d_0^2)(r/d_0)[\ (3\{1/2)(1+\cos2\theta)\}-1)\ \hat{\mathbf{r}} + (-(3/2)\sin2\theta)\hat{\boldsymbol{\theta}}\ ]$$

$$= (1/2)(mM_1G/d_0^2)(r/d_0)[\ (3\{(1+\cos2\theta)\}-2)\ \hat{\mathbf{r}} + (-3\sin2\theta)\hat{\boldsymbol{\theta}}\ ]$$

$$= (1/2)(mM_1G/d_0^2)(r/d_0)[\ (1 + 3\cos2\theta)\ \hat{\mathbf{r}}\ - 3\sin2\theta\hat{\boldsymbol{\theta}}\ ]$$

$$= (3/2)(mM_1G/d_0^2)(r/d_0)[\ ([1/3] + \cos2\theta)\ \hat{\mathbf{r}}\ - \sin2\theta\hat{\boldsymbol{\theta}}\ ]$$

This says

$\hat{\boldsymbol{\theta}}$ piece $= -(3/2)(mM_1G/d_0^2)(r/d_0)\sin2\theta$	horizontal	(5) Butikov
$\hat{\mathbf{r}}$ piece $= (3/2)(mM_1G/d_0^2)(r/d_0)(\cos2\theta +1/3)$	vertical	(6) Butikov

So far so good!

Can this force be represented by a potential? In electrostatics we have

$$\mathbf{E} = -\nabla\varphi \quad \Rightarrow \quad \text{div}\ \mathbf{E} = -\nabla^2\varphi = 0 \quad \text{region of no charge}$$

$$\text{curl}\mathbf{E} = -\nabla \times (\nabla\varphi) = 0$$

So **E** must have zero divergence and zero curl in order to be rep by a potential. So consider

$$\text{div}\ \mathbf{F}_{tid} = \text{div}(\ 2x\hat{\mathbf{x}} - y\hat{\mathbf{y}}\ ) = \partial^2_x(2x) + \partial^2_y(-y) = 0 + 0 = 0$$

$$\text{curl}\ \mathbf{F}_{tid} = [\partial_x F_y - \partial_y F_x]\ \hat{z} + \text{cyclic}$$

$$[\ 0 - 0] + 0 + 0 = 0$$

Therefore you NOT represent $\mathbf{F}_{tid}$ by a potential.

WRONG!!! First of all, div $\mathbf{E} = -\nabla^2\varphi$ = "ρ" and ≠ 0.
Second of all, conservativity only involves the curl!

So the tidal force is a conservative force.

Here then is the force for which we seek a potential

$$\mathbf{F}_{g2} = -[(GM_2m)/r^3]\ \mathbf{r} \qquad \text{Earth gravity} \qquad \text{ok}$$

$\mathbf{F}_{tid} = (mM_1G/d_0^3)\,(\,2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}\,)$ $\quad$ $\dim(F) = M\,L/T^2$ $\quad$ ok

$\dim(GMm/r^2) = M\,L/T^2$ $\quad$ $\dim(GMm) = M\,L^3/T^2$

Now consider $\quad$ $\dim(V) = M\,L^2/T^2$

$\nabla V(x,y) = -[(GM_2m)/r^3]\;\mathbf{r} + (mM_1G/d_0^3)\,(\,2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}\,)$ $\quad$ ok

$= -[(GM_2m)/r^3](\,x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}}\,) + (mM_1G/d_0^3)\,(\,2x\,\hat{\mathbf{x}} - y\,\hat{\mathbf{y}}\,)$

$= [\,-(GM_2m)/r^3) + (mM_1G/d_0^3)2\,]\;x\,\hat{\mathbf{x}} + [\,-(GM_2m)/r^3) - (mM_1G/d_0^3)\,]\;y\,\hat{\mathbf{y}}$

$= Gm\,\{\,[\,-M_2/r^3 + 2(M_1/d_0^3)\,]\;x\,\hat{\mathbf{x}} + [\,-M_2/r^3 - (M_1/d_0^3)\,]\;y\,\hat{\mathbf{y}}\}$ $\quad$ ok

Now suppose you try as ansatz, **WRONG!!!!**

$V(x,y) = k[A^2/x^2 + B^2/y^2 - 1]$ $\quad$ ok $\quad$ // an arbitrary ellipse centered at Earth center

$\nabla V(x,y) = -2kA^2/x^3\,\hat{\mathbf{x}} - 2kB^2/y^3\,\hat{\mathbf{y}}$ $\quad$ ok $\quad$ // $\dim(k) = M\,L^2/T^2$

This does not seem to work, why is that? You would need now to have

$-2kA^2/x^3\,\hat{\mathbf{x}} = Gm\,[\,-M_2/r^3 + 2(M_1/d_0^3)\,]\;x\,\hat{\mathbf{x}}$

$-2kB^2/y^3\,\hat{\mathbf{y}} = Gm\,[\,-M_2/r^3 - (M_1/d_0^3)\,]\;y\,\hat{\mathbf{y}}$

or

$-2kA^2 = Gm\,[\,-M_2/r^3 + 2\,(M_1/d_0^3)\,]\;x^4$

$-2kB^2 = Gm\,[\,-M_2/r^3 - (M_1/d_0^3)\,]\;y^4$ $\qquad$ $r^2 = x^2+y^2$

Just check dimensions please.

$\dim(kA^2) = M\,L^4/T^2$ $\quad$ $\dim(GmMx^4/r^3) = M\,L^3/T^2 * L = M\,L^4/T^2$

so dimensions are OK. I do know that

$r << d_0$ $\quad$ $x << d_0$ $\quad$ $y << d_0$

but this does not seem to help at all. I am stumped, this will take another day or two of burned time to figure out. It is so simple, what am I doing wrong here?

Paradox #1. Suppose I let $M_1 \rightarrow 0$ so the moon goes away slowly. The last two equations then become

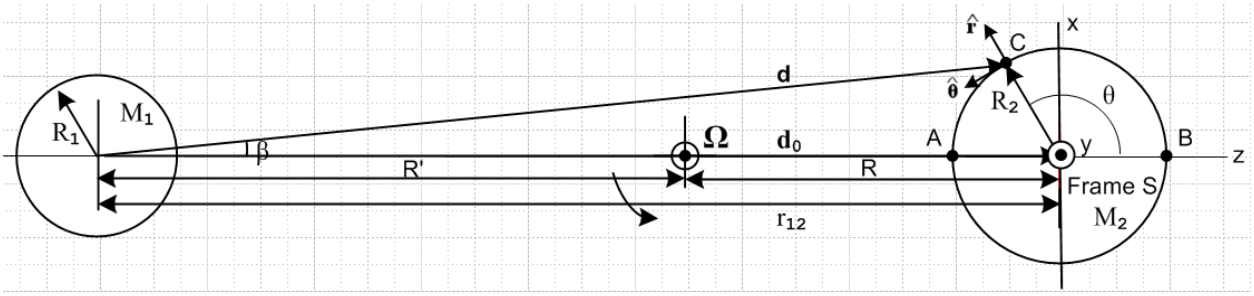
$-2kA^2 = Gm\,[\,-M_2/r^3\,]\;x^4$

$-2kB^2 = Gm\ [\ -M_2/r^3\ ]\ y^4$ $\qquad\qquad$ $r^2 = x^2+y^2$

I expect the potential surface to be a circle rather than an ellipse since there are now no tidal forces, and that means I expect to have A = B. But instead I have A ≠B. This should give A = B = R_2 somehow.

Maybe I have to do this in 3D space and not fake 2D space?

Start Over! Use spherical coordinates r,θ,φ centered on the Earth with z to the right.



$\mathbf{F}_{tid} = mM_1G\ \{\ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3\ \}$

$\mathbf{d}_0 = d_0\hat{\mathbf{z}}$

$\mathbf{C} = \mathbf{r} = r\sin\theta\cos\varphi\ \hat{\mathbf{x}} + r\sin\theta\sin\varphi\ \hat{\mathbf{y}} + r\cos\theta\ \hat{\mathbf{z}} = \mathbf{x}$

Moon center = $\mathbf{c}_m = -d_0\hat{\mathbf{z}}$

$\mathbf{d} = \mathbf{C} - \mathbf{c}_m = r\sin\theta\cos\varphi\ \hat{\mathbf{x}} + r\sin\theta\sin\varphi\ \hat{\mathbf{y}} + r\cos\theta\ \hat{\mathbf{z}} + d_0\hat{\mathbf{z}}$

$= r\sin\theta\cos\varphi\ \hat{\mathbf{x}} + r\sin\theta\sin\varphi\ \hat{\mathbf{y}} + (r\cos\theta+d_0)\hat{\mathbf{z}}$

$d^2 = (\ r\sin\theta\cos\varphi)^2 + (\ r\sin\theta\sin\varphi)^2 + (r\cos\theta+d_0)^2$

$= r^2\sin^2\theta + (d_0 + r\cos\theta)^2$

$= r^2\sin^2\theta + d_0^2(1 + [r/d_0]\cos\theta)^2$

$\approx r^2\sin^2\theta + d_0^2(1 + 2\,[r/d_0]\cos\theta)$

$= d_0^2\ [\ \ [r/d_0]^2\sin^2\theta + d_0^2(1 + 2\,[r/d_0]\cos\theta)$

$\approx d_0^2(1 + 2\,[r/d_0]\cos\theta)$ $\qquad\qquad$ // ignore quadratic term

This is same as in Section 8.8, no surprise. So still get

$$d^{-3} \approx d_0^{-3}\,[\ 1 + 2(R_2/d_0)\cos\theta]^{-3/2} \approx d_0^{-3}\,[\ 1 - 3(R_2/d_0)\cos\theta\ ]\ . \qquad (8.8.22)$$

Now:

$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$= mM_1G \{ [d_0\hat{\mathbf{z}}]/d_0^3 - [r\sin\theta\cos\varphi\ \hat{\mathbf{x}} + r\sin\theta\sin\varphi\ \hat{\mathbf{y}} + (r\cos\theta+d_0)\hat{\mathbf{z}}]/d^3 \}$

$= mM_1G \{ \hat{\mathbf{z}}\ d_0^2 - [r\sin\theta\cos\varphi\ d^{-3}\hat{\mathbf{x}} + r\sin\theta\sin\varphi d^{-3}\ \hat{\mathbf{y}} + (r\cos\theta+d_0)d^{-3}\hat{\mathbf{z}}]\}$

$= mM_1G \{ - [r\sin\theta\cos\varphi\ d^{-3}\hat{\mathbf{x}} + r\sin\theta\sin\varphi d^{-3}\ \hat{\mathbf{y}} + \{d_0^2+(r\cos\theta+d_0)d^{-3}\}\ \hat{\mathbf{z}}]\}$

STOP. Why not do this all in Cartesians.

Start over

$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$\mathbf{d}_0 = d_0\hat{\mathbf{z}}$
$\mathbf{C} = \mathbf{r} = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}}$

Moon center = $\mathbf{c}_m = -d_0\hat{\mathbf{z}}$

$\mathbf{d} = \mathbf{C} - \mathbf{c}_m = x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + z\ \hat{\mathbf{z}} + d_0\hat{\mathbf{z}}$

$= x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + (d_0+z)\ \hat{\mathbf{z}}$

$d^2 = x^2 + y^2 + (d_0+z)^2$

$= d_0^2\{ [1 + (z/d_0)]^2 + (x/d_0)^2 + (y/d_0)^2 \}$

$\approx d_0^2\{ [1 +2 (z/d_0) + (z/d_0)^2] + (x/d_0)^2 + (y/d_0)^2 \}$

$\approx d_0^2[1 +2 (z/d_0)]$ dropping all quadratic terms

$d^{-3} \approx d_0^{-3}[1 +2 (z/d_0)]^{-3/2} \approx d_0^{-3}(1- 3(z/d_0))$

Now have

$\mathbf{F}_{tid} = mM_1G \{ \mathbf{d}_0/d_0^3 - \mathbf{d}/d^3 \}$

$= mM_1G \{ d_0\hat{\mathbf{z}}/d_0^3 - [x\ \hat{\mathbf{x}} + y\ \hat{\mathbf{y}} + (d_0+z)\ \hat{\mathbf{z}}]/d^3 \}$

$= mM_1G \{ \hat{\mathbf{z}}/d_0^2 - [(x/d^3)\ \hat{\mathbf{x}} + (y/d^3)\ \hat{\mathbf{y}} + ((d_0+z)/d^3)\ \hat{\mathbf{z}}] \}$

$$= \; mM_1G \{ - (x/d^3)\,\hat{\mathbf{x}} - (y/d^3)\,\hat{\mathbf{y}} - ((d_0+z)/d^3 - 1/d_0^2)\,\hat{\mathbf{z}}] \}$$

$$= - mM_1G \{ (x/d^3)\,\hat{\mathbf{x}} + (y/d^3)\,\hat{\mathbf{y}} + ((d_0+z)/d^3 - 1/d_0^2)\,\hat{\mathbf{z}}] \}$$

Now look at each component separately

$$(x/d^3) \approx d_0^{-3}(1 - 3(z/d_0))\, x \approx x/d_0^3$$

$$(y/d^3) \approx d_0^{-3}(1 - 3(z/d_0))\, y \approx y/d_0^3$$

$$(d_0+z)/d^3 - 1/d_0^2 = [(d_0+z)/d_0^3](1 - 3(z/d_0)) - 1/d_0^2$$

$$= [(1+(z/d_0))/d_0^2](1 - 3(z/d_0)) - 1/d_0^2$$

$$= \{ [(1+(z/d_0))](1 - 3(z/d_0)) - 1 \}/d_0^2$$

$$= \{ 1 + (z/d_0) - 3(z/d_0) - 3(z/d_0)^2 - 1 \}/d_0^2$$

$$\approx \{ - 2(z/d_0) \}/d_0^2 = -2z/d_0^3$$

We then end up with

$$\mathbf{F}_{tid} = - mM_1G\{ (x/d_0^3)\hat{\mathbf{x}} + (y/d_0^3)\hat{\mathbf{y}} + (-2z/d_0^3)\hat{\mathbf{z}} \}$$

$$= - (mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \}$$

and this then is my 3D generalization of previous result where I had y = 0. Could make 3D field plot.

I also have

$$\mathbf{F}_{g2} = -[(GM_2m)/r^3]\,\mathbf{r}$$

The total force acting on a particle m of water is then

$$\mathbf{F} = \mathbf{F}_{tid} + \mathbf{F}_{g2} = - (mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \} - [(GM_2m)/r^3]\,\mathbf{r}$$

$$= mG \{ (M_1/d_0^3 - M_2/r^3)x\,\hat{\mathbf{x}} + (M_1/d_0^3 - M_2/r^3)y\hat{\mathbf{y}} + (-2M_1/d_0^3 - M_2/r^3)z\,\hat{\mathbf{z}} \}$$

Now suppose I drop the $1/d_0^3$ terms since $d_0 >> r$ and this is OK for $M_1 \rightarrow 0$ as well. then

$$\mathbf{F} = \mathbf{F}_{tid} + \mathbf{F}_{g2} = - (mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \} - [(GM_2m)/r^3]\,\mathbf{r}$$

$= mG \{ - M_2/r^3)x \hat{\mathbf{x}} + (- M_2/r^3)y\hat{\mathbf{y}} + (- M_2/r^3)z \hat{\mathbf{z}} \}$

and only gravity is left.

NOW: Here is an ansatz ellipsoid for the potential surface :

$V(x,y,z) = k[A^2/x^2 + A^2/y^2 + C^2/z^2]$

$\nabla V = -2k[(A^2/x^3)\hat{\mathbf{x}} + (A^2/y^3)\hat{\mathbf{y}} + (C^2/z^3)\hat{\mathbf{z}}]$

Now I have the same problem as before I think. I have to identify: $\nabla V = \mathbf{F}_{tid}$ and then the following must be true:

$2k[(A^2/x^3)\hat{\mathbf{x}} + (A^2/y^3)\hat{\mathbf{y}} + (C^2/z^3)\hat{\mathbf{z}}] = (mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \} - [(GM_2m)/r^3] \mathbf{r}$

$= (mM_1G/d_0^3)\{ x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - 2z\hat{\mathbf{z}} \} - [(GM_2m)/r^3] \{x \hat{\mathbf{x}} + y \hat{\mathbf{y}} + z \hat{\mathbf{z}} \}$

$= mG \{ (M_1/d_0^3 - M_2/r^3)x \hat{\mathbf{x}} + (M_1/d_0^3 - M_2/r^3)y\hat{\mathbf{y}} + (-2M_1/d_0^3 - M_2/r^3) \hat{\mathbf{z}} \}$

which then requires that

$2k(A^2/x^3) = mG (M_1/d_0^3 - M_2/r^3)x$

$2k(A^2/y^3) = mG (M_1/d_0^3 - M_2/r^3)y$

$2k(C^2/z^3) = mG (-2M_1/d_0^3 - M_2/r^3)$

So converting to 3D coordinates has done NOTHING to explain my problem.

Paradox 1 Reappears: Suppose I turn off the moon. Then I get

$2k(A^2/x^3) = mG (- M_2/r^3)x$

$2k(A^2/y^3) = mG (- M_2/r^3)y$

$2k(C^2/z^3) = mG (- M_2/r^3)$

and this is supposed to give a sphere!!!

So I just wasted 2 hours maybe seeing if going from 2D to 3D would fix this paradox. A bad hunch.

Let's try a simpler problem! There is no moon, only earth gravity so

$$\mathbf{F} = mG \{ -M_2/r^3)x\,\hat{\mathbf{x}} + (-M_2/r^3)y\hat{\mathbf{y}} + (-M_2/r^3)z\,\hat{\mathbf{z}} \}$$

This is supposed to be related to a sphere.

$$x^2+y^2+z^2 = A^2$$

How is this sphere connected? It is not hard to see in sphericals

$$\mathbf{F} = -(mG/r^2)\hat{\mathbf{r}}$$

$$V(r) = (mG/r^2)$$

So V is constant on a sphere of radius r. But how does that work in Cartesians?

$$V = (mG)\frac{1}{x^2+y^2+z^2}$$

The sphere is "on the bottom". So you don't say

$$V(r) = k\,(x^2+y^2+z^2)$$

You just need to have "V be constant on a sphere".

In this case I can integrate the force to get a potential:

$$\mathbf{F} = -(mG/r^2)\hat{\mathbf{r}}$$

$$V(r) = \int_{\mathbf{r}}^{\infty} \mathbf{F} \bullet \mathbf{d}r = -\int_{\mathbf{r}}^{\infty} (mG/r^2) = mG/r$$

If I try to do that in Cartesians, how does it go?

$$\mathbf{F} = mG \{ (-M_2/r^3)x\,\hat{\mathbf{x}} + (-M_2/r^3)y\hat{\mathbf{y}} + (-M_2/r^3)z\,\hat{\mathbf{z}} \}$$

$$V(r) = \int_{\mathbf{path}} \mathbf{F} \bullet \mathbf{d}r = \int_{\mathbf{path}} \{ \{ (-M_2/r^3)xdx + (-M_2/r^3)ydy + (-M_2/r^3)zdz \}$$

You can pick any path you want to make the integral be simple. Suppose I try a path from the divergent origin and do it in Cartesian steps. Then

$$V(r) = \int_0^x (-M_2/r^3)xdx + \int_0^y (-M_2/r^3)ydy + \int_0^z (-M_2/r^3)zdz$$

I will need:

$$\int_{\infty}^{x} \frac{x\,dx}{(x^2+y^2+z^2)^{3/2}} = \frac{1}{\sqrt{x^2+y^2+z^2}}$$ // says Maple

Somehow it integral is path independent it seems that you get

$$\frac{1}{\sqrt{x^2}} + \left\{ \frac{1}{\sqrt{x^2+y^2}} - \frac{1}{\sqrt{x^2}} \right\} + \left\{ \frac{1}{\sqrt{x^2+y^2+z^2}} - \frac{1}{\sqrt{x^2+y^2}} \right\} = 1/r$$

and there is your hard-won result.

Int (∞,0,0) to (x,0,0)] x integrand
\+ Int (x,0,0) to (x,y,0)] y integrand
\+ Int (x,y,0) to (x,y,z)] z integrand giving what I quote just above.

So in the sphere case, you just want to show that V = V(r) so it is constant on a sphere.

Another way:

$$V(r) = \int_{\infty 00}^{x00} (-M_2/r^3)x\,dx + \int_{x00}^{xy0} (-M_2/r^3)y\,dy + \int_{xy0}^{xyz} (-M_2/r^3)z\,dz$$

The three indefinite integrals are all the same!! So could write

$$V(r) = I_1(x,0,0) - I_1(\infty,0,0) + I_1(x,y,0) - I_1(x,0,0) + I_1(x,y,z) - I_1(x,y,0)$$

$$= I_1(x,y,z) - I_1(\infty,0,0) = I_1(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

What do you do for an ellipse? What is constant on an ellipse? The string distance.

I think I have stated the problem correctly, going back to 2D, $\nabla V = + \mathbf{F}$

$$\nabla V(x,y) = Gm \{ [-M_2/r^3 + 2(M_1/d_0^3)] x \hat{\mathbf{x}} + [-M_2/r^3 - (M_1/d_0^3)] y \hat{\mathbf{y}} \}$$

Compute V(x,y) and show it is constant on the desired ellipse.

How about just *guessing* the V(x,y) ! For the two $1/r^3$ terms I know that

$$V_{13}(x,y) = GmM_2/r \qquad r = \sqrt{x^2+y^2}$$

For term #2 my guess is

$$V_2(x,y) = Gm\, 2(M_1/d_0^3)\,(x^2/2)$$

For term #4 my guess is

$$V_2(x,y) = - Gm (M_1/d_0^3) (y^2/2)$$

Then my candidate potential is this

$$V(x,y) = GmM_2/r + Gm\, 2(M_1/d_0^3) (x^2/2) - Gm (M_1/d_0^3) (y^2/2) \}$$

$$= Gm \{ M_2/r + 2(M_1/d_0^3) (x^2/2) - (M_1/d_0^3) (y^2/2) \}$$

So this is constant not on a circle but on a distorted circle. We then have this equation

$$M_2/r + 2(M_1/d_0^3) (x^2/2) - (M_1/d_0^3) (y^2/2) = k \qquad \text{// some constant}$$

$$M_2/r = k - 2(M_1/d_0^3) (x^2/2) + (M_1/d_0^3) (y^2/2)$$

$$M_2^2 = (x^2+y^2)[k - 2(M_1/d_0^3) (x^2/2) + (M_1/d_0^3) (y^2/2)]^2$$

$$= (x^2+y^2)k^2 [1 - 2(M_1/kd_0^3) (x^2/2) + (M_1/kd_0^3) (y^2/2)]^2$$

$$\approx k^2(x^2+y^2) [1 + 2*\{-2(M_1/kd_0^3) (x^2/2) + (M_1/kd_0^3) (y^2/2)\}]$$

$$\approx k^2(x^2+y^2) [1 - 4 (M_1/kd_0^3) (x^2/2) + 2(M_1/kd_0^3) (y^2/2)]$$

$$\approx k^2(x^2+y^2) [1 - 2 (M_1/kd_0^3)x^2 + (M_1/kd_0^3)y^2]$$

$$M_2^2 \approx k^2(x^2+y^2) [1 - 2\varepsilon x^2 + \varepsilon y^2] \qquad \varepsilon = (M_1/kd_0^3)$$

$$(M_2/k)^2 \approx (x^2+y^2) [1 - 2\varepsilon x^2 + \varepsilon y^2]$$

Since $\varepsilon << 1$. we see that $R_2 \approx M_2/k$ = radius of the basic circle (R_2 = radius of Earth). My quartic is now

$$R_2^2 \approx (x^2+y^2) [1 - 2\varepsilon x^2 + \varepsilon y^2] \qquad k = (M_2/R_2) \qquad \varepsilon = (M_1/kd_0^3) = (M_1/M_2) R_2/d_0^3$$

Imagine plotting this thing.

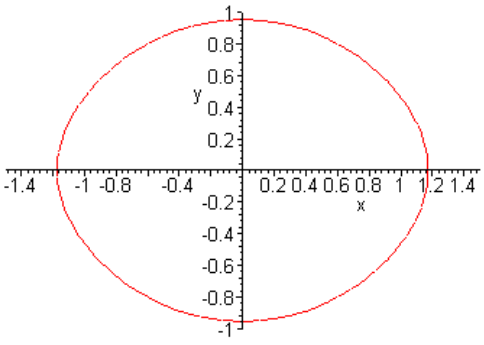
```
restart;with(plots):
e := 0.1;
```

$$e := .1$$

```
f := (x^2+y^2)*(1-2*e*x^2+e*y^2);
```

$$f := (x^2+y^2)(1-.2x^2+.1y^2)$$

```
implicitplot(f=1,x=-1.5..1.5,y=-2..2, scaling = constrained, view=[-1.5..1.5,-1..1]);
```



So there is your ellipse-like curve. How far to the right of 1 does it go? At that point we have

$$R_2^2 \approx x^2\ [\ 1 - 2\varepsilon x^2\ ]$$

which you have to solve for x. Write as

$$a \approx z\ [\ 1 - 2\varepsilon z\ ]$$

```
restart;
f := a-z+2*e*z^2;
```

$$f := a - z + 2ez^2$$

```
s := solve(f=0,z);
```

$$s := \frac{1}{4}\frac{1+\sqrt{1-8ea}}{e}, \frac{1}{4}\frac{1-\sqrt{1-8ea}}{e}$$

```
g := s[2];
```

$$g := \frac{1}{4}\frac{1-\sqrt{1-8ea}}{e}$$

```
series(g,e);
```

$$a + 2a^2e + 8a^3e^2 + 40a^4e^3 + 224a^5e^4 + O(e^5)$$

So the solution is that $z = a + 2ea^2$ which I translate to

$$x^2 = R_2^2 + 2\varepsilon R_2^4 = R_2^2(1 + 2\varepsilon R_2^2)$$

$$x = R_2\sqrt{1 + 2\varepsilon R_2^2} \approx R_2(1 + \varepsilon R_2^2) = R_2 + \varepsilon R_2^3$$

Then the little tide height is

$$h = \varepsilon R_2^3 = (M_1/M_2)\, R_2/d_0^3 * R_2^3 = R_2\ (M_1/M_2)\ (R_2/d_0)^3$$

My answer to this question from Section 8,8 is this:

$$a = (3/4)\, R_2\ (M_1/M_2)\, (R_2/d_0)^3$$

which is pretty close. So why don't I have a 3/4 sitting there?

I think I could redo this doing a better job.

You see the distortion, but this is a quartic which I have to approximate with a quadratic somehow. General problem:

$$f(x,y) = x^2+y^2 + \varepsilon_1(x^2+y^2)x^2 + \varepsilon_2(x^2+y^2)y^2$$

$$= x^2+y^2 + \varepsilon_1 x^4 + (\varepsilon_1+\varepsilon_2)x^2y^2 + \varepsilon_2 y^4$$

The equation of an ellipse is this

$$A^2y^2+B^2x^2 = x^2y^2$$

Could do a least squares fit to estimate A and B. Go back to

$$M_1^2 \approx k^2(x^2+y^2)\ [\ 1 + 2\varepsilon x^2 - \varepsilon y^2\ ] \qquad \varepsilon = 2(M_1/kd_0^3)$$

Accept this as a quartic, not an ellipse. Have

$$(x^2+y^2)\ [\ 1 + 2\varepsilon x^2 - \varepsilon y^2\ ] = (M_1/k)^2$$

At the left edge how far is it displaced from the circle :

$$x_{circle} = (M_1/k)$$

$$x_q^2\ [\ 1 + 2\varepsilon x_q^2\ ] = (M_1/k)^2$$

$$2\varepsilon x_q^4 + x_q^2 - (M_1/k)^2 = 0$$

$$x_q^2 = [\ -1 \pm\sqrt{1+4(2\varepsilon)(M_1/k)^2}\ ]/(4\varepsilon)$$

$$\approx [-1 \pm(1 + 4\varepsilon(M_1/k)^2 -(1/8)\, 8^2\varepsilon^2(M_1/k)^4)] /(4\varepsilon)$$

$$\approx [-1 +(1 + 4\varepsilon(M_1/k)^2 -(1/8)\, 8^2\varepsilon^2(M_1/k)^4)] /(4\varepsilon)$$

$$\approx [4\varepsilon(M_1/k)^2 -8\varepsilon^2(M_1/k)^4)] /(4\varepsilon)$$

$$= [(M_1/k)^2 -2\varepsilon(M_1/k)^4$$

$$= (M_1/k)^2 -2\varepsilon(M_1/k)^4$$

$$= (M_1/k)^2 \ (1 -2\varepsilon(M_1/k)^2)$$

$$= x_{circle}^2 \ (1 -2\varepsilon(M_1/k)^2)$$

$$x_q = x_{circle} \sqrt{1 -2\varepsilon(M_1/k)^2}$$

Sign is wrong, but there is a plan here where you evaluate the thing "a".

I am not sure why this method is so horrible! It gives a quartic to which you have to fit an ellipse, and then you can get A and B for the ellipse, incredibly ugly!

Taylor never mentions "ellipse" but just does what I am doing above to get the tide height!

Go back to the surface which I think is exact

$$M_1^2 \approx k^2(x^2+y^2) \ [1 - 2\varepsilon x^2 - \varepsilon y^2] \qquad ` \qquad \varepsilon = 2(M_1/kd_0^3)$$