Question regarding Frames doc and Tensor doc 2.15.17

This doc has been reviewed on 3.18.17 and I am happy with it.

Question: How do $\mathbf{e}_n$ and $\mathbf{e}'_n$ in frames doc relate to basis vectors in tensor doc? [a good question]

In tensor doc there are 4 basis vector types.

$\mathbf{u}_n$ axis aligned in x-space
$\mathbf{e}_n$ tangent base vectors in x-space
$\mathbf{e}'_n$ axis aligned in x'-space
$\mathbf{u}'_n$ I guess inverse tangent base vectors in x'-space

Linear differentials are S and R.

tensor doc	frames doc	
$\mathbf{e}'_n = R(\mathbf{x})\, \mathbf{e}_n$	$\mathbf{e}_n = R\mathbf{e}'_n$	$\mathbf{e}'_n = S\mathbf{e}_n$

[all data above I think is correct, no conclusions yet]

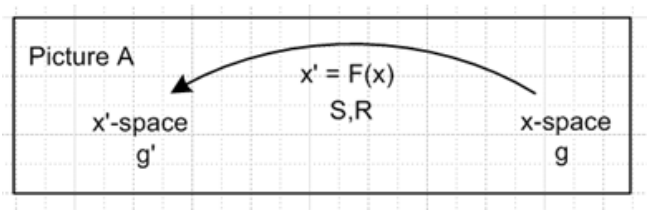
I have to start somewhere. Suppose I make this connection

tensor doc: $\mathbf{u}_n$ = axis-aligned in x-space, expansion $\mathbf{V} = V^i\, \mathbf{u}_i$, g_{ij} [ok]

frames doc: $\mathbf{e}_n$ = axis-aligned in Frame S, expansion $\mathbf{V} = V_i\, \mathbf{e}_i$, δ_{ij} [ok]

Let us now probe into what this comparison implies. If we restrict to rotations, then $\mathbf{x}$ is a vector, and we have the transformation $\mathbf{x}' = R\mathbf{x}$ in tensor doc, similar to $\mathbf{v}' = R\mathbf{v}$ in frames doc.

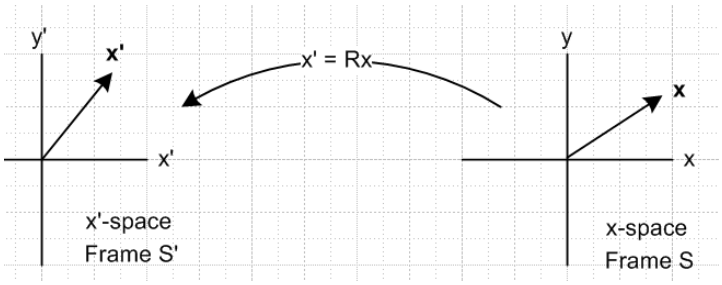
In tensor doc, I say that $\mathbf{x}$ is in x-space while $\mathbf{x}'$ is in x'-space.



What exactly does that mean? [good question]I *use* it in the sense that $\mathbf{x}'$ = curvilinear coordinates, $\mathbf{x}$ = Cartesian. The axes in x'-space might be θ and r for polar coordinates.

You would *think* that for a simple rotation as transformation, $\mathbf{x}' = R\mathbf{x}$, the axes of x'-space would be something like x' and y' [ok]. So you would *think* that x-space was Frame S with axes x and y [ok], while x'-space was Frame S' with axes x' and y' [wrong!]. So in this view, x-space is Frame S with y going up, and x'-space is Frame S' with y' going up. [wrong!]

Here is what at least seems to be a reasonable picture



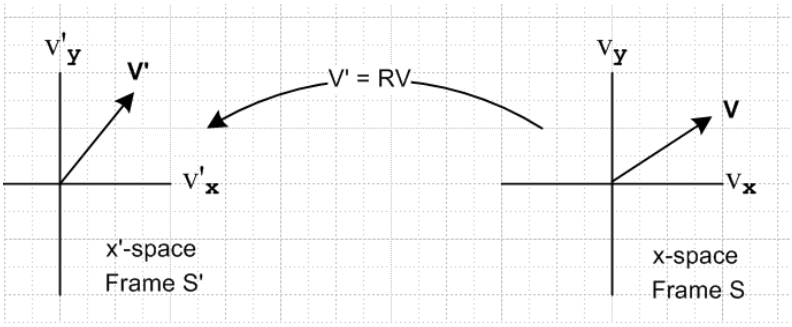
$\mathbf{x}' = x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}'$ $\mathbf{x} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ [wrong]

This above picture is then only about coordinates **x** and **x'**. There is no V and V'. I guess I would say

$$x'^{\mu} = R^{\mu}{}_{\nu} x^{\nu}$$

so show components of the mapping.

Question: In tensor doc, does **V'** = R**V** have anything to do with the above pictures? How about this picture as a candidate:



As in the previous picture, here each drawing shows the appropriate *components* of the vector .

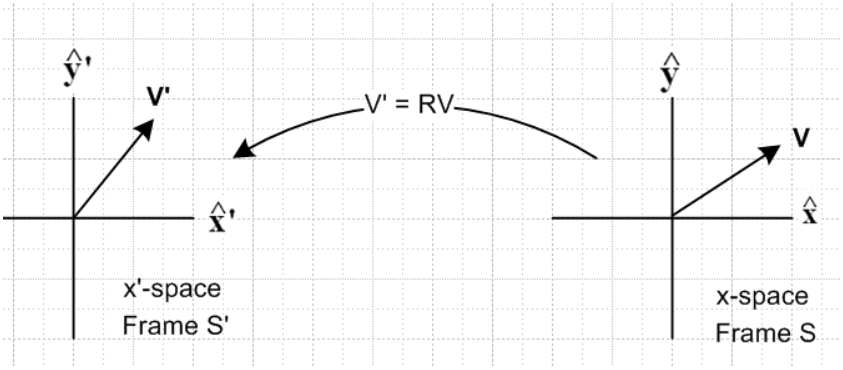
In Frame S the components of the velocity vector V are $v_{\mathbf{x}}$ and $v_{\mathbf{y}}$. [ok]

In Frame S' the components of the velocity vector **V'** are $v'_{\mathbf{x}}$ and $v'_{\mathbf{y}}$. [ok]

This seems to be an Active View situation since we have two vectors **V'** and **V**. We write

$$V'^{\mu} = R^{\mu}{}_{\nu} V^{\nu}$$

Actually, I think the next picture is better



$\mathbf{V'} = v'_x\hat{\mathbf{x}}' + v_{y'}\hat{\mathbf{y}}'$ $\qquad\qquad$ $\mathbf{V} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$

This is compatible with my tensor doc expansions

$\mathbf{V'} = V'_1\,\mathbf{e}'_1 + V'_2\,\mathbf{e}'_2 +$ $\qquad\qquad$ $\mathbf{V} = V^1\,\mathbf{u}_1 + V^2\,\mathbf{u}_2 + \ldots$

But this is not really Active View because in Active View, **V** and **V'** are drawn *in the same space/Frame* which has the same basis vectors. You have

$V = V_i e_i \;\; = V'$

[the above picture is reasonable if you say that **V** is a vector in x-space and **V'** is a different vector in x'-space with components there of V'_i , but we really need Frame S and Frame S' in the same space.]

**********saved from bottom of Question 2 doc **********

Now of the following I am not so sure:

Conjecture: Maybe the $\mathbf{e}'_n$ could serve as the Cartesian basis vectors of x'-space and maybe we can associate x'-space with Frame S'. I think I would like things to work out that way, but right now I have no idea. [again, for Active View need both Frames in the same space, with Frame S' axes back rotated .]

What are the $\mathbf{e}_n$? According to 3.2.4,

$\mathbf{e}_n \equiv S\mathbf{e}'_n$. (3.2.4)

Therefore

$\mathbf{e}_n = (R^{-1})\mathbf{e}'_n \qquad \Rightarrow \qquad \mathbf{e}'_n = R\mathbf{e}_n$

Now I will use

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

My frames doc basis theorem (1.1.29) says, for a rotation Q (in this application. $Q = R^{-1}$ local vars)

$$\mathbf{e'}_n = Q\mathbf{e}_n \quad \Leftrightarrow \quad \mathbf{e}_n = Q_{nm}\,\mathbf{e'}_m \qquad \begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \\ \mathbf{e'}_3 \end{pmatrix} = Q^{-1}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \qquad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = Q\begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \\ \mathbf{e'}_3 \end{pmatrix}$$

Applying that idea here I would use $Q = R = R_z(\alpha)$ and conclude that

$$\mathbf{e'}_n = R\mathbf{e}_n \quad \Leftrightarrow \quad \mathbf{e}_n = R_{nm}\,\mathbf{e'}_m \qquad \begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \end{pmatrix} = R^{-1}\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} \qquad \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = R\begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \end{pmatrix}$$

Then explicitly we have

$$\begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} \mathbf{e'}_1 \\ \mathbf{e'}_2 \end{pmatrix}$$

Then

$$\mathbf{e}_1 = \cos\alpha\,\mathbf{e'}_1 - \sin\alpha\,\mathbf{e'}_2$$
$$\mathbf{e}_2 = \sin\alpha\,\mathbf{e'}_1 + \cos\alpha\,\mathbf{e'}_2$$

Now I have always said that the $\mathbf{e}_n$ are vectors in x-space. [that is correct] So this equation seems a little strange since it says that $\mathbf{e}_1$ is a sum of two x'-space vectors. So I am at once confused. The $\mathbf{u}_n$ were supposed to be the Cartesian basis vectors in x-space = Frame S. You see why this is a "hard question".

Let's go back to the original equation $\mathbf{e}_n \equiv S\mathbf{e'}_n$ quoted above.

What is the *meaning* of this equation? It appears early in tensor doc. I start there by *defining* the $\mathbf{e'}_n$ to be axis aligned vectors in x'-space. For the polar coordinates case, these are $\mathbf{e'}_r$ and $\mathbf{e'}_\theta$ in x'-space which is the space whose axes are labeled by r and θ.

[I think all these matters are resolved in Appendix J ! There I have no need really for the $\mathbf{e'}_n$ and $\mathbf{u'}_n$ vectors in x'-space because we can completely ignore x'-space.]