**Question 2** 2.15.17

This doc seems confused. I was at a confused stage on this topic. I had not yet understood that Frame S and Frame S' both exist in x-space and x'-space is superfluous.

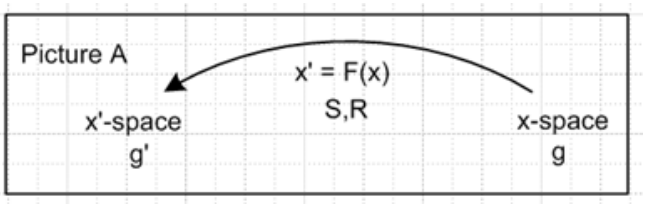
**Question**: How do you apply the full machinery of tensor doc to a transformation which is a simple rotation, such as perhaps R = $R_z(\alpha)$ for some fixed α value. [ good question ]

Comment: I never added a section to tensor doc addressing this question (I don't think). [ correct ]

When I say "machinery", I mean $\mathbf{x}' = \mathbf{F}(\mathbf{x})$ becomes $\mathbf{x}' = R_z(\alpha)\mathbf{x}$ which is a linear transformation.

• The R matrix is therefore R = $R_z(\alpha)$ according to (2.8.7).

The underpinning concept is this Picture A




• Both metric tensors are $\delta_{ij}$ which means up and down indices are the same in both "spaces".

• Contravariant and covariant vectors are the same, so we talk just about a "vector".

• A tensorial vector by definition is something that transforms according to $\mathbf{V}' = R\mathbf{V}$ or $V'_i = R_{ij}V_j$

• In this case, a position vector like $\mathbf{r}' = R\mathbf{r}$ is in fact a tensorial vector.

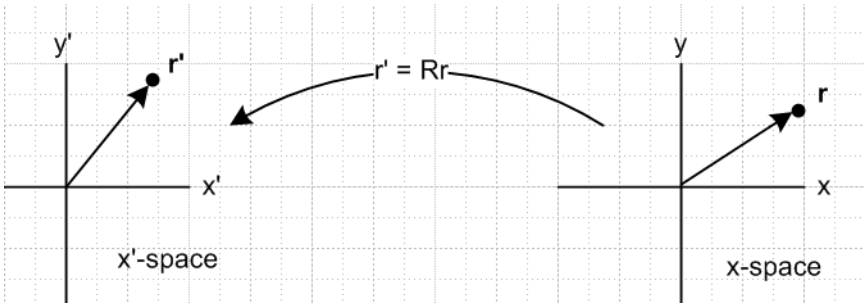
• The basis vectors in $\mathbf{u}_n$ are Cartesian basis vectors for x-space. I *think* we can treat x-space as Frame S in the language of frames doc. [ correct ]

• The covariant $\mathbf{a} \bullet \mathbf{b}$ dot product is just the regular geometric dot product.

[ all the above are true, granted ]

**What is the *meaning* of Picture A?**

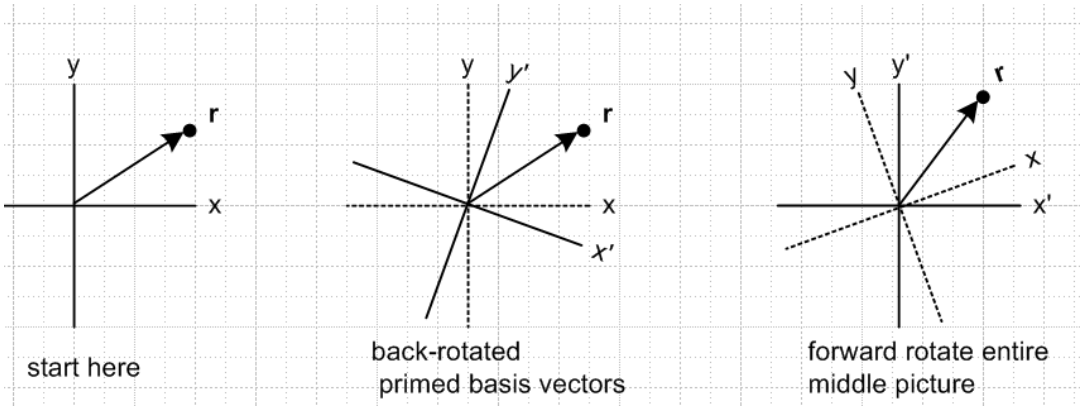
If you apply it to $\mathbf{r}' = R\mathbf{r}$ it seems to map a vector **r** in x-space into a vector **r'** in x'-space. For polar coordinates the x'-space has axes labeled r and θ since $\mathbf{x}' = (r,\theta)$. You then map a point (x,y) in x-space into a point (r,θ) in x'-space. In Picture A, x-space is always the *domain* of a mapping or transformation, and x'-space is the *range* of that mapping. For our z rotation, let's try to draw a picture, showing that a point in the domain maps into a point in the range:




The arrow on the left is rotated by amount α relative to the one on the right. Note all axis labels. I have not discussed the word "Frame" so far.

Let us now digress to look at the infamous passive discussion pictures:

PASSIVE VIEW




Frame S Frame S'

For the first time *the word Frame appears*. In Frame S, the vector r has coordinates (x,y). On the right in Frame S' it has different coordinates (x',y') that I can calculate by doing $\mathbf{r}' = R_z(\alpha)\mathbf{r}$ . Note that

$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} = x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}'$ $\quad x = \hat{\mathbf{x}} \bullet \mathbf{r}$ $\quad x' = \hat{\mathbf{x}}' \bullet \mathbf{r}$ and so on

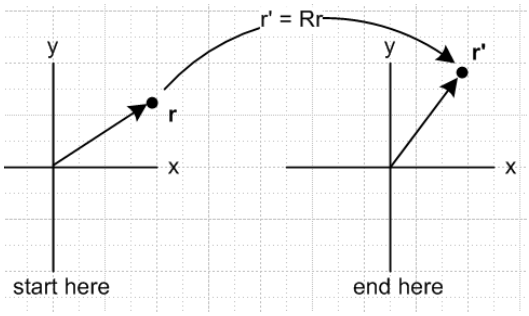
There is only vector **r**, and it has different coordinates in the two coordinate systems.

Confusion: When I say "calculate $\mathbf{r}' = R_z(\alpha)\mathbf{r}$ ", I must mean $(\mathbf{r})'_i = R_{ij}(\mathbf{r})_j$ since the only vector anywhere has the name **r**. I cannot very well say $\mathbf{r} = R_z(\alpha)\mathbf{r}$ .

Observation: One cannot help but notice the strong similarity between the rightmost picture of the set of 3 and the x'-space picture above. The big difference is that the vector is called **r**' in one case, and **r** in the other.

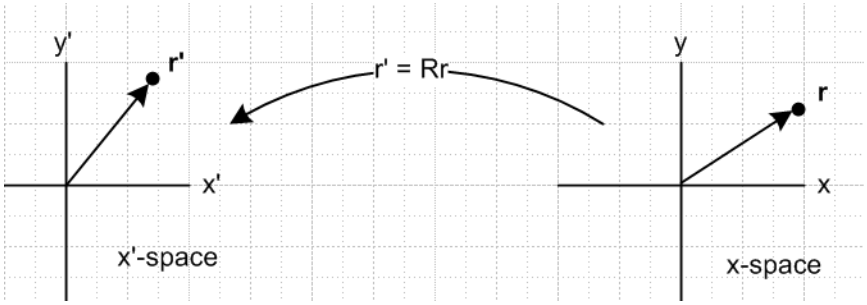
Question: How do we make sense of this **r** and **r**' labeling confusion?

What would the active view pictures look like? Rotate the apparatus within Frame S:




ACTIVE VIEW

Now we have a new vector **r'** = R**r** but it is still in Frame S. So this is a mapping x-xpace → x-space. The apparatus was rotated. Now compare this to Picture A



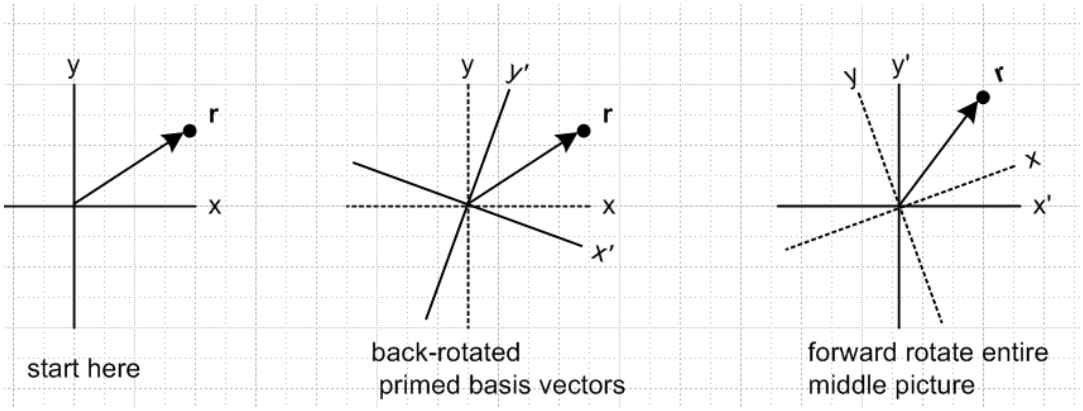

This is the exact same mapping, but instead of R: X→X, it is R: X→X' so the domain in Picture A has a different name X' from the domain X of the active view business.

For polar coordinates, a mapping like the active view does not make much sense. That mapping has to go from (x,y) space to (r,θ) space which must have axes of different labels.

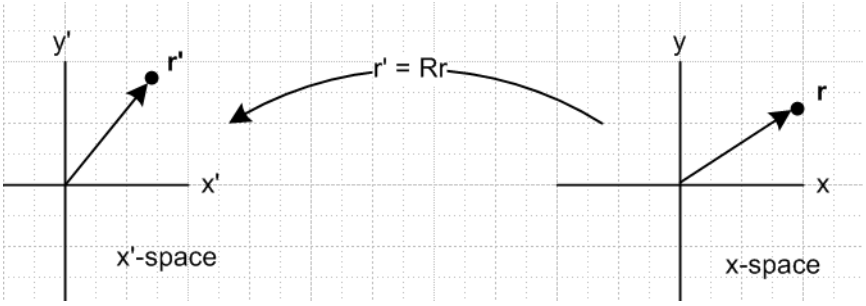
Question: Is Picture A closer to the Passive View, or to the Active View? I think the Passive View wins here hands down. The only problem there is the **r** and **r'** discrepancy. In the Active View we have no space anywhere whose axes are labeled x' and y'.

So let's give up on Active view and do side-by-side with the passive view stuff:

PASSIVE VIEW



PICTURE A




In both these ideas, the x'-space coordinates of the destination point are *exactly the same*. So that is another positive feature of the comparison. The problem is the NAME of the destination point.

$\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} = x'\,\hat{\mathbf{x}}' + y'\,\hat{\mathbf{y}}'$ $\quad x = \hat{\mathbf{x}} \bullet \mathbf{r}$ $\quad x' = \hat{\mathbf{x}}' \bullet \mathbf{r}$ and so on

I think the only resolution is to say that in **r'** is a shorthand for (**r**)' which is a notation I was considering in frames doc, and then you have (**r**)' = R**r** as the means to compute the x'-space coordinates of the only vector in town which is **r**. You then have $(\mathbf{r})'_i = R_{ij}(\mathbf{r})_j$ which is exactly what you want to say. This gives you the x'-space coordinates of the vector **r**. Then $(\mathbf{r})'_1 = x'$ and $(\mathbf{r})'_2 = y'$ and then everything seems to work.

You could create a *new* vector **r'** in x'-space such that $(\mathbf{r}')_i = (\mathbf{r})'_i$ , that would not violate any laws of the universe. Then for that new vector you would say **r'** = R**r** so that $(\mathbf{r}')_i = R_{ij}(\mathbf{r})_j$ .

I think this passive view interpretation of Picture A must be correct! If so, we then have

Implications:

- The basis vectors called $\mathbf{e}'_n$ in frames doc are also called $\mathbf{e}'_n$ in this tensor doc interp. ($\mathbf{e}'_1 = \hat{\mathbf{x}}'$ )
- The basis vectors called $\mathbf{e}_n$ in frames doc are called $\mathbf{u}_n$ in this tensor doc interp. ($\mathbf{u}_1 = \hat{\mathbf{x}}$ ) .
- Frame S is associated with x-space
- Frame S' is associated with x'-space.

Error noted: Somewhere I said (it may be gone) that "both Frame S and Frame S' are in x-space, and the basis vectors in x-space are the $\mathbf{u}_n$ for Frame S and tangent base vectors $\mathbf{e}_n$ for Frame S. " This idea vaguely applies to the center picture of the triple passive set below (as is shown below in the discussion of the $\mathbf{e}_n$), but I think I have to reject the entire idea with regard to Picture A. I much prefer the new interpretation arrived at above.

Question: In this tensor doc interp, **what has become of the tangent base vectors $\mathbf{e}_n$** ???

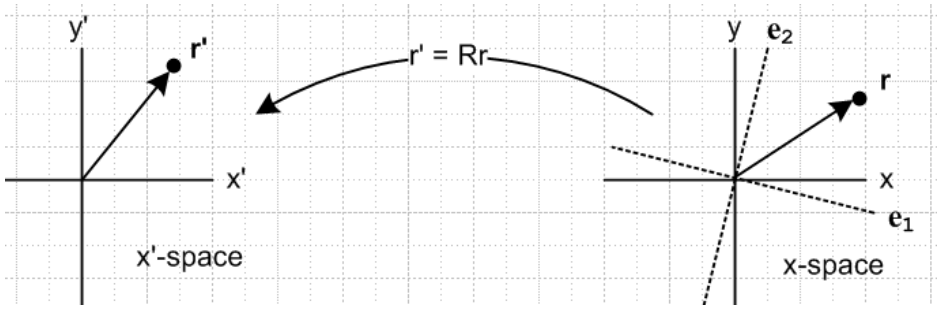
According to tensor doc (3.2.4),

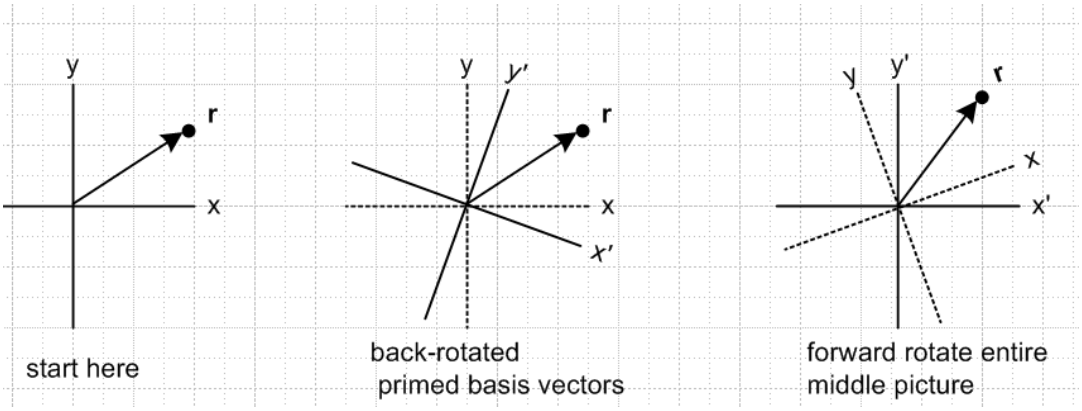
$$\mathbf{e_n} \equiv S\mathbf{e'_n} = R^{-1}\mathbf{e'_n} = R_z(-\alpha)\mathbf{e'_n} \qquad (3.2.4)$$

What exactly does the equation $\mathbf{e_n} = R_z(-\alpha)\mathbf{e'_n}$ mean? In the Picture A world, I have to consider this to be a mapping from x'-space to x-space which agrees with my repeated comment that the $\mathbf{e_n}$ are in x-space. Then for example

$$\mathbf{e_1} = R_z(-\alpha)\hat{\mathbf{x}}'$$

I will add these vectors to the Picture A picture above




Coordinate line interpretation. If I *vary* y' while holding x' = K = constant, that is a vertical line in x'-space (parallel to the y' axis) . Into what does this map in x-space? Whatever it is, it is called the y' coordinate line. The tangent to this y' coordinate line will be $\mathbf{e_2}$ . So consider,

$$R_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \qquad \mathbf{r}' = R_z(\alpha)\mathbf{r}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$x' = \cos\alpha\, x - \sin\alpha\, y \qquad x = \cos\alpha\, x' + \sin\alpha\, y'$$
$$y' = \sin\alpha\, x + \cos\alpha\, y \qquad y = -\sin\alpha\, x' + \cos\alpha\, y'$$

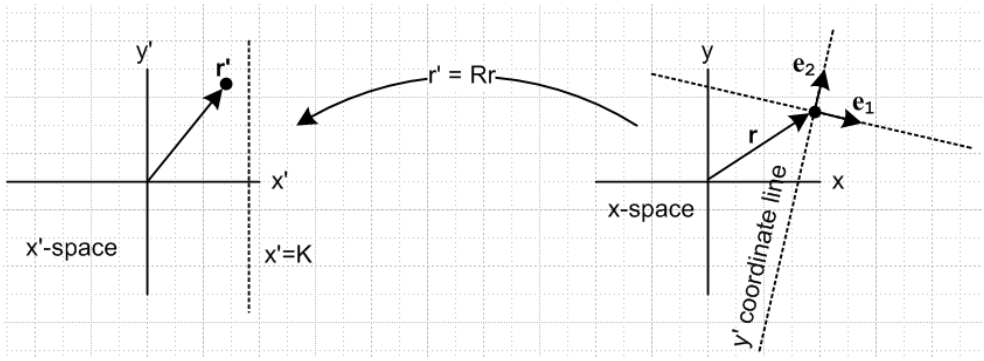
If x' = K we then have

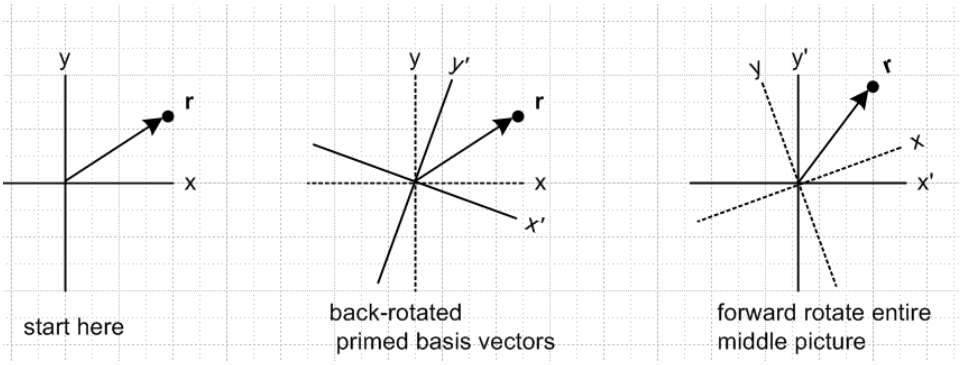
$K = \cos\alpha\, x - \sin\alpha\, y$

$\sin\alpha\, y = \cos\alpha\, x - K$

$y = (\cot\alpha)\, x - (K/\sin\alpha) = mx + b$

For small α, this indicates a large positive slope and perhaps a large negative intercept. Here I show two coordinate lines passing through the point **r .** The directions of the $\mathbf{e_n}$ agree with the previous picture.




So then, I asked "what has become of the tangent base vectors $\mathbf{e_n}$" ? They exist in x-space and I have drawn them. They of course depend on α and so are NOT axis aligned in x-space. They do line up with the directions of the primed axes in my middle picture above from the passive view procedure.

Conclusion: We do *not* associate the tangent base vectors $\mathbf{e_n}$ of tensor doc with either the x-space or the x'-space Cartesian axis-aligned basis vectors in Picture A. However, they are associated with the Frame S' axes in the middle picture above where we have back-rotated the Frame S basis vectors to get the Frame S' ones. But this is not a tensor doc Picture A situation .

Next step: how does this apply to vectors other than $\mathbf{r}' = R_z(\alpha)\mathbf{r}$ ? Like $\mathbf{V}' = R_z(\alpha)\mathbf{V}$ ?
Related question: this require Special Case #4 ?