About L = I ω PhL 1.11.15

Question Set. We have **L** = I **ω** .

Is this equation "in a frame"? Is this the Frame S equation and there is a Frame S' equation that reads **L'** = I' **ω'** like **F** = m**a** and **F'** = m**a'** ?

For the case that Frame S' is the body frame, it seems that **L'** = 0 for the rigid body or any particle of it, since the body is at rest in this body frame. Then **L'** = I' **ω'** would read 0 = I' **ω'**. What does *that* mean? In one sense **ω'** = 0 in Frame S' since nothing is moving, but in another sense, Frame S' is moving relative to Frame S at some **ω'** and perhaps **ω'** = **-ω**.

Comment added 3.12.17. In the body frame, a particle has **v'** = 0 and **p'** = 0 so **p'** = m**v'** says 0 = 0. This does not mean that **v** = 0 for that particle in Frame S.

If **L** = I **ω** is a Frame S equation with Frame S being "the table on which the top is spinning" then how can you say I is diagonal? I think of the principle axes being the body Frame S' axes, maybe that is wrong.

Maybe **L** = I **ω** is *not* an equation that lives within a frame? -

So let's take it on one piece at a time.

Question 2a: We have **L** = I **ω** . Is this equation "in a frame"? Is this the Frame S equation and there is a Frame S' equation that reads **L'** = I' **ω'** like **F** = m**a** and **F'** = m**a'** ?

Conjecture A: Holding off on the question for a bit, this **L** = I **ω** is a vector equation with a matrix involved and like any other vector equation with a matrix it can be evaluated either in Frame S or in Frame S' Here is how that would work

$(L)_i = I_{ij}(\omega)_j$ Frame S

$(L)'_i = I'_{ij}(\omega')_j$ Frame S'

Comment added 3.12.17 What is **ω** here? In my Fig 1, **ω** is **ω** for the Frame S' relative to Frame S. In that Figure I show a particle of mass m. This particle could be doing any motion you want. But if the particle is at rest relative to Frame S', then I claim below 4.1.1 that such a particle also rotates at **ω** about the same **ω** axis, It has a a different rotation radius, but the same dφ in dt. So if this is a particle of a rigid body at rest in Frame S', then that particle is doing **ω** in Frame S about the same **ω** axis that describes the Frame S' axes. In Frame S', however, this particle is at rest relative to the Frame S' axes. The particle is static, there is no axis fixed in Frame S' about which the particle has some **ω'** which is non-zero.

Question: How does this fit in with my claim: If Frame S' rotates at **ω** relative to Frame S, then Frame S rotates at **-ω** relative to Frame S' ? These **ω**'s are what Observers static in each Frame would see. I guess you could in some sense call one of these two **ω**'s by the name **ω'** then you have **ω'** = **-ω**. But this is different from talking about the **ω** of a particle as measured in two frames? I think this is a bad use of the symbol **ω'** because **ω** describes the relation *between* two frames, it is not a property of one frame or the other frame. In the example above, I would reserve **ω** to mean that of a rigid body particle at rest in Frame S and then **ω'**= 0 for that particle. Then **ω** is a property of a particle about some axis. I think no matter now a particle is moving, at some time t there is some unique axis and it has some unique **ω**. I show this

in Fig 8.7.5. When Frame S' is the body-frame, **ω** is always refer to the Frame S ang velox of a particle in that rigid body.

I think I could write this in Dirac notation and it would all make sense with an operator I.

$$(L)_i = \langle e_i | \mathbf{L} \rangle = \langle e_i | I\,\boldsymbol{\omega} \rangle = \langle e_i | \mathcal{I} | \boldsymbol{\omega} \rangle = \langle e_i | \mathcal{I} | e_j \rangle \langle e_j | \boldsymbol{\omega} \rangle = I_{ij}(\omega)_j$$

Like any operator we will find that

$$I' = RIR^{-1}$$

and since I is not R, we will NOT find that I' = I.

There is no non-zero vector called **L**' and no vector called **ω**'. We take the Passive View because the notation **L**' already has a separate meaning! It is **L**' = m**r**' x **v**'. The symbols **r**' and **v**' also have the "special meaning" and are "used up" symbols, so we cannot "create" new vectors such as **L**' ≡ R**L**.

In terms of the natural **v** and **v**' we can define **p** ≡ m**v** and **p**' ≡ m**v**'. The law **F** = m**a** in Frame S' becomes $\mathbf{F}'_{\mathrm{eff}} = m\mathbf{a}'$, however. Now consider **L** = I **ω** . We already have a meaning for **L**' , so it might not work to try do define **L'** ≡ I' **ω'**. Also, I don't think **ω**' had any meaning. The rotation of Frame S' seen from Frame S is **ω**, and going the other way it is **-ω**. It is like **b**, there is no **b**'. So my tentative answer to Question 2a is that no, there is no equation **L'** = I' **ω'** which is similar to **p'** ≡ m**v**'. The latter is a definition and I cannot define **L'** ≡ I' **ω'** . [not quite sure of this]

Question 2b : For the case that Frame S' is the body frame, it seems that **L'** = 0 for the rigid body or any particle of it, since the body is at rest in this body frame. Then **L'** = I' **ω'** would read 0 = I' **ω'**. What does *that* mean? In one sense **ω'** = 0 in Frame S' since nothing is moving, but in another sense, Frame S' is moving relative to Frame S at some **ω'** and perhaps **ω'** = **-ω**. [bad use of symbol **ω**']

First of all, I guess I accept that **L**' ≡ sum of **r**' x **p**' (in body frame Frame S') really is 0 since all the **p'** are zero. Recall that,

$$\mathbf{L}^{(c)}{}_S \equiv (\mathbf{r}\text{-}\mathbf{c}) \times m\mathbf{v}_S = (\mathbf{r}\text{-}\mathbf{c}) \times m\mathbf{v} = (\mathbf{r}\text{-}\mathbf{c}) \times \mathbf{p} \equiv \mathbf{L}^{(c)}$$

$$\mathbf{L}'^{(c')}{}_{S'} \equiv (\mathbf{r}'\text{-}\mathbf{c}') \times m\mathbf{v}'_{S'} = (\mathbf{r}'\text{-}\mathbf{c}') \times m\mathbf{v}' = (\mathbf{r}'\text{-}\mathbf{c}') \times \mathbf{p}' \equiv \mathbf{L}'^{(c')} . \qquad (1.9.3)$$

For common origins special case 4 we get

$$\mathbf{L} \equiv \mathbf{r}\text{-}\mathbf{c}) \times m\mathbf{v} = \mathbf{r} \times \mathbf{p} \qquad \text{// for each rigid body particle}$$

$$\mathbf{L}' \equiv \mathbf{r}' \times m\mathbf{v}' = \mathbf{r}' \times \mathbf{p}'$$

Later in Section 11 I show that

$$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + m(\mathbf{r}'\text{-}\mathbf{c}') \times [\,(\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S] \qquad (11.2.14)$$

so in Special Case 4 we get

$$\mathbf{L} = \mathbf{L'} + m\mathbf{r'} \times (\boldsymbol{\omega} \times \mathbf{r'})$$

Then since we just said above that for rigid body Frame S' we have $\mathbf{L'} = 0$, we get

$$\mathbf{L} = m\mathbf{r'} \times (\boldsymbol{\omega} \times \mathbf{r'})$$

I think this agrees with **Marion** page 365 where α just labels particles, and he has no primes. In my version r' is a location in the body of a particle, whereas **L** is the Frame S natural **L**.

Now let's roll out that old vector identity and do the double cross product ourselves right here.

$\mathbf{L} = -m\mathbf{r'} \times (\mathbf{r'} \times \boldsymbol{\omega})$ $\qquad$ $\mathbf{A} \times (\mathbf{A} \times \mathbf{C}) = (\mathbf{A}\bullet\mathbf{C})\mathbf{A} - A^2\mathbf{C}$

$= -m [(\mathbf{r'}\bullet\boldsymbol{\omega})\mathbf{r'} - r'^2\boldsymbol{\omega}]$ $\qquad$ // **L** is the Frame S **L** !

$= m [r'^2\boldsymbol{\omega} - (\mathbf{r'}\bullet\boldsymbol{\omega})\mathbf{r'}]$ $\qquad$ // Marion page 365 (12-18)

$L_i = m [r'^2(\omega)_i - (r')_j(\omega)_j(r')_i]$ $\qquad$ // Frame S evaluation

$= m [r'^2(\omega)_j\delta_{ij} - (r')_j(\omega)_j(r')_i]$

$= m [r'^2\delta_{ij} - (r')_j(r')_i] (\omega)_j$

$= I_{ij} (\omega)_j$ $\qquad$ $I_{ij} = m[r'^2\delta_{ij} - (r')_j(r')_i]$ $\qquad$ but sum over all particles!

$(L)'_i = m [r'^2(\omega)'_i - (r')'_j(\omega)'_j(r')'_i]$ $\qquad$ // Frame S' evaluation

$= m [r'^2(\omega)'_j\delta_{ij} - (r')'_j(\omega)'_j(r')'_i]$

$= m [r'^2\delta_{ij} - (r')'_j(r')'_i] (\omega)'_j$

$= I'_{ij} (\omega)'_j$ $\qquad$ $I'_{ij} \equiv m[r'^2\delta_{ij} - (r')'_j(r')'_i]$ $\qquad$ but sum over all particles!

Now components of vector **r'** in Frame S' could be called x'_i for short.

Now it is this thing I'_{ij} that we would like to be diagonal! How could we arrange that to happen?

Question 2c : How could you arrange that the above I'_{ij} be diagonal? If you arrange so that the axes of Frame S' line up with the principal axes of the rotating object, then I'_{ij} will be diagonal. For an object with a symmetry axis, that will always be a principal axis. But object need not have a symmetry axis. For ANY object , you could arrange to have Frame S' axes line up with the principal axis of that object. The

idea here is that you can always diagonalize a symmetric matrix, I could probably write that up to make it very explicit, but maybe don't need to do that here.

So I had a wrong impression of things I think. I was imagining that I_{ij} was non-diagonal in Frame S and diagonal in Frame S'. Probably that is true IF you select Frame S' as I just said.

So now ASSUME that Frame S' has its axes lined up with principal axes. Then we get

$$I'_{ij} = I'_i \delta_{ij}$$

and we then get

$$(L)'_i = I'_{ij} (\omega)'_j = I'_i \delta_{ij} (\omega)'_j = I'_i (\omega)'_i$$

Note that this is the Frame S L on the left!

Question 2d : How do I get from here to an equation of motion? I do know that

$$\dot{\mathbf{L}} = \mathbf{N} = \partial_{\mathbf{S}} \mathbf{L} \qquad \text{in Frame S which is inertial.}$$

I also know from the G Rule that

$$\dot{\mathbf{L}} = (\partial_{\mathbf{S}} \mathbf{L}) = (\partial_{\mathbf{S}'} \mathbf{L}) + \boldsymbol{\omega} \times \mathbf{L}$$

where $(\partial_{\mathbf{S}'} \mathbf{L})$ is "unnatural". So we then have

$$\mathbf{N} = (\partial_{\mathbf{S}'} \mathbf{L}) + \boldsymbol{\omega} \times \mathbf{L}$$

Take a component in Frame S' :

$$(N)'_i = [(\partial_t \mathbf{L})_{\mathbf{S}'}]'_i + [\boldsymbol{\omega} \times \mathbf{L}]'_i$$

Now use commutation theorem which says

$$[(\partial_t \mathbf{L})_{\mathbf{S}'}]'_i = \partial_t (L)'_i \qquad \text{// think this is right}$$

Then you get

$$(N)'_i = \partial_t (L)'_i + [\boldsymbol{\omega} \times \mathbf{L}]'_i$$

NOW use the result from above

$$(L)'_i = I'_i (\omega)'_i \qquad \text{// no implied sum on i}$$

and then we have

$$(N)'_i = \partial_t[\, I'_i(\omega)'_i] \; + \; [\boldsymbol{\omega} \times \mathbf{L}]'_i$$

$$= \; I'_i \partial_t[(\omega)'_i] \; + \; [\boldsymbol{\omega} \times \mathbf{L}]'_i$$

I would like now to show that

$$\partial_t[(\boldsymbol{\omega})'_i] \;\; = (\partial_t \boldsymbol{\omega})'_i$$

This is the commutation theorem again which says you can interchange if both same. Then we get

$$(\partial_t \boldsymbol{\omega})'_i \;\; = (\dot{\boldsymbol{\omega}})'_i$$

Meanwhile, in the other term we have

$$[\boldsymbol{\omega} \times \mathbf{L}]'_i = \varepsilon_{ijk}(\omega)'_j(L)'_k \;\; = \;\; \varepsilon_{ijk}(\omega)'_j[\;\; I'_k(\omega)'_k \;]$$

Then our equations are

$$(N)'_i = I'_i \, (\dot{\boldsymbol{\omega}})'_i + \;\; \varepsilon_{ijk}(\omega)'_j(\omega)'_k \, I'_k$$

And we can now just write these out

$$(N)'_1 = I'_1 \, (\dot{\boldsymbol{\omega}})'_1 + \;\; \varepsilon_{123}(\omega)'_2(\omega)'_3 \, I'_3 + m \, \varepsilon_{132}(\omega)'_3(\omega)'_2 \, I'_2$$

$$= I'_1 \, (\dot{\boldsymbol{\omega}})'_1 + \;\; (\omega)'_2(\omega)'_3 \, I'_3 - \;\; m \, (\omega)'_3(\omega)'_2 \, I'_2$$

$$= I'_1 \, (\dot{\boldsymbol{\omega}})'_1 + \;\; (\omega)'_2(\omega)'_3 \, [\, I'_3 - \;\; I'_2 \,]$$

The equations are cyclic, so we write them all at once

$$(N)'_1 = I'_1 \, (\dot{\boldsymbol{\omega}})'_1 + \;\; (\omega)'_2(\omega)'_3 \, [\, I'_3 - \;\; I'_2 \,]$$
$$(N)'_2 = I'_2 \, (\dot{\boldsymbol{\omega}})'_2 + \;\; (\omega)'_3(\omega)'_1 \, [\, I'_2 - \;\; I'_3 \,]$$
$$(N)'_3 = I'_2 \, (\dot{\boldsymbol{\omega}})'_3 + \;\; (\omega)'_1(\omega)'_2 \, [\, I'_1 - \;\; I'_1 \,]$$

This finally is **Goldstein p 158 A**. We know of course in terms of Euler angles that

$$(\omega)'_1 = \dot{\varphi} \sin\theta\sin\psi + \dot{\theta} \cos\psi$$
$$(\omega)'_2 = \dot{\varphi} \sin\theta\cos\psi \;\; - \dot{\theta} \sin\psi$$
$$(\omega)'_3 = \dot{\varphi} \cos\theta \;\; + \dot{\psi} \qquad\qquad \text{// Frame S'} \qquad\qquad (G.6.11)$$

I guess that to get $(\dot{\boldsymbol{\omega}})'_i$ you do all the messy derivatives!

$$(\dot{\omega})'_1 = \ddot{\varphi}\sin\theta\sin\psi + \dot{\varphi}\cos\theta\dot{\theta}\sin\psi + \dot{\varphi}\sin\theta\cos\psi\dot{\psi} + \ddot{\theta}\cos\psi - \dot{\theta}\sin\psi\dot{\psi}$$

and so on. But I don't think you install these at this point, you just work with the three equations as shown.

$$(N)'_1 = I'_1(\dot{\omega})'_1 + (\omega)'_2(\omega)'_3[\,I'_3 - I'_2\,]$$
$$(N)'_2 = I'_2(\dot{\omega})'_2 + (\omega)'_3(\omega)'_1[\,I'_2 - I'_3\,]$$
$$(N)'_3 = I'_2(\dot{\omega})'_3 + (\omega)'_1(\omega)'_2[\,I'_1 - I'_1\,]$$

Everything here is in Frame S' components.

Comment: I think I have finally cleared up these nasty equations about $\mathbf{L} = I\,\boldsymbol{\omega}$.

Question 2d: If $\mathbf{L} = I\,\boldsymbol{\omega}$ is a Frame S equation with Frame S being "the table on which the top is spinning" then how can you say I is diagonal? I think of the principle axes being the body Frame S' axes, maybe that is wrong.

Maybe $\mathbf{L} = I\,\boldsymbol{\omega}$ is *not* an equation that lives within a frame?

Answer: Well, the equation *is* $\mathbf{L} = I\,\boldsymbol{\omega}$, and we evaluate it in Frame S' components and this gives $(L)'_i = I'_{ij}(\omega)'_j$ where L is indeed the Frame S $\mathbf{L}$. ($\mathbf{L}' = 0!$) Then it is I'_{ij} which is diagonal, and yes that is in Frame S'.

$\mathbf{L} = I\,\boldsymbol{\omega}$ is an equation which can be evaluated in either Frame. it does not live within a Frame.

Continue 3.12.17. OK, I guess the above is all fine and I can write it up. But how then to you solve a problem like a top spinning on a table? Let's do a little Goldstein review on this right now. It is Chapter 5 that is of interest here.

Question 2e: What is the meaning of saying $I_{\mathbf{n}} = \mathbf{n} \bullet I \bullet \mathbf{n}$ as on Goldstein p 155?

Well think of $\mathbf{n} = n_i\mathbf{e}_i$ then maybe $I_{\mathbf{n}} = \langle \mathbf{n} \mid \mathcal{I} \mid \mathbf{n} \rangle = \mathbf{n}^{\mathbf{T}}\, I\, \mathbf{n} = n_i I_{ij} n_j = \langle n_i\mathbf{e}_i \mid \mathcal{I} \mid n_j\mathbf{e}_j \rangle$
$= n_i n_j \langle \mathbf{e}_i \mid \mathcal{I} \mid \mathbf{e}_j \rangle = n_i n_j I_{ij}$. So mechanically this is what $\mathbf{n} \bullet I \bullet \mathbf{n}$ means. But what is the physics meaning? "moment of inertia about the $\mathbf{n}$ axis" I could just define it this way. Below 5.15 Gold shows that $T = (1/2)\,\boldsymbol{\omega} \bullet I \bullet \boldsymbol{\omega} = (1/2)\,\omega^2\,\hat{\boldsymbol{\omega}} \bullet I \bullet \hat{\boldsymbol{\omega}} = (1/2)\,\omega^2\,I_{\boldsymbol{\omega}}$. So then the physics meaning is that $I_{\boldsymbol{\omega}}$ is the thing that makes the formula $T = (1/2)\,\omega^2\,I_{\boldsymbol{\omega}}$ work, analogous to $T = (1/2)\,v^2\,m$ for linear motion. That seems OK to me. So really $\mathbf{n} = \hat{\boldsymbol{\omega}}$ as he states under 5-15 page 149. This fact is derived on page 149.

On page 157 Goldstein writes the Lagrangian in Frame S' (but no primes displayed) in 5-31, where these ω_i are the body-frame guys I derive in Appendix G. I agree with page 157 A, looking at those body frame formulas. Gold then writes the Euler-Lagrange equation for coordinate ψ and gets (5-33). I obtain this result above without using the Lagrange equations. Gold says "clearly we can permute the indices" to get the other equations. I suppose that would correspond to doing the other two Euler Lagrange equations.

Gold rederives 5-34 using my method above on page 158. It is the next sections that I now want to study up on from my G notes and from his book. Start with Section 5.6.