**Question 4** PhL 2.27.17

This resulted in a rewrite of the section in question within Section 8.8. It is all done and installed. I made a stupid error with the rotations as noted below due to lack of clarity on the Passive View concept.

How do I clean up my claims around (8.8.44) which I think are wrong! I say :

We have now redefined the Earth frame on the right to be Frame S (formerly it was Frame S') and we have drawn new x,y,z axes for this new Frame S so the z axis points away from mass $M_1$. In Frame S then the angle $\theta$ is the usual spherical-coordinates polar angle.

We now assume that the Earth rotates about some axis $\hat{\mathbf{z}}'$ (new Frame S') which is obtained by rotating the $\hat{\mathbf{z}}$ axis by angles $\theta_1$ and $\varphi_1$ as follows (see (E.2.2) for the matrix),

$$\hat{\mathbf{z}}' = R_z(\varphi_1)\, R_y(\theta_1)\, \hat{\mathbf{z}} \equiv R_1\, \hat{\mathbf{z}} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \hat{\mathbf{z}}$$

$$= \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta_1\cos\varphi_1 \\ \sin\theta_1\sin\varphi_1 \\ \cos\theta_1 \end{pmatrix} .\text{// Frame S' components (8.8.44)}$$

I think this is correct. In more detail, I could say this

$$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}') = R_z(\varphi_1)R_x(\theta_1)R_z(\psi_1)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$$

which is exactly (G.5.15). One of these equations indeed then is $\hat{\mathbf{z}}' = R_z(\varphi_1)\, R_y(\theta_1)\, \hat{\mathbf{z}}$ where the $R_z(\psi_1)$ has no effect on $\hat{\mathbf{z}}$ .

Now given this rule for basis vectors being back-rotated, I know that for any Kinematic Vector like r,

$$(\mathbf{r})' = R_1^{-1}\,\mathbf{r}$$

In the Passive View, this gives the coordinates of the vector **r** in Frame S'. The corrected version of this is then

$$(\mathbf{r})' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \text{(Frame S' coordinates of vector } \mathbf{r}) = R_1^{-1}\,\mathbf{r}$$

$$R_1 = R_z(\varphi_1)R_x(\theta_1) = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\varphi_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix}$$

$$R_1^{-1} = R_1^{T} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & \cos\theta_1\sin\varphi_1 & -\sin\varphi_1 \\ -\sin\theta_1 & \cos\varphi_1 & 0 \\ \sin\theta_1\cos\varphi_1 & \sin\theta_1\sin\varphi_1 & \cos\theta_1 \end{pmatrix}$$

so then

$$(\mathbf{r})' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = R_1^{-1}\,\mathbf{r} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & \cos\theta_1\sin\varphi_1 & -\sin\theta_1 \\ -\sin\theta_1 & \cos\varphi_1 & 0 \\ \sin\theta_1\cos\varphi_1 & \sin\theta_1\sin\varphi_1 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \text{new}$$

and

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = R_1(\mathbf{r})' = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \qquad \text{new (8.8.45)}$$

Suppose we define spherical coordinates in both Frame S and Frame S' we then have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \sin\theta'\cos\varphi' \\ \sin\theta'\sin\varphi' \\ \cos\theta' \end{pmatrix}$$

THEN we have

$$\begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta_1\cos\varphi_1 & -\sin\theta_1 & \sin\theta_1\cos\varphi_1 \\ \cos\theta_1\sin\varphi_1 & \cos\varphi_1 & \sin\theta_1\sin\varphi_1 \\ -\sin\theta_1 & 0 & \cos\theta_1 \end{pmatrix} \begin{pmatrix} \sin\theta'\cos\varphi' \\ \sin\theta'\sin\varphi' \\ \cos\theta' \end{pmatrix}$$

and from this I can write (I did this totally wrong, amazing!

$$\cos\theta = \cos\theta_1\cos\varphi_1\sin\theta'\cos\varphi' - \sin\theta_1\sin\theta'\sin\varphi' + \sin\theta_1\cos\varphi_1\cos\theta'$$

$$\cos 2\theta = 2\cos^2\theta - 1\ . \qquad \text{new (8.8.48)}$$

Comment: Here I am so much tempted to say that, since $\hat{\mathbf{z}}' = R_z(\varphi_1)\,R_y(\theta_1)\,\hat{\mathbf{z}} \equiv R_1\,\hat{\mathbf{z}}$, I can apply this not just to the vector $\hat{\mathbf{z}}$ but to any vector $\mathbf{r}$ to get $\mathbf{r}' = R_1\mathbf{r}$. But I just wrote Section 1.3 where I just got done saying that in the Passive View, the basis vectors back-rotate $\mathbf{e}'_n = R^{-1}\mathbf{e}_n$, then Kinematic Vectors stay put and must do $(\mathbf{V})' = R\mathbf{V}$. I say this directly in (1.3.3). So if $\mathbf{e}'_n = R_1\mathbf{e}_n$ , then $(\mathbf{V})' = R_1^{-1}\mathbf{V}$ . So I think my corrected results above are right.

Let's now continue with the corrected equations: First we get :

Recall the equation of the water surface from (8.8.33),

$$r(\theta) = R_2 + a\cos 2\theta\ . \qquad a > 0 \qquad \text{ok} \qquad \textit{(8.8.33)}$$

This implies a tide height of

$$h(\theta) = a\cos 2\theta\ . \qquad \text{ok} \qquad (8.8.49)$$

Therefore on our idealized Earth which rotates about an axis $(\theta_1,\varphi_1)$ relative to Fig (8.8.43) we obtain the following tide height during the day

$$h(t) = a\cos 2\theta(t) = a\,[\,2\cos^2\theta(t) - 1\,]$$

$$= a\,[\,2\,(\cos\theta_1\cos\varphi_1\sin\theta'\cos\omega t - \sin\theta_1\sin\theta'\sin\omega t + \sin\theta_1\cos\varphi_1\cos\theta')^2 - 1\,]\,. \qquad \text{new} \qquad (8.8.50)$$

Here $\theta'$ indicates the line of latitude at which our Observer is positioned.

I then do Example 1:

Example 1: $(\hat{\mathbf{z}}' = \hat{\mathbf{y}})$

Suppose the Earth's rotation axis were in the $\hat{\mathbf{y}}$ direction in Fig (8.8.43) (pointing out of the plane of paper). In that case one has $\theta_1 = \pi/2$ and $\varphi_1 = \pi/2$, since

$$\hat{\mathbf{z}}' = R_z(\pi/2)\,R_y(\pi/2)\,\hat{\mathbf{z}} = \begin{pmatrix} \sin\theta_1\cos\varphi_1 \\ \sin\theta_1\sin\varphi_1 \\ \cos\theta_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \hat{\mathbf{y}}\,. \qquad (8.8.51)$$

so far this seems right. But now continue with corrections

Then from (8.8.50),

$$h(t) = a\,[\,2(\cos\theta_1\cos\varphi_1\sin\theta'\cos\omega t - \sin\theta_1\sin\theta'\sin\omega t + \sin\theta_1\cos\varphi_1\cos\theta')^2 - 1\,]$$

$$= a\,[\,2(-\sin\theta'\sin\omega t)^2 - 1\,]$$

$$= a\,[\,2(\sin\theta'\sin\omega t)^2 - 1\,]$$

$$= a\,[\,2\sin^2\theta'\sin^2\omega t - 1\,]\,. \qquad \text{// same answer as wrong } \cos\theta \text{ gave!!!} \qquad (8.8.52)$$

OK. **I will now go for a full rewrite of this frames doc section**! It was a good catch! It means all my plots are wrong. The code for this section was in rotation matrices.mws and I will now make a new version of it with a 1 suffix.

Got this done today 2.27.17.