Mystery 1 PhL 2.12.17

Notes 2.24.17. Here I ask about meaning of $[Bc]'_i$ and this leads to clarity on this question and also leads one to Dirac notation. This is now all understand and encoded into frames doc Section 1.1.

Complaint: I try to make my rotation notation as clear, clean, sharp, unambiguous, laser-focused as I can possibly make it. But as soon as I ask the simplest question, I cannot answer the question because I don't understand the notation, or it is incomplete or unclear or ambiguous.

Assume that we have Frame S and Frame S' as described in frames doc where $\mathbf{e}_n = R\ \mathbf{e}'_n$ for a fixed R.

Example 1: In frames doc I write in (1.5.7) :

$$\mathbf{a}(t+dt) = R_n(d\varphi)\ \mathbf{a}(t) \qquad \text{where} \qquad R_n(d\varphi) = \exp(-i\, d\varphi\ \mathbf{n} \bullet \mathbf{J})$$

where **a** is an arbitrary vector.

Question #1: Presumably the above equation is written in Frame S. What does the equation look like in Frame S' ?

Question #1 Answer Attempt A:

If the equation is "covariant" under rotations, maybe it has this form in some Frame S'

$$\mathbf{a}'(t+dt) = R_{n'}(d\varphi)\ \mathbf{a}'(t)$$

If **a** is an arbitrary vector, then we should have

$$\mathbf{a}'(t+dt) = R\mathbf{a}(t+dt)$$
$$\mathbf{a}'(t) = R\mathbf{a}(t)$$
$$\mathbf{n}' = R\mathbf{n}$$

If the above conjectured equation in Frame S' is valid, then it says

$$R\mathbf{a}(t+dt) = R_{n'}(d\varphi)R\mathbf{a}(t)$$

Apply R^{-1} to both sides to get

$$\mathbf{a}(t+dt) = [R^{-1} R_{n'}(d\varphi)R]\ \mathbf{a}(t) = R_n(d\varphi)\ \mathbf{a}(t)$$

If this is all correct, then we must have

$$[R^{-1} R_{n'}(d\varphi)R] = R_n(d\varphi)$$

and

$$[R\, R_n(d\varphi)R^{-1}] = R_{n'}(d\varphi)$$

If the above is correct, this is how you would know how to compute $R_{n'}(d\varphi)$. I am not sure this is all correct, but it might be correct. One lesson here is that you cannot just think of R_n as a dumb matrix of constants which has no transformation rule. It really is a rank-2 tensor and transforms like one! [that fact is now quite clear in Section 1.1 where we have matrices like R and R'. It happens they are the same.]

Question #2: How do you take components of the above vector equation in Frame S and Frame S'

This has to do with when do matrices have primes on them!

Question #2 Answer Attempt A: Start with

$\mathbf{a}(t+dt) = R_n(d\varphi)\mathbf{a}(t)$

$[\mathbf{a}(t+dt)]_i = [R_n(d\varphi)\mathbf{a}(t)]_i = [R_n(d\varphi)]_{ij}\ [\mathbf{a}(t)]_i$ Frame S'

$[\mathbf{a}(t+dt)]'_i = [R_n(d\varphi)\mathbf{a}(t)]'_i$ = ??? [good question!]

Here we have one of my high-ambiguity issues? What happens on the above line? Suppose we had

$\mathbf{a} = B\mathbf{c}$ B = rank-2 tensor

$a_i = [Bc]_i = B_{ij}c_j$ Frame S

$\mathbf{a}' = B'\mathbf{c}'$ covariant, Frame S'

$(a')_i = [B'\mathbf{c}']_i = B'_{ij}(c')_j$ Frame S components of vector equation $\mathbf{a}' = B'\mathbf{c}'$

$(a')'_i = [B'\mathbf{c}']'_i$ = ???

So despite years of tensor doc work, I run aground on the above simple situation.

[This was a good question. When you do $[Bc]'_i$ do you get $B'_{ij}(c')_j$? Answer is yes.]

Let's try Dirac notation

Question #3: How do the basis vectors work in Dirac Notation?

[In order to understand primes on matrices, it is helpful to go into Dirac notation. All this stuff and more is now installed into Section 1.1.]

This topic all by itself is so completely confusing that a whole doc is warranted just on it!!!
[is there some other doc?]

Answer:

$\mathbf{e}_n = R\,\mathbf{e}'_n \qquad |\mathbf{e}_n\rangle = |R\,\mathbf{e}'_n\rangle = \mathcal{R}|\,\mathbf{e}'_n\rangle$

$\langle e'_m|\mathbf{e}_n\rangle = \langle e'_m|\mathcal{R}|\,\mathbf{e}'_n\rangle \equiv R'_{mn} = R^{(p,p)}{}_{mn}$

$R_{mn} = \langle e'_m|\mathcal{R}|\,\mathbf{e}_i\rangle\langle e_i|\,\mathbf{e}'_n\rangle = R^{(p,n)}{}_{mi}\,R_{ni} = R^{(p,n)}{}_{mi}\,R^{T}{}_{in}$

so $R_{mn} = R^{(p,n)}{}_{mi}\,R^{T}{}_{in}$

$R_{mn}R_{nk} = R^{(p,n)}{}_{mi}\,R^{T}{}_{in}\,R_{nk} = R^{(p,n)}{}_{mk} = (R^2)_{mk}$

$\mathbf{e}'_n = R^{-1}\mathbf{e}_n \qquad |\mathbf{e}'_n\rangle = |R^{-1}\,\mathbf{e}_n\rangle = \mathcal{R}^{-1}|\,\mathbf{e}_n\rangle$

$\langle e_m|\mathbf{e}'_n\rangle = \langle e_m|R^{-1}\,\mathbf{e}_n\rangle = \langle e_m|\mathcal{R}^{-1}|\,\mathbf{e}_n\rangle = R^{-1}{}_{mn} = R_{nm} = R^{(n,n)}{}_{nm}$

transpose this last line to get (since real) : $\langle \mathbf{e}'_n|\mathbf{e}_m\rangle = R_{nm}$

compare to earlier line then to get: $\langle \mathbf{e}'_n|\mathbf{e}_m\rangle = R_{nm} = R'_{nm}$ so matrix R' = R or $R^{(p,p)} = R^{(n,n)}{}_{nm}$

Do I really believe this last result?

$R^{(p,p)}{}_{mn} = \langle e'_m|\mathcal{R}|\,\mathbf{e}'_n\rangle = \langle e'_m|e_i\rangle\langle e_i|\mathcal{R}|\,\mathbf{e}_j\rangle\langle e_j|\mathbf{e}'_n\rangle$ [p means prime]

$= R^{(p,p)}{}_{mi}\;R^{(n,n)}{}_{ij}\,R^{(p,p)}{}_{nj}$ [n means noprime]

or simply

$R_{mn} = R_{mi}\;R_{ij}\,R_{nj} = R_{mi}\,R_{ij}\,R^{-1}{}_{jn} = R_{mi}\,\delta_{in} = R_{mn}$ so OK

Start over with

$|\mathbf{e}_n\rangle = |R\,\mathbf{e}'_n\rangle = \mathcal{R}|\,\mathbf{e}'_n\rangle$
$\langle e_m|\mathbf{e}_n\rangle = \langle e_m|R\,\mathbf{e}'_n\rangle = \langle e_m|\mathcal{R}|\,\mathbf{e}'_n\rangle$
$\delta_{m,n} = \langle e_m|\mathcal{R}|\,\mathbf{e}'_n\rangle = R^{(n,p)}{}_{mn} = R^{(n,p)}{}_{nm}$ (n,p) = (noprime, prime) mixed basis

$|\mathbf{e}'_n\rangle = |R^{-1}\,\mathbf{e}_n\rangle = \mathcal{R}^{-1}|\,\mathbf{e}_n\rangle$
$\langle \mathbf{e}'_m|\mathbf{e}'_n\rangle = \langle \mathbf{e}'_m|R^{-1}\,\mathbf{e}_n\rangle = \langle \mathbf{e}'_m|\mathcal{R}^{-1}|\,\mathbf{e}_n\rangle = \langle \mathbf{e}'_m|\mathcal{R}^{T}|\,\mathbf{e}_n\rangle = \langle e_n|\mathcal{R}\,|\mathbf{e}'_m\rangle = R^{(n,p)}{}_{nm}$
$\delta_{m,n} = R^{(n,p)}{}_{nm} = R^{(n,p)}{}_{mn}$

Notice the this mixed-basis matrix $R^{(n,p)}$ is symmetric! Also

$R^{(n,p)}{}_{nm} = \langle e_m|\mathcal{R}|\,\mathbf{e}'_n\rangle = \langle e'_n|\mathcal{R}^{T}\,|e_m\rangle = R^{T}$

But since everything is real, we must have

$$\langle e'_m | e_n \rangle = R'_{mn} = R_{mn}$$

Then:

$$\mathbf{a} = B\mathbf{c}$$

$$\langle e_i | a \rangle = \langle e_i | \mathcal{B} | e_j \rangle \langle e_j | c \rangle \qquad \text{// components in Frame S}$$

$$a_i = B_{ij} \; c_j$$

Alternately we could say

$$\langle e'_i | a \rangle = \langle e'_i | \mathcal{B} | e'_j \rangle \langle e'_j | c \rangle$$

$$(\mathbf{a})'_i = B'_{ij} \; (\mathbf{c})'_j \qquad \text{// components in Frame S'}$$

where

$$B'_{ij} = \langle e'_i | \mathcal{B} | e'_j \rangle = \langle e'_i | e_j \rangle \langle e_j | \mathcal{B} | e_k \rangle \langle e_k | e'_j \rangle = R'_{ij} \, B_{jk}$$

OK, early fiddlings with Dirac, now all encoded into Section 1.1.