Paradox 1 of 2.19.17 PhL 2.19.17

Notes added 2.24.17. This was quite a severe paradox and set me back probably 2 days. It is all clear now and it is written up at the end of Section 1.1 in frames doc (so I won't ever forget!) It is subtle.

This is a good one, at 8 AM I am totally 100% mystified by this Paradox!

STATEMENT OF PARADOX

1. Suppose we know that

$$(\hat{\xi},\hat{\eta},\hat{\zeta}) = R_z(\varphi)\,(\hat{x},\hat{y},\hat{z}) \qquad (\hat{\xi}',\hat{\eta}',\hat{\zeta}') = R_\xi(\theta)\,(\hat{\xi},\hat{\eta},\hat{\zeta}) \quad \text{(G.5.7)} \quad (1) \quad \text{ok}$$

$$e'_n = R^{-1}\, e_n \qquad e''_n = S^{-1}\, e'_n$$

It then follows from the (1.1.1) and (1.1.2) theorem that

$$\begin{pmatrix}\hat{\xi}\\ \hat{\eta}\\ \hat{\zeta}\end{pmatrix} = R_z(-\varphi)\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix} \qquad \begin{pmatrix}\hat{\xi}'\\ \hat{\eta}'\\ \hat{\zeta}'\end{pmatrix} = R_\xi(-\theta)\begin{pmatrix}\hat{\xi}\\ \hat{\eta}\\ \hat{\zeta}\end{pmatrix} \quad \text{(G.5.9)} \quad (2) \quad \text{ok}$$

$$e'_n = R_{nm}\, e_m \qquad e''_n = S'_{nm}\, e'_m$$

2. Combine the transformations above to get

$$(\hat{\xi}',\hat{\eta}',\hat{\zeta}') = R_\xi(\theta)\,(\hat{\xi},\hat{\eta},\hat{\zeta}) = [R_\xi(\theta)R_z(\varphi)]\,(\hat{x},\hat{y},\hat{z}) \quad \text{(G.5.7)} \quad (3) \quad \text{ok}$$

$$S^{-1}\ R^{-1}$$

It then follows from the Basis Rule that

$$\begin{pmatrix}\hat{\xi}'\\ \hat{\eta}'\\ \hat{\zeta}'\end{pmatrix} = [R_\xi(\theta)R_z(\varphi)]^{-1}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix} = R_z(-\varphi)\,R_\xi(-\theta)\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix} \quad \text{(G.5.10)} \quad (4) \quad \text{ok}$$

$$R \qquad S$$

3. *On the other hand* , from (2) it certainly seems that we can say

$$\begin{pmatrix}\hat{\xi}'\\ \hat{\eta}'\\ \hat{\zeta}'\end{pmatrix} = R_\xi(-\theta)\begin{pmatrix}\hat{\xi}\\ \hat{\eta}\\ \hat{\zeta}\end{pmatrix} = R_\xi(-\theta)\ \{\ R_z(-\varphi)\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}\} = [\,R_\xi(-\theta)\,R_z(-\varphi)]\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix} \quad (5)$$

$$e''_n = S'_{nm}\, e'_m \qquad S'_{nm}\ R_{mk}\ e_k = [S'\,R]_{nm}\ e_k$$

BUT (5) contradicts (4). Paradox! Which one is right and which one is wrong and why?

Resolution: I have written up this case in Section 1 Concatenated Two Transformations near the end. One key fact is that S' = S" and you do not have S' = S. The equality of (4) and (5) follows from

$R\,S = S'\,R$ or $S' = RSR^{-1}$

as noted on the very last line of this little subsection. I agree, this was a fairly subtle paradox and could not be cleared up basically until I wrote that section!!!

Plan A Maybe it has to do with the fact that

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{pmatrix} = R_z(-\varphi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

is not really a normal "matrix equation". [no, this was not the problem]

It is these three vector equations

$$\begin{aligned}
\hat{\xi} &= [R_z(-\varphi)]_{11}\,\hat{x} + [R_z(-\varphi)]_{12}\,\hat{y} + [R_z(-\varphi)]_{13}\,\hat{z} \\
\hat{\eta} &= [R_z(-\varphi)]_{21}\,\hat{x} + [R_z(-\varphi)]_{22}\,\hat{y} + [R_z(-\varphi)]_{23}\,\hat{z} \\
\hat{\zeta} &= [R_z(-\varphi)]_{31}\,\hat{x} + [R_z(-\varphi)]_{32}\,\hat{y} + [R_z(-\varphi)]_{33}\,\hat{z}
\end{aligned}$$

And we also then have

$$\begin{aligned}
\hat{\xi}' &= [R_\xi(-\theta)]_{11}\,\hat{\xi} + [R_\xi(-\theta)]_{12}\,\hat{\eta} + [R_\xi(-\theta)]_{13}\,\hat{\zeta} \\
\hat{\eta}' &= [R_\xi(-\theta)]_{21}\,\hat{\xi} + [R_\xi(-\theta)]_{22}\,\hat{\eta} + [R_\xi(-\theta)]_{23}\,\hat{\zeta} \\
\hat{\zeta}' &= [R_\xi(-\theta)]_{31}\,\hat{\xi} + [R_\xi(-\theta)]_{32}\,\hat{\eta} + [R_\xi(-\theta)]_{33}\,\hat{\zeta}
\end{aligned}$$

Now manually solve for $\hat{\xi}'$

$$\begin{aligned}
\hat{\xi}' &= [R_\xi(-\theta)]_{11}\,([R_z(-\varphi)]_{11}\,\hat{x} + [R_z(-\varphi)]_{12}\,\hat{y} + [R_z(-\varphi)]_{13}\,\hat{z}) \\
&\quad + [R_\xi(-\theta)]_{12}\,([R_z(-\varphi)]_{21}\,\hat{x} + [R_z(-\varphi)]_{22}\,\hat{y} + [R_z(-\varphi)]_{23}\,\hat{z}) \\
&\quad + [R_\xi(-\theta)]_{13}\,([R_z(-\varphi)]_{31}\,\hat{x} + [R_z(-\varphi)]_{32}\,\hat{y} + [R_z(-\varphi)]_{33}\,\hat{z}) \\
&= \{\,[R_\xi(-\theta)]_{11}[R_z(-\varphi)]_{11} + R_\xi(-\theta)]_{12}[R_z(-\varphi)]_{21} + [R_\xi(-\theta)]_{13}[R_z(-\varphi)]_{31}\,\}\,\hat{x} \\
&+ \{\,[R_\xi(-\theta)]_{11}[R_z(-\varphi)]_{12} + R_\xi(-\theta)]_{12}[R_z(-\varphi)]_{22} + [R_\xi(-\theta)]_{13}[R_z(-\varphi)]_{23}\,\}\,\hat{y} \\
&+ \{\,[R_\xi(-\theta)]_{11}[R_z(-\varphi)]_{13} + R_\xi(-\theta)]_{12}[R_z(-\varphi)]_{23} + [R_\xi(-\theta)]_{13}[R_z(-\varphi)]_{33}\,\}\,\hat{z}
\end{aligned}$$

This sure looks like;

$$\begin{pmatrix} \hat{\xi}' \\ \hat{\eta}' \\ \hat{\zeta}' \end{pmatrix} = [\, R_{\xi}(-\theta)\, R_{z}(-\varphi)] \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$$= \begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix} \begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

$R_{\xi}(-\theta)$ $R_{z}(-\varphi)$

The upper left element of the product matrix is

$$11\ 11 + 12\ 21 + 13\ 31$$

so we are going to get

$$\hat{\xi}' = [11\ 11 + 12\ 21 + 13\ 31\ ]\ \hat{x} + ...$$

and that is exactly what I see above.

So this argues that (5) is correct, so (4) must be wrong.

But (4) is a trivial application of my theorem of Section 1.1 :

$$(\hat{\xi}', \hat{\eta}', \hat{\zeta}') = [R_{\xi}(\theta) R_{z}(\varphi)]\ (\hat{x}, \hat{y}, \hat{z})$$

$$\Leftrightarrow \qquad \begin{pmatrix} \hat{\xi}' \\ \hat{\eta}' \\ \hat{\zeta}' \end{pmatrix} = [R_{\xi}(\theta) R_{z}(\varphi)]^{-1} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R_{z}(-\varphi) R_{\xi}(-\theta) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix}$$

In other words

$$\mathbf{e}'_{n} = R^{-1}\, \mathbf{e}_{n} \qquad R^{-1} = [R_{\xi}(\theta) R_{z}(\varphi)] \Rightarrow R = R_{z}(-\varphi) R_{\xi}(-\theta)$$

$$\Leftrightarrow \qquad \mathbf{e}'_{n} = R_{nm}\, \mathbf{e}_{m} \quad \text{or} \quad \begin{pmatrix} \mathbf{e}'_{1} \\ \mathbf{e}'_{2} \\ \mathbf{e}'_{3} \end{pmatrix} = R \begin{pmatrix} \mathbf{e}_{1} \\ \mathbf{e}_{2} \\ \mathbf{e}_{3} \end{pmatrix}$$

So this theory does not explain the Paradox! [nice try, no cigar]

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Plan B. Go back to (1.1.1) and (1.1.2). Suppose

$$\mathbf{e}_{n} = R^{1}\, \mathbf{e}'_{n} \qquad \text{and} \qquad \mathbf{e}'_{n} = R^{2}\, \mathbf{e}''_{n}$$

Then surely

$$\mathbf{e}_n = R^1\,\mathbf{e}'_n = R^1 R^2\,\mathbf{e}''_n \qquad \text{(1), I have no doubts about this}$$

Invert to get

$$\mathbf{e}''_n = (R^2)^{-1}(R^1)^{-1}\,\mathbf{e}_n \qquad \text{(2) no doubts}$$

Now look at the claims of theorem (1.1.2) :

$$\mathbf{e}'_n = R^1{}_{nm}\,\mathbf{e}_m \quad \text{and} \quad \mathbf{e}''_n = R^{2\prime}{}_{nm}\,\mathbf{e}'_m \qquad \text{(3) no doubts if THM is OK}$$

prime add later! but this line was my error!

Then surely

$$\mathbf{e}''_n = R^{2\prime}{}_{nm}\,\mathbf{e}'_m = R^{2\prime}{}_{nm}\,[\,R^1{}_{mk}\,\mathbf{e}_k\,] = [R^{2\prime}R^1]_{nk}\,\mathbf{e}_k \qquad \text{(4) no doubts}$$

Write this as

$$\begin{pmatrix}\mathbf{e}''_1\\ \mathbf{e}''_2\\ \mathbf{e}''_3\end{pmatrix} = [R^{2\prime}R^1]\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \qquad \text{(5) no doubts as lin comb equation}$$

This does NOT say that

$$\mathbf{e}''_n = [R^2R^1]\,\mathbf{e}_n \qquad \text{// this would contradict (2) above}$$

The previous is NOT a set of three component matrix equations! So beware that illusion!!

Now where is my paradox here? I do have from (3) that

$$\begin{pmatrix}\mathbf{e}'_1\\ \mathbf{e}'_2\\ \mathbf{e}'_3\end{pmatrix} = R^1\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\mathbf{e}''_1\\ \mathbf{e}''_2\\ \mathbf{e}''_3\end{pmatrix} = R^{2\prime}\begin{pmatrix}\mathbf{e}'_1\\ \mathbf{e}'_2\\ \mathbf{e}'_3\end{pmatrix} \qquad \text{(6) no doubt as lin comb eqs}$$

It is tempting then to say that

$$\begin{pmatrix}\mathbf{e}''_1\\ \mathbf{e}''_2\\ \mathbf{e}''_3\end{pmatrix} = R^{2\prime}\begin{pmatrix}\mathbf{e}'_1\\ \mathbf{e}'_2\\ \mathbf{e}'_3\end{pmatrix} = R^{2\prime}\left\{R^1\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}\right\} = R^{2\prime}R^1\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix} \qquad \text{(7) agrees with (5)}$$

So far no paradox since this does not contradict anything yet written.

Now go back to (1)

$\mathbf{e}_n = R^1 R^2 \mathbf{e}''_n = [R^1 R^2]\, \mathbf{e}''_n \equiv R\, \mathbf{e}''_n$ where $R = R^1 R^2$ (8) no doubts

Now use theorem (1.1.1) + (1.1.2) to say

$\mathbf{e}''_n = (R)_{nm}\, \mathbf{e}_m = (R^1 R^2)_{nm}\, \mathbf{e}_m$ (9) no doubts

and we would write this as

$$\begin{pmatrix} \mathbf{e}''_1 \\ \mathbf{e}''_2 \\ \mathbf{e}''_3 \end{pmatrix} = [R^1 R^2] \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$ (10) no doubts as lincom eq

The **paradox** is that this disagrees with (5) !!

Resolution: $R^{2\prime}R^1 = R^1 R^2$ because $R^{2\prime} = R^1 R^2 (R^1)^{-1}$. This is unfriendly notation but that is it! See frames doc clear writeup near end of Section 1.1. The red prime makes all the difference!

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Plan C . Try to make the paradox presentation as brief as possible. Omit lin comb matrix notation.

Write (1.1.2) for two transformations (these are lin comb equations) [prime added in red]

$\mathbf{e}'_n = A_{nm}\, \mathbf{e}_m$ and $\mathbf{e}''_n = B'_{nm}\, \mathbf{e}'_m$ $A = R$ $B = S$ (1)
($\mathbf{e}'_m = A_{mk}\, \mathbf{e}_k$)

Then

$\mathbf{e}''_n = B'_{nm}\, \mathbf{e}'_m = B'_{nm}\, [A_{mk}\, \mathbf{e}_k] = [B'A]_{nk}\, \mathbf{e}_k$ (2)

Now write the (1.1.1) versions of the two items in (1) (these are comp equations)

$\mathbf{e}_n = A\, \mathbf{e}'_n$ and $\mathbf{e}'_n = B\mathbf{e}''_n$ (3)

Combine to get (this is a comp equation)

$\mathbf{e}_n = [AB]\, \mathbf{e}''_n \equiv R\mathbf{e}''_n$ $R \equiv AB$ (4)

Apply the 1.1.1 and 1.1.2 theorem to (4) to get

RS S'R

$\mathbf{e}''_n = R_{nm}\, \mathbf{e}_m = [AB]_{nm}\, \mathbf{e}_m = [B'A]_{nk}\, \mathbf{e}_k$ (5)

This contradicts (2) !

Resolution: Same as previous, see end of frames doc Sec 1.1.

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Plan D. When all else fails, activate Dirac notation! $1 = |e_n\rangle\langle e_n|$

Let's try to state Theorem 1.1.1 + 1.1.2 in Dirac notation

$$\mathbf{e}_n = R\,\mathbf{e}'_n \quad \Leftrightarrow \quad \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m$$

It will take a moment to convert this. Turn the gears:

$$|\mathbf{e}_n\rangle = |R\,\mathbf{e}'_n\rangle = \boldsymbol{\mathcal{R}}\,|\,\mathbf{e}'_n\rangle = |e'_m\rangle\langle e'_m|\boldsymbol{\mathcal{R}}\,|\,\mathbf{e}'_n\rangle = |e'_m\rangle R'_{mn} = R'_{mn}\,|e'_m\rangle = (R'^{-1})_{nm}|e'_m\rangle$$

Apply $(R')_{kn}$ to both sides

$$(R')_{kn}|\mathbf{e}_n\rangle = (R')_{kn}(R'^{-1})_{nm}|e'_m\rangle = \delta_{km}\,|e'_m\rangle = |e'_k\rangle$$

or

$$(R')_{nm}|\mathbf{e}_m\rangle = |e'_n\rangle$$

or

$$|e'_n\rangle = (R')_{nm}|\mathbf{e}_m\rangle$$

So here is the comparison

$$\mathbf{e}_n = R\,\mathbf{e}'_n \quad \Leftrightarrow \quad \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m$$

$$|\mathbf{e}_n\rangle = |R\,\mathbf{e}'_n\rangle \quad \Leftrightarrow \quad |e'_n\rangle = (R')_{nm}|\mathbf{e}_m\rangle$$

So I learn (I guess for the first time), that the matrix which appears here $(R')_{nm}$ is in the prime basis!

Now recall from "essay" that

$$(\mathbf{e}'_n)_i = R_{ni} \qquad (\mathbf{e}_n)_i = \delta_{ni} \qquad \text{Frame S components}$$

$$(\mathbf{e}_n)'_i = R_{in} \qquad (\mathbf{e}'_n)'_i = \delta_{ni} \qquad \text{Frame S' components}$$

Fact:

$$(R')_{mn} = \langle e'_m|\boldsymbol{\mathcal{R}}\,|\,\mathbf{e}'_n\rangle = \langle e'_m|\,|e_i\rangle\langle e_i|\boldsymbol{\mathcal{R}}\,|\;|e_j\rangle\langle e_j|\mathbf{e}'_n\rangle = \langle e'_m|e_i\rangle\, R_{ij}\langle e_j|\mathbf{e}'_n\rangle$$

$$= (e'_m)_i\, R_{ij}\, (e'_n)_j = R_{mi}\, R_{ij}\, R_{nj} = R_{mi}\, R_{ij}\, R^{T}_{jn} = R_{mi}\delta_{in} = R_{mn}$$

This shows that $(R')_{mn} = R_{mn}$ so the matrix in the no-prime basis is the same as in the prime basis. [ok]

NOW let's do the paradox again: Repeat the above and just translate.

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[Here is where I found the resolution for the first time.]

Write (1.1.2) for two transformations (these are lin comb equations)

$|e'_n\rangle = \Sigma_m A'_{nm} |e_m\rangle$ $\qquad$ $|e''_n\rangle = \Sigma_m B'_{nm} |e'_m\rangle$

We know that in terms of matrices and their bases

$A'_{nm} = A_{nm}$ $\qquad$ $B''_{nm} = B'_{nm}$ $\qquad$ [I see the light!]

Note that the top right equation would be wrong if you wrote it as $|e''_n\rangle = \Sigma_m B_{nm} |e'_m\rangle$!!!

Then

$|e''_n\rangle = B'_{nm} |e'_m\rangle = B'_{nm} [A'_{mk} |e_k\rangle] = [BA]'_{nk} |e_k\rangle$ $\qquad$ [ok] $\qquad$ (2)

Now write the (1.1.1) versions of the two items in (1) (these are comp equations)

$|e_n\rangle = |A\, e'_n\rangle = \mathcal{A} \,|\, e'_n\rangle$ $\qquad$ and $\qquad$ $|e'_n\rangle = |B\, e''_n\rangle = \mathcal{B} \,|\, e''_n\rangle$ $\qquad$ (3)

Combine to get (this is a comp equation)

$|e_n\rangle = \mathcal{AB} \,|\, e''_n\rangle \equiv \mathcal{R} \,|\, e''_n\rangle$ $\qquad$ $\mathcal{R} \equiv \mathcal{AB}$ $\qquad$ (4)

Apply the 1.1.1 and 1.1.2 theorem to (4) to get

$|e''_n\rangle = R_{nm} |e_m\rangle = [AB]_{nm} |e_m\rangle = [AB]_{nk} |e_k\rangle$ $\qquad$ (5)

The choices of basis here are no-prime or double prime, and single prime would be wrong!

So we end up with

$|e''_n\rangle = B'_{nm} A'_{mk} |e_k\rangle = [BA]'_{nk} |e_k\rangle$ $\qquad$ (2)

[ok]

$|e''_n\rangle = A_{nm} B_{mk} |e_k\rangle = [AB]_{nk} e_k$ $\qquad$ (5)

Now *maybe* we no longer have a contradiction due to the basis difference! Recall

$A'_{nm} = A_{nm}$ $\qquad$ $B''_{nm} = B'_{nm}$

Maybe I can show that

$[BA]'_{nk} = [AB]_{nk}$

which would require

$\langle e'_n|\mathcal{BA}|e'_k\rangle = \langle e_n|\mathcal{AB}|e_k\rangle$?

$\langle e'_n|\mathcal{B}|e'_i\rangle\langle e'_i|\mathcal{A}|e'_k\rangle = \langle e_n|\mathcal{A}|e_i\rangle\langle e_i|\mathcal{B}|e_k\rangle$?

$B'_{ni}A'_{ik} = A_{ni}\,B_{ik}$?

$B'A = AB$?

$B' = ABA^{-1}$ looks like tensor but what exactly is it?

How are Frame S and Frame S' related?

$\mathbf{e}_n = A\,\mathbf{e}'_n$

where A is our rotation matrix. Therefore for any tensor we have

$T' = ATA^{-1}$

Recall G.4.10 that $RTR^{-1} = T'$

So THAT is the resolution! I need to get this clarified EARLY in frames doc. perhaps Section 1! Perhaps another appendix to encapsulate this ? [I worked it into the end of Section 1.1]

What is my rule for a combination of two transformations? Start with the individual transformations :

$\mathbf{e}_n = R\,\mathbf{e}'_n \quad \Leftrightarrow \quad \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \quad R_{ij} = R'_{ij}$

$\mathbf{e}'_n = S\,\mathbf{e}''_n \quad \Leftrightarrow \quad \mathbf{e}''_n = S'_{nm}\,\mathbf{e}'_m \quad S'_{ij} = S''_{ij}$

So there are the two single rules. We can combine the above to get

$\mathbf{e}''_n = S'_{nm}R'_{nk}\,\mathbf{e}_k = (SR)'_{nk}\,\mathbf{e}_k$

What then is the combined rule?

$\mathbf{e}_n = (RS)\mathbf{e}''_n \quad \Leftrightarrow \quad \mathbf{e}''_n = (RS)_{nm}\,\mathbf{e}_m \quad (RS)_{nm} = (RS)''_{nm}$

We then end up with two *seemingly different statements*

$\mathbf{e}''_n = (SR)'_{nk}\,\mathbf{e}_k$

$$\mathbf{e''}_n = (RS)_{nk}\,\mathbf{e}_k$$

One must be very careful with the basis of a matrix, here indicated by no-prime, ' or ". The two statements above which look very different are in fact the same because (these equations are for matrices)

(SR)' = RS ?

S'R' = RS ?

S'R = RS ? // since R = R'

S' = RSR^{-1} yes! This is how a rank-2 tensor transforms under R.

Let's do a vector notation now

$$\begin{pmatrix} \mathbf{e''}_1 \\ \mathbf{e''}_2 \\ \mathbf{e''}_3 \end{pmatrix} = (SR)' \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} = (RS) \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \qquad \text{// use either one!}$$

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OK, so in the original paradox problem, which one was wrong and which was right? I will just copy it down from above and make changes as needed

1. Suppose we know that

$$\mathbf{e'}_n = R^{-1}\,\mathbf{e}_n \qquad\qquad \mathbf{e'}_n = R^{-1}\,\mathbf{e}_n$$

$$(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) = R_z(\varphi)\,(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}) \qquad (\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}') = R_\xi(\theta)\,(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}}) \qquad (1)$$

Lets denote the three coordinate systems this way

$(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ = Frame S

$(\hat{\boldsymbol{\xi}},\hat{\boldsymbol{\eta}},\hat{\boldsymbol{\zeta}})$ = Frame S^

$(\hat{\boldsymbol{\xi}}',\hat{\boldsymbol{\eta}}',\hat{\boldsymbol{\zeta}}')$ = Frame S~

$(\hat{\mathbf{x}}',\hat{\mathbf{y}}',\hat{\mathbf{z}}')$ = Frame S'

It then follows from the (1.1.1) and (1.1.2) theorem that for lincoms,

$$\begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{pmatrix} = R_z(-\varphi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \qquad \begin{pmatrix} \hat{\xi}' \\ \hat{\eta}' \\ \hat{\zeta}' \end{pmatrix} = R^{\wedge}_{\xi}(-\theta) \begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{pmatrix} \tag{2}$$

where

$$R_z(-\varphi) = R^{\wedge}_z(-\varphi) \qquad R^{\wedge}_{\xi}(-\theta) = R^{\sim}_{\xi}(-\theta)$$

2. Combine the transformations above to get

$$(\hat{\xi}',\hat{\eta}',\hat{\zeta}') = R_{\xi}(\theta)\ (\hat{\xi},\hat{\eta},\hat{\zeta}) = [R_{\xi}(\theta)R_z(\varphi)]\ (\hat{x},\hat{y},\hat{z}) \tag{3}$$

It then follows from the (1.1.1) and (1.1.2) theorem that

$$\begin{pmatrix} \hat{\xi}' \\ \hat{\eta}' \\ \hat{\zeta}' \end{pmatrix} = [R_{\xi}(\theta)R_z(\varphi)]^{-1} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} = R_z(-\varphi)\ R_{\xi}(-\theta) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{4}$$

where $R_{\xi}(\theta) = R^{\sim}_{\xi}(\theta)$ and $R_z(\varphi) = R^{\sim}_z(\varphi)$

3. On the other hand , from (2) it certainly seems that we can say

$$\begin{pmatrix} \hat{\xi}' \\ \hat{\eta}' \\ \hat{\zeta}' \end{pmatrix} = R^{\wedge}_{\xi}(-\theta) \begin{pmatrix} \hat{\xi} \\ \hat{\eta} \\ \hat{\zeta} \end{pmatrix} = R^{\wedge}_{\xi}(-\theta)\ \{\ R^{\wedge}_z(-\varphi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \} = [\ R_{\xi}(-\theta)\ R_z(-\varphi)]^{\wedge} \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{5}$$

Here we have $R^{\wedge}_{\xi}(-\theta) = R^{\sim}_{\xi}(-\theta)$ and $R_z(-\varphi) = R^{\wedge}_z(-\varphi)$.

So we have these two equations

$$\begin{pmatrix} \hat{\xi}' \\ \hat{\eta}' \\ \hat{\zeta}' \end{pmatrix} = R_z(-\varphi)\ R_{\xi}(-\theta) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} \hat{\xi}' \\ \hat{\eta}' \\ \hat{\zeta}' \end{pmatrix} = R^{\wedge}_{\xi}(-\theta)\ R^{\wedge}_z(-\varphi) \begin{pmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{pmatrix} \tag{2}$$

If we want to use the standard Frame S matrix, we must choose (1) !!!!

In order to always use rule (1) with regular matrices in Frame S, you must make a link all the way from your coordinates of interest over to the Frame S basis vectors.

THAT was a pretty good paradox (!!) and will have update implications.
[update is that I wrote this all up at the end of Section 1.1 AND added lots of Dirac stuff]