14. Rotating Frames in Curvilinear Coordinates

The solution equations for our Forward Problem are summarized in Section 12.1 and 12.2 above, and those for the Inverse Problem are summarized in Section 13.3 and 13.4. All equations are stated in bolded vector notation. Such equations may be projected onto (dotted with) any complete set of basis vectors, such as the $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\varphi}}$ used in spherical coordinates. Every orthogonal curvilinear coordinate system has such a set of orthonormal unit basis vectors which we shall call $\hat{\boldsymbol{e}}_i$, orthonormality meaning $\hat{\boldsymbol{e}}_i \bullet \hat{\boldsymbol{e}}_j = \delta_{i,j}$. In general, curvilinear basis vectors like $\hat{\boldsymbol{e}}_i = \hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\varphi}}$ are different at different points in space, so one can think of them as $\hat{\boldsymbol{e}}_i(\mathbf{r})$. It is appropriate then to use them as basis vectors for a vector *field* $\mathbf{V}(\mathbf{r})$ or for a vector associated with a discrete Particle located at position $\mathbf{r}$ such as the velocity or acceleration of that Particle.

We might want to use one curvilinear system of coordinates ξ_i with basis unit vectors $\hat{\boldsymbol{e}}_i$ for Frame S, and an entirely different system ξ'_i with basis unit vectors $\hat{\boldsymbol{e}}'_i$ for Frame S'. We might, for example, have ξ_i be spherical coordinates and ξ'_i be toroidal coordinates. Here then is the situation,

	Cartesian coords and basis vectors		Curvilinear coords and basis vectors		
Frame S	r_i	$\mathbf{e}_i$	ξ_i	$\hat{\boldsymbol{e}}_i$	
Frame S'	$(r')_i$	$\mathbf{e}'_i$	$(\xi')_i$	$\hat{\boldsymbol{e}}'_i$	(14.1)

$$\mathbf{e}_i \bullet \mathbf{e}_j = \mathbf{e}'_i \bullet \mathbf{e}'_j = \hat{\boldsymbol{e}}_i \bullet \hat{\boldsymbol{e}}_j = \hat{\boldsymbol{e}}'_i \bullet \hat{\boldsymbol{e}}'_j = \delta_{i,j} \quad \text{// orthonormality of all bases} \qquad (14.2)$$

There must exist some matrix $R(\xi)$ such that $\hat{\boldsymbol{e}}_i(\mathbf{r}) = R(\boldsymbol{\xi})\mathbf{e}_i$ for any given curvilinear system. Recall now the Basis Theorem (1.1.29),

$$\mathbf{e}_n = R\mathbf{e}'_n \quad n = 1,2,3 \quad \text{or} \quad (\mathbf{e}_n)_i = R_{ij}(\mathbf{e}'_n)_j \qquad (1.1.29)$$
$$\mathbf{e}_n = (R^{-1})_{nm}\,\mathbf{e}'_m \quad \text{or} \quad \mathbf{e}'_n = R_{nm}\,\mathbf{e}_m \quad . \qquad (1.1.30)$$

In these equations replacing $\mathbf{e}_n \rightarrow \hat{\boldsymbol{e}}_n$ and $\mathbf{e}'_n \rightarrow \mathbf{e}_n$ and $R \rightarrow R(\xi)$ gives,

$$\hat{\boldsymbol{e}}_n = R(\xi)\,\mathbf{e}_n \quad n = 1,2,3 \quad \text{or} \quad (\hat{\boldsymbol{e}}_n)_i = R(\xi)_{ij}(\mathbf{e}_n)_j$$
$$\hat{\boldsymbol{e}}_n = (R(\xi)^{-1})_{nm}\,\mathbf{e}_m \quad \text{or} \quad \mathbf{e}_n = R(\xi)_{nm}\,\hat{\boldsymbol{e}}_m \qquad (14.3)$$

which we summarize in the first line below. The second line is for then some other curvilinear coordinate system in Frame S'.

$$\hat{\boldsymbol{e}}_n = R(\xi)\,\mathbf{e}_n \Rightarrow \mathbf{e}_n = R(\xi)_{nm}\hat{\boldsymbol{e}}_m \quad \text{or} \quad \hat{\boldsymbol{e}}_n = [R(\xi)]^{-1}_{nm}\,\mathbf{e}_m$$
$$\hat{\boldsymbol{e}}'_n = R'(\xi')\,\mathbf{e}'_n \Rightarrow \mathbf{e}'_n = R'(\xi')_{nm}\hat{\boldsymbol{e}}'_m \quad \text{or} \quad \hat{\boldsymbol{e}}'_n = [R'(\xi')]^{-1}_{nm}\,\mathbf{e}'_m \quad . \qquad (14.4)$$

Example of an R(ξ) matrix. In spherical coordinates with ordering 1,2,3 = r,θ,φ, where θ is the polar angle and φ the azimuth, the matrix R(ξ) is given by (note that $R^{-1} = R^{T}$),

$$R(\xi) = \begin{pmatrix} \cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ \sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \qquad [R(\xi)]^{-1} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \tag{14.5}$$

This matrix is derived in (A.9) and (A.11). We can then use (14.3) that $\hat{\boldsymbol{e}}_n = [R(\xi)\,]^{-1}_{nm}\,\mathbf{e}_m$ to write, using the alternate notation of (1.1.32),

$$\begin{pmatrix} \hat{\boldsymbol{e}}_1 \\ \hat{\boldsymbol{e}}_2 \\ \hat{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\boldsymbol{\varphi}} \end{pmatrix} = \begin{pmatrix} \cos\varphi\sin\theta & \sin\varphi\sin\theta & \cos\theta \\ \cos\varphi\cos\theta & \sin\varphi\cos\theta & -\sin\theta \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)\,]^{-1} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = [R(\xi)\,]^{-1} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix} \tag{14.6}$$

or

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\varphi\sin\theta\,\hat{\mathbf{x}} + \sin\varphi\sin\theta\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\varphi\cos\theta\,\hat{\mathbf{x}} + \sin\varphi\cos\theta\,\hat{\mathbf{y}} - \sin\theta\,\hat{\mathbf{z}} \\ \hat{\boldsymbol{\varphi}} &= -\sin\varphi\,\hat{\mathbf{x}} + \cos\varphi\,\hat{\mathbf{y}} \quad . \end{aligned} \tag{14.7}$$

Expansions and naming. If **V** is an arbitrary vector, we then have these four expansions of interest :

$$\begin{aligned} &\mathbf{V} = V_i\mathbf{e}_i && V_i = \mathbf{V}\bullet\mathbf{e}_i \\ &\mathbf{V} = (V)'_i\,\mathbf{e}'_i && (V)'_i = \mathbf{V}\bullet\mathbf{e}'_i \\ &\mathbf{V} = (\mathit{V})_i\,\hat{\boldsymbol{e}}_i && (\mathit{V})_i = \mathbf{V}\bullet\hat{\boldsymbol{e}}_i \\ &\mathbf{V} = (\mathit{V})'_i\,\hat{\boldsymbol{e}}'_i && (\mathit{V})'_i = \mathbf{V}\bullet\mathbf{e}_i \end{aligned} \tag{14.8}$$

where we use italics to denote curvilinear vector components. It is common practice, once a curvilinear system is selected, to make these replacements so the italics are no longer needed,

$$(\mathit{V})_i \to V_{\xi_i} \qquad (\mathit{V})'_i \to V_{\xi_i'} \tag{14.9}$$

In cylindrical coordinates r,θ,z and r',θ',z' this would mean, for example,

$$\begin{aligned} &(\mathit{V})_1 \to V_r && (\mathit{V})'_1 \to V_{r'} \\ &(\mathit{V})_2 \to V_\theta && (\mathit{V})'_1 \to V_{\theta'} \\ &(\mathit{V})_3 \to V_z && (\mathit{V})'_2 \to V_{z'} \quad . \end{aligned} \tag{14.10}$$

Equation Example. Consider now this equation taken from the Section 12.1 summary,

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega}\times\mathbf{r} + \dot{\mathbf{b}}_{S'} \quad . \qquad \textit{(6.6c)} \tag{14.11}$$

We can view such an equation in any of our four bases as just discussed above,

$$
\begin{aligned}
(v)_i &= (v')_i + \varepsilon_{ijk}(\omega)_j(r)_k + (\dot{b}_{S'})_i && \text{components in basis } \mathbf{e}_i \\
(v)'_i &= (v')'_i + \varepsilon_{ijk}(\omega)'_j(r)'_k + (\dot{b}_{S'})'_i && \text{components in basis } \mathbf{e}'_i \\
(v)_i &= (v')_i + \varepsilon_{ijk}(\omega)_j(r)_k + (\dot{b}_{S'})_i && \text{components in basis } \hat{\boldsymbol{e}}_i \\
(v)'_i &= (v')'_i + \varepsilon_{ijk}(\omega)'_j(r)'_k + (\dot{b}_{S'})'_i \quad . && \text{components in basis } \hat{\boldsymbol{e}}'_i
\end{aligned}
\tag{14.12}
$$

For example, in r,θ,z cylindrical coordinates if we have $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$, then $(\omega)_j = \delta_{j3}\omega$, so in the third line above we get

$$
\varepsilon_{ijk}(\omega)_j(r)_k = \varepsilon_{ijk}\,\omega\,\delta_{j3}\,(r)_k = \omega\,\varepsilon_{i3k}(r)_k = -\,\omega\,\varepsilon_{ik3}(r)_k
$$

so that line becomes

$$
(v)_i = (v')_i - \omega\,\varepsilon_{ik3}(r)_k + (\dot{b}_{S'})_i \qquad \text{components in basis } \hat{\boldsymbol{e}}_i
$$

or

$$
\begin{aligned}
(v)_1 &= (v')_1 - \omega\,\varepsilon_{123}(r)_2 + (\dot{b}_{S'})_1 \\
(v)_2 &= (v')_2 - \omega\,\varepsilon_{213}(r)_1 + (\dot{b}_{S'})_2 \\
(v)_3 &= (v')_3 - \omega\,\varepsilon_{3k3}(r)_k + (\dot{b}_{S'})_3 = (v')_3 + (\dot{b}_{S'})_3 \quad .
\end{aligned}
\tag{14.13}
$$

This then translates into (since $\mathbf{r} = r\hat{\mathbf{r}} + z\hat{\mathbf{z}} = r_r\hat{\mathbf{r}} + r_z\hat{\mathbf{z}}$ and $r_\theta = 0$)

$$
\begin{aligned}
v_r &= v'_r - \omega\, r_\theta + (\dot{b}_{S'})_r = v'_r + (\dot{b}_{S'})_r \\
v_\theta &= v'_\theta + \omega\, r_r + (\dot{b}_{S'})_\theta = v'_\theta + \omega\, r + (\dot{b}_{S'})_\theta \\
v_z &= v'_z + (\dot{b}_{S'})_z \quad .
\end{aligned}
\tag{14.14}
$$

For any Special Case #1 problem (see Section 4.4) one has $\dot{\mathbf{b}}_{S'} = 0$ and the above equations become extremely simple

$$
\begin{aligned}
v_r &= v'_r \\
v_\theta &= v'_\theta + \omega\, r \\
v_z &= v'_z \quad .
\end{aligned}
\tag{14.15}
$$