10. Notation comparison with Goldstein (1950) and Goldstein, Poole and Safko (2001)

In this Section compare our notation for rotating-frame kinematics and non-inertial-frame physics to that of Goldstein on (1950) and Goldstein, Poole and Safko (GFS 2001).

Their notation is close to our "swap" notation, so before making any comparisons, we restate various of our equations in swap notation. A swap notation equation has an s subscript on the equation number and is obtained from the corresponding non-swap equation by prime ↔ noprime (**b** and **ω** do not change):

$$\mathbf{r}' = \mathbf{b} + \mathbf{r} \qquad (6.1)_s$$

$$\mathbf{v}_{S'} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} \quad . \qquad \mathbf{r} = \text{short vector} \qquad (6.5)_s$$

$$\mathbf{a}_{S'} = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \qquad S = \text{rotating frame} \qquad (7.4)_s$$

$$m\mathbf{a} = \mathbf{F}_{\mathbf{eff}} = \mathbf{F}' - m\ddot{\mathbf{b}}_{S'} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad (8.1.5)_s$$

In our table (1.8.4) $\mathbf{v}_{S'} = (d\mathbf{r}/dt)_{S'}$ is not one of the two "natural" velocities because **r** is the particle position relative to the Frame S origin, whereas we have a Frame S' time derivative. It is nevertheless a totally acceptable object of interest, something that could be measured by an Observer in Frame S'.

Similarly $\mathbf{a}_{S'} = (d^2\mathbf{r}/dt^2)_{S'}$ is not one of the natural acceleration in table (1.8.6), but again it is a perfectly reasonable object of interest, and of course $\mathbf{a}_{S'} = (d\mathbf{v}_{S'}/dt)_{S'}$.

We start by comparing these equations :

$$\mathbf{v}_{S'} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} \quad . \qquad \mathbf{r} = \text{short vector} \qquad (6.5)_s$$

$$\mathbf{v}_s = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} \quad . \qquad \mathbf{r} = \text{short vector} \qquad \text{// Goldstein p 135 (4-104)}$$

A careful reading of Goldstein page 135 shows that **r** is in fact the "short vector" whose tail lies at the origin of the rotating Frame S. He refers to this frame, which we think of as attached to the surface of the rotating Earth, as being a "terrestrial" coordinate system "fixed in the earth, which therefore rotates uniformly relative to the inertial system". He does not discuss the origin location of his inertial frame.

We next compare our $(7.4)_s$ above to a Goldstein equation,

$$\mathbf{a}_{S'} = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad (7.4)_s$$

$$\mathbf{a}_s = \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad \text{// Goldstein p 135 (4-105)}$$
$$\text{// GPS p 175 (4.89)}$$

Here is another pair for comparison where we must now assume that $\ddot{\mathbf{b}}_{S'} = 0$:

$$m\mathbf{a} = \mathbf{F}_{\mathbf{eff}} = \mathbf{F}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad (8.1.5)_{\mathbf{s}}$$

$$m\mathbf{a}_{\mathbf{r}} = \mathbf{F}_{\mathbf{eff}} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_{\mathbf{r}} - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad .$$

// Goldstein p 135 (4-106,7)
// GPS p 175 (4.90)

The implication of $\ddot{\mathbf{b}}_{\mathbf{S}'} = 0$ is that Goldstein's fixed inertial "space" frame must be a gimbaled frame whose origin coincides with his Earth-surface rotating frame, but whose axes are fixed relative to the stars. In this case we have $\mathbf{b} = 0$ all the time so all time derivatives of $\mathbf{b}$ are also 0.

Based on these comparisons, we make the following translation table:

our swap notation	Goldstein authors	
S	r	name of the rotating frame
S'	s	name of the fixed frame (space)
$\partial_{\mathbf{S}'}$	$(d/dt)_{\mathbf{s}}$	time derivative in the fixed frame
$\partial_{\mathbf{S}}$	$(d/dt)_{\mathbf{r}}$	time derivative in the rotating frame
$\mathbf{r}$	$\mathbf{r}$	position in rotating frame (short vector)
$\mathbf{v}$	$\mathbf{v}_{\mathbf{r}}$	velocity in rotating frame
$\mathbf{a}$	$\mathbf{a}_{\mathbf{r}}$	acceleration in rotating frame
$\mathbf{r}'$		position in fixed frame (long vector)
$\mathbf{v}_{\mathbf{S}'}$	$\mathbf{v}_{\mathbf{s}}$	velocity in fixed frame
$\mathbf{a}_{\mathbf{S}'}$	$\mathbf{a}_{\mathbf{s}}$	acceleration in fixed frame
$\mathbf{F}'$	$\mathbf{F}$	force in fixed frame = true force in rotating frame
$\mathbf{F}_{\mathbf{eff}}$	$\mathbf{F}_{\mathbf{eff}}$	total effective force in the rotating frame
$\mathbf{F}_{\mathbf{fict}}$		total fictitious force in the rotating frame

ATTEMPT USING MY SWAP NOTATION

Recall our two equations in swap notation.

$$\mathbf{a}' = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \ddot{\mathbf{b}}_{S'} \qquad S = \text{rotating frame} \qquad (7.6a)_S$$

S' S Euler Coriolis centripetal frame (**r** = short vector, **r'** = long vector)

$$\ddot{\mathbf{b}}_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}). \qquad \text{// Special Case \#1} \qquad (7.13)_S$$

Inserting the second into the first and using **r'** = **b**+**r** gives

$$\mathbf{a}' = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \qquad \text{// } \mathbf{r}' = \text{long vector} \qquad (*)$$

We may compare this to the Goldstein equation,

$$\mathbf{a}_S = \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad \text{// Goldstein p 135 (4-105)} \qquad (**)$$
$$\text{// GPS p 175 (4.89)}$$

In my equation (*) I have **r'** = long vector. But in (**) I know that **r** is a short vector because he says this on page 135. This means that (*) and (**) are not compatible.

Meanwhile, let's compare these two items

$$\mathbf{v}_{S'} = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} \quad . \qquad \mathbf{r} = \text{short vector} \qquad (6.5)_S$$

$$\mathbf{v}_S = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} \quad . \qquad \mathbf{r} = \text{short vector} \qquad \text{// Goldstein p 135 (4-104)}$$
$$\text{// GPS p 175 (4.88)}$$

I am OK with $\mathbf{v} = \mathbf{v}_r$ in the rotating frame. And **r** looks right. But in my equation I have

$$\mathbf{v}_{S'} = (d\mathbf{r}/dt)_{S'} = (d\mathbf{r}/dt)_{space}$$

I refer to this as a cross velocity because I am looking at the motion of the short vector r in the rotating frame using a time derivative from the space frame. That is certainly OK to do. With this connection that we have $v_S = \mathbf{v}_{S'}$, we are 100% OK here, which favors this approach.

Maybe he does the same thing with acceleration. He would have

$$\mathbf{a}_S = \partial_{S'}\mathbf{v}_{S'} = \partial^2_{S'}\mathbf{r} = (\partial^2\mathbf{r}/dt^2)_{space} \qquad \text{exactly what G says!}$$

So I want then to connect $\mathbf{a}_S = \mathbf{a}_{S'S'} = \mathbf{a}_{S'}$. What do I know about this accel?

$$\mathbf{a}'_S = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') \quad . \qquad (7.4)$$

$$\mathbf{a}_{S'} = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad (7.4)_S$$

Compare this then to

$$\mathbf{a_s} = \mathbf{a_r} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v_r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad \text{// Goldstein p 135 (4-105)} \qquad (**)$$
$$\text{// GPS p 175 (4.89)}$$

ATTEMPT USING MY NON-SWAP NOTATION

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \qquad S' = \text{rotating frame} \qquad (7.6a)$$

S S' Euler Coriolis centripetal frame ($\mathbf{r}'$ = short vector, $\mathbf{r}$ = long vector)

$$\ddot{\mathbf{b}}_S = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}) \quad . \qquad \text{// Special Case \#1} \qquad (7.13)$$

Inserting the second into the first and using $\mathbf{r} = \mathbf{b}+\mathbf{r}'$ gives

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \; . \qquad (*) \qquad (10.1)$$

This may be compared with the Goldstein equation (to which we have added a $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ term)

$$\mathbf{a_s} = \mathbf{a_r} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v_r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad . \qquad \text{// Goldstein p 135 (4-105)} \qquad (**)$$
$$\text{// GPS p 175 (4.89)}$$

In my equation (*) I have $\mathbf{r}$= long vector. But in (**) I know that $\mathbf{r}$ is a short vector because he says this on page 135. This means that (*) and (**) are not compatible.

I am unable to find a way to compare the two notations!!

Meanwhile, let's compare these two items

$$\mathbf{v}'_S = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \quad . \qquad (6.5)$$

$$\mathbf{v_s} = \mathbf{v_r} + \boldsymbol{\omega} \times \mathbf{r} \quad . \qquad \text{// Goldstein p 135 (4-104)}$$
$$\text{// GPS p 175 (4.88)}$$

$$\mathbf{v_s} = \mathbf{v_r} + \boldsymbol{\omega} \times \mathbf{r} \quad . \qquad \text{// Goldstein p 135 (4-104)}$$
$$\text{// GPS p 175 (4.88)}$$

All my equations above in swap-notation are these

($\mathbf{r}$ = short vector, $\mathbf{r}'$ = long vector)

$$\mathbf{a}' = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \ddot{\mathbf{b}}_{S'} \qquad \text{S = rotating frame} \qquad (7.6a)_S$$

S' S Euler Coriolis centripetal frame

$$\ddot{\mathbf{b}}_{S'} = \dot{\boldsymbol{\omega}} \times \mathbf{b} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{b}). \qquad \text{// Special Case \#1} \qquad (7.13)_S$$

Inserting the second into the first and using $\mathbf{r}' = \mathbf{b} + \mathbf{r}$ gives

$$\mathbf{a}' = \mathbf{a} + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')$$

Now compare this to

$$\mathbf{a}_S = \mathbf{a}_r + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v}_r + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad .$$

// Goldstein p 135 (4-105)
// GPS p 175 (4.89)

So this is IT!!

our swap notation	Goldstein authors	
S	r	name of the rotating frame
S'	s	name of the fixed frame (s = space)
$\partial_{S'}$	$(d/dt)_s$	time derivative in the fixed frame
∂_S	$(d/dt)_r$	time derivative in the rotating frame
r		position in rotating frame (short vector)
v	$\mathbf{v}_r$	velocity in rotating frame
a	$\mathbf{a}_r$	acceleration in rotating frame
r'	**r**	position in fixed frame (long vector)
v'	$\mathbf{v}_f$	velocity in fixed frame
a'	$\mathbf{a}_s$	acceleration in fixed frame
F'	**F**	force in fixed frame = true force in rotating frame
$\mathbf{F}_{eff}$	$\mathbf{F}_{eff}$	total effective force in the rotating frame
$\mathbf{F}_{fict}$		total fictitious force in the rotating frame
b	**R**	location of the rotating frame origin (measured in the fixed frame)
$\dot{\mathbf{b}}_{S'}$	$\dot{\mathbf{R}}_f$,**V**	velocity of the rotating frame origin (measured in the fixed frame)
$\ddot{\mathbf{b}}_{S'}$	$\ddot{\mathbf{R}}_f$	acceleration of the rotating frame origin (measured in the fixed frame)