First rewrite Text after (1.34)

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Angular momentum. We set mass m = 1, so momentum **p** = m**v** = **v**. One can then think of all **L** objects below as really being **L**/m and use that fact to reinsert m's at the end.

The drawing of interest is a reoriented Fig 1, to which we have added arbitrary points **c** and **c'**,

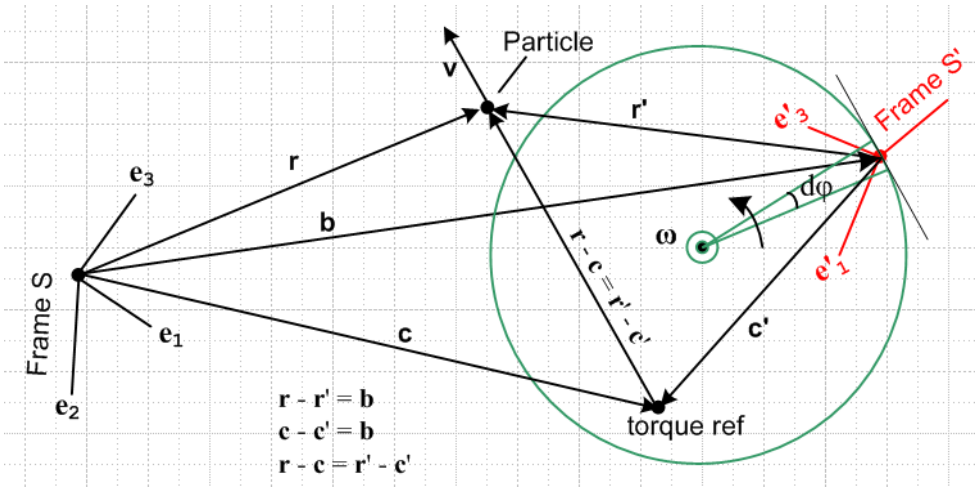


Fig 1.4

Whereas the linear momentum **p** of a particle does not require a reference point, angular momentum **L** does require such a reference point. For example, the Particle in the above figure has many different values of **L** in Frame S, some of which we might denote as follows,

$\mathbf{L}^{(0)} = \mathbf{r} \times \mathbf{v}$ // **L** in Frame S with respect to Frame S origin
$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{v}$ // **L** in Frame S with respect to Frame S point **c**
$\mathbf{L}^{(b)} = (\mathbf{r}-\mathbf{b}) \times \mathbf{v}$ // **L** in Frame S with respect to Frame S point **b**
$\mathbf{L}^{(b)} = \mathbf{r}' \times \mathbf{v}$. // **L** in Frame S with respect to Frame S' origin

The last two lines are exactly the same since **b** is a vector between the two origins.

Using the general form $\mathbf{L}^{(c)}$, we may identify the following two "natural" angular momenta in Frames S and S',

$$\mathbf{L}^{(c)}{}_S \equiv (\mathbf{r}-\mathbf{c}) \times \mathbf{v}_S = (\mathbf{r}-\mathbf{c}) \times \mathbf{v} \equiv \mathbf{L}^{(c)}$$
$$\mathbf{L}'^{(c')}{}_{S'} \equiv (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}'_{S'} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}' \equiv \mathbf{L}'^{(c')} \quad . \qquad (1.35)$$

The time derivative of the first of these objects is given by

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}^{(c)}{}_S = \partial_S \mathbf{L}^{(c)}{}_S = \partial_S [(\mathbf{r}-\mathbf{c}) \times \mathbf{v}_S] = (\mathbf{v}_S - \dot{\mathbf{c}}_S) \times \mathbf{v}_S + (\mathbf{r}-\mathbf{c}) \times \partial_S \mathbf{v}_S = -\dot{\mathbf{c}}_S \times \mathbf{v}_S + (\mathbf{r}-\mathbf{c}) \times \mathbf{a}_S$$
$$= -\dot{\mathbf{c}} \times \mathbf{v} + (\mathbf{r}-\mathbf{c}) \times \mathbf{a} \quad .$$

A similar result is obtained by priming everything on the above line, so we end up with these four equations:

$$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{v}$$

$$\dot{\mathbf{L}}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{a} - \dot{\mathbf{c}} \times \mathbf{v}$$

$$\mathbf{L}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}'$$

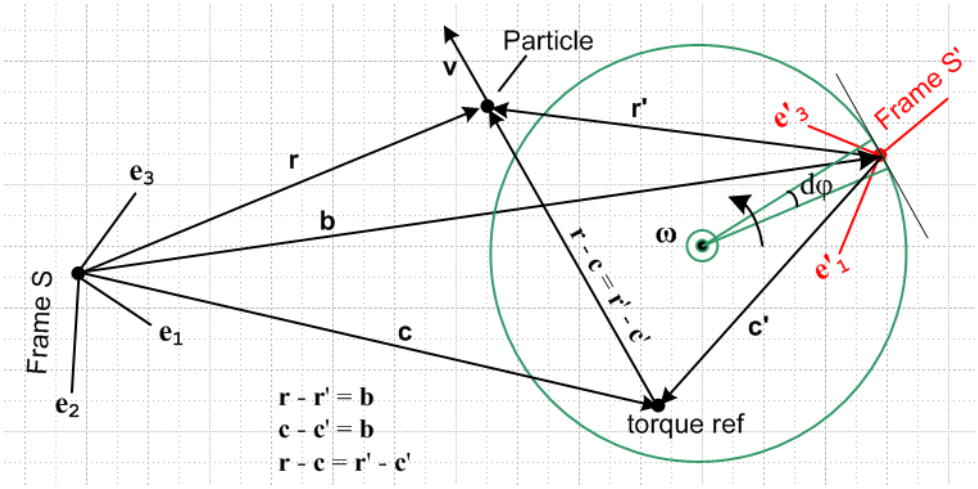
$$\dot{\mathbf{L}}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{a}' - \dot{\mathbf{c}}' \times \mathbf{v}' \quad . \tag{1.36}$$

The discussion of angular momentum continues in Section 11.

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11.2 Expression of $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ in terms of Frame S' objects // was (b)

We replicate Fig 1.4 which now includes a single torque reference point which is **c** is Frame S and **c'** in Frame S'.



(11.2.1)

where the three equations shown are obvious from the drawing,

$$\mathbf{r} - \mathbf{r}' = \mathbf{b} \tag{11.2.2}$$
$$\mathbf{c} - \mathbf{c}' = \mathbf{b} \tag{11.2.3}$$
$$\mathbf{r} - \mathbf{c} = \mathbf{r}' - \mathbf{c}' \tag{11.2.4}$$

Again mass m = 1 so momentum $\mathbf{p} = m\mathbf{v} = \mathbf{v}$. For arbitrary reference point **c** one has from (1.36),

$$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{v} \tag{11.2.5}$$

$$\dot{\mathbf{L}}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{a} - \dot{\mathbf{c}} \times \mathbf{v} \tag{11.2.6}$$

$$\mathbf{L}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}' \tag{11.2.7}$$

$$\dot{\mathbf{L}}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{a}' - \dot{\mathbf{c}}' \times \mathbf{v}' \quad . \qquad \textit{(1.36)} \qquad (11.2.8)$$

The reference point **c** might be moving. We know from Sections 6 and 7 how **r**,**v**,**a** and **r'**,**v'**,**a'** are related,

$$\mathbf{r} = \mathbf{r}' + \mathbf{b} \qquad \textit{(6.1)} \qquad (11.2.9)$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad \textit{(6.6a)} \qquad (11.2.10)$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \quad . \qquad \textit{(7.6a)} \qquad (11.2.11)$$

The middle equation above can be written,

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad (11.2.12)$$

The torque reference point **c** in Fig 11.3 is just like the particle point **r**, so the above for **c** becomes

$$\dot{\mathbf{c}} = \dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S \qquad (11.2.13)$$

Part of the goal of Section 5 is to express $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ in terms of Frame S' quantities. In the following all algebra is shown to provide an easily traceable path since there won't be any result verifications:

$$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{v} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{v} = (\mathbf{r}'-\mathbf{c}') \times [\ \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S\ ] \qquad \text{// (11.2.4), (11.2.10)}$$

$$= (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}' + (\mathbf{r}'-\mathbf{c}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]$$

$$= \mathbf{L}'^{(c')} + (\mathbf{r}'-\mathbf{c}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]\ . \qquad \text{// (11.2.7)}$$

$$\dot{\mathbf{L}}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{a} - \dot{\mathbf{c}} \times \mathbf{v} \qquad \text{// (11.2.6)}$$

$$= (\mathbf{r}'-\mathbf{c}') \times [\ \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] - \dot{\mathbf{c}} \times \mathbf{v} \qquad \text{// (11.2.4), (11.2.11)}$$

$$= (\mathbf{r}'-\mathbf{c}') \times \mathbf{a}' - \dot{\mathbf{c}}' \times \mathbf{v}' + (\mathbf{r}'-\mathbf{c}') \times [\ \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] - \dot{\mathbf{c}} \times \mathbf{v} + \dot{\mathbf{c}}' \times \mathbf{v}'$$

$$= \dot{\mathbf{L}}'^{(c')} + (\mathbf{r}'-\mathbf{c}') \times [\ \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \qquad \text{// (11.2.8)}$$
$$- \dot{\mathbf{c}} \times \mathbf{v} + \dot{\mathbf{c}}' \times \mathbf{v}'$$

$$= \dot{\mathbf{L}}'^{(c')} + (\mathbf{r}'-\mathbf{c}') \times [\ \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S]$$
$$- (\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + \dot{\mathbf{c}}' \times \mathbf{v}' \qquad \text{// (11.2.13), (11.2.10)}$$

$$= \dot{\mathbf{L}}'^{(c')} - (\mathbf{r}'-\mathbf{c}') \times \mathbf{F}_{fict} - (\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + \dot{\mathbf{c}}' \times \mathbf{v}' \qquad \text{// (8.6)}$$

These results may now be summarized:

$$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + (\mathbf{r}'-\mathbf{c}') \times [(\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S] \qquad (11.2.14)$$

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} - (\mathbf{r}'-\mathbf{c}') \times \mathbf{F}_{fict} - (\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + \dot{\mathbf{c}}' \times \mathbf{v}' \,. \qquad (11.2.15)$$

These equations express $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ entirely in terms of Frame S' objects, for the general case where the angular momentum reference point **c** is arbitrarily selected. The second equation can be written

$$\dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{F}_{fict} + (\dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - \dot{\mathbf{c}}' \times \mathbf{v}' \qquad (11.2.16)$$

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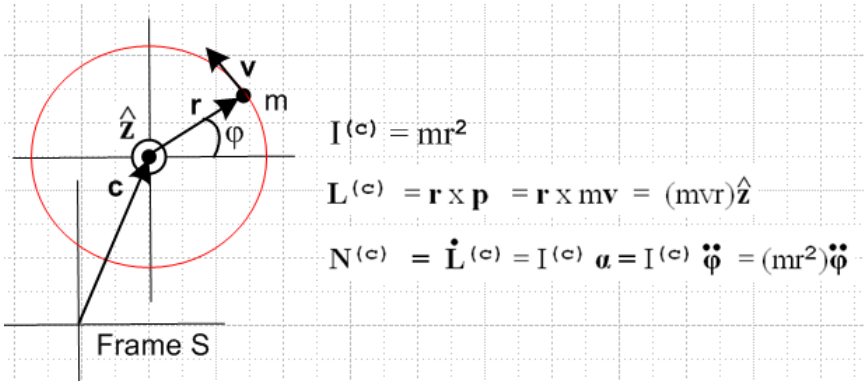
11.3 Fictitious Torques and Newton's Rotational Law in a non-inertial frame **// was (c)**

We now compare Newton's (2nd) Law for linear motion with that for circular motion, both in Frame S,

$$\mathbf{F} = \dot{\mathbf{p}} = m\mathbf{a} = m\ddot{\mathbf{r}} \qquad \text{// } \dot{\mathbf{p}} \text{ and } \ddot{\mathbf{r}} \text{ "natural" in Frame S} \qquad (11.3.1)$$

$$\mathbf{N}^{(c)} = \dot{\mathbf{L}}^{(c)} = I^{(c)}\,\boldsymbol{\alpha} = I^{(c)}\,\ddot{\boldsymbol{\varphi}} = (mr^2)\ddot{\boldsymbol{\varphi}} \qquad \text{// } \dot{\mathbf{L}}^{(c)} \text{ and } \ddot{\boldsymbol{\varphi}} \text{ "natural" in Frame S} \qquad (11.3.2)$$

Here is a drawing showing the various parameters :



(11.3.3)

Here $\mathbf{L}^{(c)}$ is the angular momentum of our Particle in Frame S relative to reference point **c**, $\mathbf{N}^{(c)}$ is some externally applied torque about that same reference point acting on the Particle. Quantity $\boldsymbol{\alpha} = \ddot{\boldsymbol{\varphi}}$ is the angular acceleration about the point **c** and $I^{(c)}$ is the moment of inertia about point **c**.

In Frame S' we want to find some *effective* Newton's Rotational Law,

$$\mathbf{N}'^{(c')}_{eff} = \dot{\mathbf{L}}'^{(c')} \qquad \text{// eff. Newton's Rot Law in Frame S'} \qquad (11.3.4)$$

$$\mathbf{N'^{(c')}_{eff}} = \mathbf{N^{(c)}} + \mathbf{N^{(c)}_{fict}} \qquad \text{// } \textit{defines } \mathbf{N^{(c)}_{fict}} \tag{11.3.5}$$

where $\mathbf{N^{(c)}_{fict}}$ is the fictitious torque that mysteriously appears in non-inertial Frame S', so then

$$\mathbf{N^{(c)}} + \mathbf{N^{(c)}_{fict}} = \dot{\mathbf{L}}'^{(c')} \ . \tag{11.3.6}$$

We can then apply this bogus Newton's Rotational Law in non-inertial Frame S' *as long as* we include the fictitious torques $\mathbf{N'^{(c')}_{fict}}$ along with the real torques $\mathbf{N^{(c)}}$ which are just those present in inertial Frame S. This is in complete analogy with the use of fictitious *forces* as reviewed in Section 8.

Solve (11.3.6) for $\mathbf{N^{(c)}_{fict}}$ and then replace $\mathbf{N^{(c)}}$ by $\dot{\mathbf{L}}^{(c)}$ in (11.3.2) to get

$$\mathbf{N^{(c)}_{fict}} = \dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)} \quad , \tag{11.3.5}$$

Replacing the right side of the above using (11.2.16) gives

$$\mathbf{N^{(c)}_{fict}} = (\mathbf{r'-c'}) \times \mathbf{F_{fict}} + (\ddot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c}' + \dot{\mathbf{b}}_S) \times (\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) - \ddot{\mathbf{c}}' \times \mathbf{v}' \tag{11.3.6}$$

where

$$\mathbf{F_{fict}} = \underbrace{-m\ddot{\mathbf{b}}_S}_{\text{frame}} \underbrace{- m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}')}_{\text{centrifugal}} \underbrace{- 2m\,\boldsymbol{\omega} \times \mathbf{v}'}_{\text{Coriolis}} \underbrace{- m\dot{\boldsymbol{\omega}} \times \mathbf{r}'}_{\text{Euler}} \ . \tag{8.6}$$
