This material has been installed and updated within frames doc, do not view again.

(b) Expression of $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ in terms of Frame S' objects

We replicate Fig 1.4 which now includes a single torque reference point which is **c** is Frame S and **c'** in Frame S'.

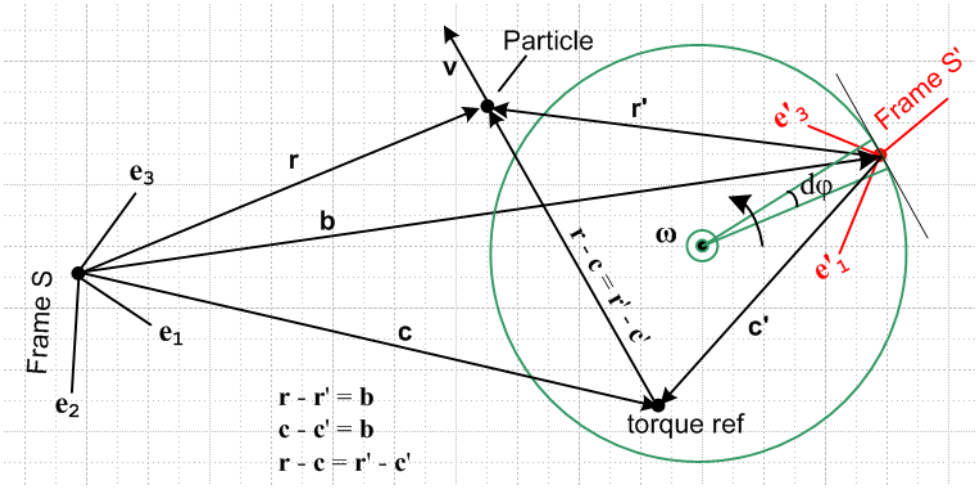


Fig 11.3

where the three equations shown are obvious from the drawing.

Again mass m = 1 so momentum **p** = m**v** = **v**. For arbitrary reference point **c** one has from (1.36),

$$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{v}$$

$$\dot{\mathbf{L}}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{a} - \dot{\mathbf{c}}_S \times \mathbf{v}$$

$$\mathbf{L}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}'$$

$$\dot{\mathbf{L}}'^{(c')} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{a}' - \dot{\mathbf{c}}'_{S'} \times \mathbf{v}' \quad . \qquad (1.36) \qquad (11.1)$$

The reference point **c** might be moving. We know from Sections 6 and 7 how **r**,**v**,**a** and **r'**,**v'**,**a'** are related,

$$\mathbf{r} = \mathbf{r}' + \mathbf{b} \qquad (6.1)$$

$$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S \qquad (6.6a)$$

$$\mathbf{a} = \mathbf{a}' + \dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S \quad . \qquad (7.6a) \qquad (11.2)$$

Part of the goal of Section 5 is to express $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c)}$ in terms of Frame S' quantities. In the following all algebra is shown to provide an easily traceable path since there won't be any result verifications:

$$\mathbf{L}^{(c)} = (\mathbf{r}-\mathbf{c}) \times \mathbf{v} = (\mathbf{r}'-\mathbf{c}') \times \mathbf{v} = (\mathbf{r}'-\mathbf{c}') \times [\ \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S\ ]$$

$$= (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}' + (\mathbf{r}'-\mathbf{c}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]$$

$$= \mathbf{L}'^{(c')} + (\mathbf{r}'-\mathbf{c}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]\ .$$

Here was my previous result:

$$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + (\mathbf{r}' - \mathbf{c} + \mathbf{b}\ ) (\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + (\mathbf{c}' - \mathbf{c} + \mathbf{b}) \times \mathbf{v}'$$

In this previous result replace $\mathbf{c}' - \mathbf{c} + \mathbf{b} = \mathbf{0}$ and $\mathbf{b}-\mathbf{c} = -\mathbf{c}'$ to get

$$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + (\mathbf{r}' - \mathbf{c}'\ ) (\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S)$$

and this agrees with the result above obtained much more quickly.

Next:

$$\dot{\mathbf{L}}^{(c)} = \partial_S \mathbf{L}^{(c)} = \partial_{S'} \mathbf{L}^{(c)} + \boldsymbol{\omega} \times \mathbf{L}^{(c)} \qquad \text{G-rule}$$

$$= \partial_{S'} \{\ \mathbf{L}'^{(c')} + (\mathbf{r}'-\mathbf{c}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]\ \} + \boldsymbol{\omega} \times \mathbf{L}^{(c)}$$

$$= \dot{\mathbf{L}}'^{(c')} + (\dot{\mathbf{r}}'-\dot{\mathbf{c}}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S] + (\mathbf{r}'-\mathbf{c}') \times [\ (\dot{\boldsymbol{\omega}} \times \mathbf{r}') + (\boldsymbol{\omega} \times \dot{\mathbf{r}}') + \partial_{S'} \dot{\mathbf{b}}_S] + \boldsymbol{\omega} \times \mathbf{L}^{(c)}$$

Now I use

$$\partial_{S'} \dot{\mathbf{b}}_S = \partial_S \dot{\mathbf{b}}_S - \boldsymbol{\omega} \times \dot{\mathbf{b}}_S = \ddot{\mathbf{b}}_S - \boldsymbol{\omega} \times \dot{\mathbf{b}}_S$$

so that

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} + (\dot{\mathbf{r}}'-\dot{\mathbf{c}}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S] + (\mathbf{r}'-\mathbf{c}') \times [\ (\dot{\boldsymbol{\omega}} \times \mathbf{r}') + (\boldsymbol{\omega} \times \dot{\mathbf{r}}') + \ddot{\mathbf{b}}_S - \boldsymbol{\omega} \times \dot{\mathbf{b}}_S]$$

$$+ \boldsymbol{\omega} \times \{\ (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}' + (\mathbf{r}'-\mathbf{c}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]\ \}$$

$$= \dot{\mathbf{L}}'^{(c')} + (\mathbf{v}'-\dot{\mathbf{c}}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S] + (\mathbf{r}'-\mathbf{c}') \times [\ \dot{\boldsymbol{\omega}} \times \mathbf{r}' + \boldsymbol{\omega} \times \mathbf{v}' + \ddot{\mathbf{b}}_S - \boldsymbol{\omega} \times \dot{\mathbf{b}}_S]$$

$$+ \boldsymbol{\omega} \times \{\ (\mathbf{r}'-\mathbf{c}') \times \mathbf{v}' + (\mathbf{r}'-\mathbf{c}') \times [\ (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\mathbf{b}}_S]\ \} \qquad = \text{LA (checked)}$$

Notice that two of the terms are

$(\mathbf{r}'-\mathbf{c}') \times [\boldsymbol{\omega} \times \mathbf{v}'] + \boldsymbol{\omega} \times [(\mathbf{r}'-\mathbf{c}') \times \mathbf{v}']$ = not cyclic so cannot combine

A B C B A C

$$A \times (B \times C) + B \times (A \times C) = (\mathbf{A}\bullet\mathbf{C})\mathbf{B} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C} + (\mathbf{B}\bullet\mathbf{C})\mathbf{A} - (\mathbf{A}\bullet\mathbf{B})\mathbf{C}$$

$$= [(\mathbf{r'}-\mathbf{c'})\bullet\mathbf{v'}]\,\boldsymbol{\omega} - [\boldsymbol{\omega}\bullet\mathbf{v'}]\,(\mathbf{r'}-\mathbf{c'}) - 2\,[(\mathbf{r'}-\mathbf{c'})\bullet\boldsymbol{\omega}]\,\mathbf{v'} \quad = \text{does not help much}$$

In a special case where $\mathbf{c'} = 0$ and $\ddot{\mathbf{c}}' = \mathbf{0}$ This says

$$\dot{\mathbf{L}}^{(b)} = \dot{\mathbf{L}}'^{(0)} + \mathbf{v'} \times [\,(\boldsymbol{\omega} \times \mathbf{r'}) + \dot{\mathbf{b}}_S] + \mathbf{r'} \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r'} + \boldsymbol{\omega} \times \mathbf{v'} + \ddot{\mathbf{b}}_S - \boldsymbol{\omega} \times \dot{\mathbf{b}}_S]$$

$$+ \boldsymbol{\omega} \times \{\,\mathbf{r'} \times \mathbf{v'} + \mathbf{r'} \times [\,(\boldsymbol{\omega} \times \mathbf{r'}) + \dot{\mathbf{b}}_S]\,\} \qquad = \text{LA (checked)}$$

I just do NOT see any good way to simplify this. But maybe ....

Note that $\mathbf{v} = \mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_S \Rightarrow \dot{\mathbf{c}} = \dot{\mathbf{c}}' + \boldsymbol{\omega} \times \mathbf{c'} + \dot{\mathbf{b}}_S$. I like the idea of $\mathbf{c'}$ and $\dot{\mathbf{c}}'$ showing up in the results because we wanted to have things expressed in Frame S' variables.

My previous result was

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} + (\mathbf{c'}- \mathbf{c} + \mathbf{b}) \times \mathbf{a'} - \dot{\mathbf{c}}_S \times [\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_S] + \dot{\mathbf{c}}'_{S'} \times \mathbf{v'}$$
$$+ (\mathbf{r'} - \mathbf{c} + \mathbf{b}) \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + \ddot{\mathbf{b}}_S] \quad .$$

In this previous result replace $\mathbf{c'}- \mathbf{c} + \mathbf{b} = \mathbf{0}$ and $\mathbf{b}-\mathbf{c} = -\mathbf{c'}$ to get

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{c}}_S \times [\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_S] + \dot{\mathbf{c}}'_{S'} \times \mathbf{v'}$$
$$+ (\mathbf{r'} - \mathbf{c'}) \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + \ddot{\mathbf{b}}_S] \quad .$$

Now replace $\dot{\mathbf{c}}_S = \dot{\mathbf{c}}'_{S'} + \boldsymbol{\omega} \times \mathbf{c'} + \dot{\mathbf{b}}_S$ to get

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} - [\,\dot{\mathbf{c}}'_{S'} + \boldsymbol{\omega} \times \mathbf{c'} + \dot{\mathbf{b}}_S\,] \times [\mathbf{v'} + \boldsymbol{\omega} \times \mathbf{r'} + \dot{\mathbf{b}}_S] + \dot{\mathbf{c}}'_{S'} \times \mathbf{v'}$$
$$+ (\mathbf{r'} - \mathbf{c'}) \times [\,\dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + \ddot{\mathbf{b}}_S] \quad . \qquad \equiv \text{LB}$$

Let's try Maple to show the two expressions are the same. I need to have names

$\mathbf{v'}, \dot{\mathbf{c}}', \boldsymbol{\omega}, \mathbf{r'}, \dot{\mathbf{b}}_S, \mathbf{c'}, \dot{\boldsymbol{\omega}}, \ddot{\mathbf{b}}_S$ = vp,cpd,w,rp,bd,cp,wd,bdd

After I fixed many things, Maple finally says LA and LB are the same!

These results may now be summarized:

$$\mathbf{L}^{(c)} = \mathbf{L}'^{(c')} + (\mathbf{r}' - \mathbf{c} + \mathbf{b})\,(\boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S) + (\mathbf{c}' - \mathbf{c} + \mathbf{b}) \times \mathbf{v}' \tag{11.3}$$

$$\dot{\mathbf{L}}^{(c)} = \dot{\mathbf{L}}'^{(c')} + (\mathbf{c}' - \mathbf{c} + \mathbf{b}) \times \mathbf{a}' - \dot{\mathbf{c}}_S \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S] + \dot{\mathbf{c}}'_S \times \mathbf{v}'$$
$$+ (\mathbf{r}' - \mathbf{c} + \mathbf{b}) \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \quad . \tag{11.4}$$

Equations (11.3) and (11.4) express $\mathbf{L}^{(c)}$ and $\dot{\mathbf{L}}^{(c')}$ entirely in terms of Frame S' objects, for the general case where angular momentum reference points **c** and **c'** are arbitrarily selected. Equation (11.4) can be rewritten as

$$\dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)} = -(\mathbf{c}' - \mathbf{c} + \mathbf{b}) \times \mathbf{a}' + \dot{\mathbf{c}}_S \times [\mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S] - \dot{\mathbf{c}}'_S \times \mathbf{v}'$$
$$- (\mathbf{r}' - \mathbf{c} + \mathbf{b}) \times [\dot{\boldsymbol{\omega}} \times \mathbf{r}' + 2\,\boldsymbol{\omega} \times \mathbf{v}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \ddot{\mathbf{b}}_S] \quad . \tag{11.5}$$

(c) Fictitious Torques and Newton's Rotational Law in a non-inertial frame

In analogy with Newton's Law $\mathbf{F} = (d\mathbf{p}/dt)_S$, Newton's Rotational Law in *inertial* Frame S is given by,

$$\mathbf{N}^{(c)} = (d\mathbf{L}^{(c)}/dt)_S = \dot{\mathbf{L}}^{(c)}{}_S = \dot{\mathbf{L}}^{(c)} \qquad \text{// true Newton's Rot Law in Frame S} \tag{11.6}$$

where $\mathbf{L}^{(c)}$ is the angular momentum of our Particle in Frame S relative to reference point **c**, and $\mathbf{N}^{(c)}$ is some externally applied torque about that same reference point acting on the Particle.

In Frame S' we want to find some effective ("bogus") Newton's Rotational Law,

$$\mathbf{N}'^{(c')}{}_{\mathbf{eff}} = \dot{\mathbf{L}}'^{(c')} \qquad \text{// bogus Newton's Rot Law in Frame S'} \tag{11.7}$$

$$\mathbf{N}'^{(c')}{}_{\mathbf{eff}} = \mathbf{N}^{(c)} + \mathbf{N}'^{(c')}{}_{\mathbf{fict}} \qquad \text{// } \textit{defines } \mathbf{N}'^{(c')}{}_{\mathbf{fict}} \tag{11.8}$$

where $\mathbf{N}'^{(c')}{}_{\mathbf{fict}}$ is the fictitious torque that mysteriously appears in non-inertial Frame S', so then

$$\mathbf{N}^{(c)} + \mathbf{N}'^{(c')}{}_{\mathbf{fict}} = \dot{\mathbf{L}}'^{(c')} \quad . \tag{11.9}$$

We can then apply this bogus Newton's Rotational Law in non-inertial Frame S' *as long as* we include the fictitious torques $\mathbf{N}'^{(c')}{}_{\mathbf{fict}}$ along with the real torques $\mathbf{N}^{(c)}$ which are just those present in inertial Frame S. This is in complete analogy with the use of fictitious forces as reviewed in Section 8.

Subtracting (11.6) from (11.8) gives,

$$\mathbf{N}'^{(c')}{}_{\mathbf{fict}} = \dot{\mathbf{L}}'^{(c')} - \dot{\mathbf{L}}^{(c)} \ , \tag{11.10}$$

and using (11.5) ,

$$\mathbf{N'}^{(\mathbf{c'})}_{\mathbf{fict}} = -(\mathbf{c'} - \mathbf{c} + \mathbf{b}) \times m\mathbf{a'} + \dot{\mathbf{c}}_{\mathbf{S}} \times [m\mathbf{v'} + m\boldsymbol{\omega} \times \mathbf{r'} + m\dot{\mathbf{b}}_{\mathbf{S}}] - \dot{\mathbf{c}}'_{\mathbf{S}'} \times m\mathbf{v'}$$
$$- (\mathbf{r'} - \mathbf{c} + \mathbf{b}) \times [\, m\dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2m\boldsymbol{\omega} \times \mathbf{v'} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) + m\ddot{\mathbf{b}}_{\mathbf{S}}] \qquad (11.11)$$

where we have quietly reinstalled the mass m. Recalling now the fictitious force from (8.6),

$$\mathbf{F'}_{\mathbf{fict}} = -m\ddot{\mathbf{b}}_{\mathbf{S}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) - 2m\,\boldsymbol{\omega} \times \mathbf{v'} - m\dot{\boldsymbol{\omega}} \times \mathbf{r'} \qquad \textit{(8.6)} \qquad (11.12)$$
frame centrifugal Coriolis Euler

the fictitious torque can be written as

$$\mathbf{N'}^{(\mathbf{c'})}_{\mathbf{fict}} = -(\mathbf{c'} - \mathbf{c} + \mathbf{b}) \times m\mathbf{a'} + \dot{\mathbf{c}}_{\mathbf{S}} \times [m\mathbf{v'} + m\boldsymbol{\omega} \times \mathbf{r'} + m\dot{\mathbf{b}}_{\mathbf{S}}] - \dot{\mathbf{c}}'_{\mathbf{S}'} \times m\mathbf{v'}$$
$$+ (\mathbf{r'} - \mathbf{c} + \mathbf{b}) \times \mathbf{F'}_{\mathbf{fict}} \ . \qquad (11.13)$$

As noted earlier, a common situation is to assume $\mathbf{c} \equiv 0$ and $\mathbf{c}' \equiv 0$ so that

$$\mathbf{N'}^{(\mathbf{0})}_{\mathbf{fict}} = -\mathbf{b} \times m\mathbf{a'} + (\mathbf{r'} + \mathbf{b}) \times \mathbf{F'}_{\mathbf{fict}} \ . \qquad // \ \mathbf{c} = \mathbf{c}' = 0 \qquad (11.14)$$

If in addition the origins of frames S and S' coincide, then $\mathbf{b} = 0$ and the result is simply

$$\mathbf{N'}^{(\mathbf{0})}_{\mathbf{fict}} = \mathbf{r'} \times \mathbf{F'}_{\mathbf{fict}} \ . \qquad // \ \mathbf{c} = \mathbf{c}' = \mathbf{b} = 0 \qquad (11.15)$$