archive old Goldstein and Marion sections 9 and 10 PhL 2.2.17

9. Comparison with Marion (1970), Thornton & Marion (2003) and Taylor (2005)

In this Section and the next we wish to compare our notation for rotating-frame kinematics and non-inertial-frame physics to the notation of two groups of well-known and well-read textbook authors. What symbols do *they* use to denote the two frames of reference and the various positions, velocities and accelerations?

Comparison between Marion and T&M notation and our "non-swap" notation

Marion and Thornton & Marion (T&M) both swap the primes on the frames relative to us, S ↔ S', which then includes **r** ↔ **r'**. The following notations are used (us → Marion/T&M)

$$\mathbf{r}' \to \mathbf{r} \qquad \mathbf{v} \to \mathbf{v}_f \qquad \mathbf{v}' \to \mathbf{v}_r \qquad \mathbf{b} \to \mathbf{R} \qquad \dot{\mathbf{b}}_S \to \dot{\mathbf{R}}_f = \mathbf{V}$$
$$\mathbf{a} \to \mathbf{a}_f \qquad \mathbf{a}' \to \mathbf{a}_r \qquad \ddot{\mathbf{b}}_S \to \ddot{\mathbf{R}}_f \qquad (9.1)$$

where subscript r means "rotating frame" and f means "fixed frame". Thus, Marion's and T&M's version of (8.1.3) reads ($\dot{\boldsymbol{\omega}} = 0$ in Marion but is retained in T&M)

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{b}}_S + m\mathbf{a}' + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad (8.1.3)$$

$$\mathbf{F} = m\mathbf{a}_f = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad . \qquad \text{// Marion p 344 (11.17)}$$
// T&M p 392 (10.23)

Note that for Marion **r** is a vector to the Particle from the *rotating* frame origin. Marion's version of our (8.1.9) is then

$$\mathbf{F} = m\mathbf{a} \approx m\mathbf{a}' + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + 2m\,\boldsymbol{\omega} \times \mathbf{v}' + m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad (8.1.9)$$

$$\mathbf{F} = m\mathbf{a}_f \approx m\mathbf{a}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r} \qquad \text{// Marion p 344 (11.18)}$$

and his version of (8.1.11) is

$$m\mathbf{a}' = \mathbf{F}'_{eff} \approx m\mathbf{a} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad (8.1.11)$$

$$m\mathbf{a}_r = \mathbf{F}_{eff} \approx m\mathbf{a}_f - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad . \qquad \text{// Marion p 344 (11.19)}$$

Meanwhile, translation of our (8.1.5) is done this way

$$m\mathbf{a}' = \mathbf{F}'_{eff} = \mathbf{F} - m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad (8.1.5)$$

$$m\mathbf{a}_r = \mathbf{F}_{eff} = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad . \qquad \text{// T\&M p 392 (10.25)}$$

Finally, our velocity equation (6.6a) translates, according to rules (9.1), as follows,

$\mathbf{v} = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' + \dot{\mathbf{b}}_S$ *(6.6a)*

$\mathbf{v}_f = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{V}$. // Marion p 344 (11.12)

// T&M p 392 (10.17)

Comparison between Taylor notation and our "non-swap notation"

These are the translation rules from us to Taylor :

S,S'	→	$\mathfrak{S}_0, \mathfrak{S}$	
$\mathbf{r},\mathbf{r}'$	→	$\mathbf{r}_0, \mathbf{r}$	(9.2)

so he ends up with vector **r** being the natural position vector in the rotating frame, the same as Marion and Goldstein in the next section.

Comparison between Marion and T&M notation and our "swap" notation

Recall that in our swap notation, Frame S is the rotating frame and Frame S' is the inertial frame. We put primes on forces in inertial frame S' whereas Marion and T&M do not.

True Newton's Law in Inertial Frame S':

$\mathbf{F}' = m\mathbf{a}' = m\ddot{\mathbf{b}}_S + m\mathbf{a} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ *(8.1.3)$_S$*

$\mathbf{F} = m\mathbf{a}_f = m\ddot{\mathbf{R}}_f + m\mathbf{a}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ // Marion p 344 (11.17)

// T&M p 392 (10.23)

$\mathbf{F}' = m\mathbf{a}' \approx m\mathbf{a} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v} + m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ *(8.1.9)$_S$*

$\mathbf{F} = m\mathbf{a}_f \approx m\mathbf{a}_r + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + 2m\,\boldsymbol{\omega} \times \mathbf{v}_r + m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ // Marion p 344 (11.18)

Bogus Newton's Law in non-inertial Frame S:

$m\mathbf{a} = \mathbf{F}_{eff} \approx m\mathbf{a}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ *(8.1.11)$_S$*

$m\mathbf{a}_r = \mathbf{F}_{eff} \approx m\mathbf{a}_f - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ // Marion p 344 (11.19)

$m\mathbf{a} = \mathbf{F}_{eff} = \mathbf{F}' - m\ddot{\mathbf{b}}_S - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v} - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ *(8.1.5)$_S$*

$m\mathbf{a}_r = \mathbf{F}_{eff} = \mathbf{F} - m\ddot{\mathbf{R}}_f - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r}$ // T&M p 392 (10.25)

Velocity relation:

$\mathbf{v}' = \mathbf{v} + \boldsymbol{\omega} \times \mathbf{r} + \dot{\mathbf{b}}_S$ *(6.6a)$_S$*

$\mathbf{v}_f = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{V}$ // Marion p 344 (11.12)

10. Comparison with Goldstein (1950) and Goldstein, Poole and Safko (2001)

Goldstein is a bit of a conundrum and requires careful decoding.
GPS refers to Goldstein, Poole and Safko.

10.1 The meaning of r

Goldstein also has S↔S' including **r**↔**r'** relative to our "non-swap" notation. We know this because he says on page 135 that his **r** is a vector "from the origin of the terrestrial system to the given particle". Earlier he says "terrestrial measurements are usually made with respect to a coordinate system fixed in the Earth, which therefore rotates uniformly with a constant angular velocity **ω** relative to the inertial system". Presumably "fixed in the Earth" means "fixed on the surface of the Earth". Therefore surely the vector he calls **r** is the one we call **r'**. This is consistent with a general S↔ S' swap and agrees with Marion's use of vector **r**, but mainly it is consistent with Goldstein's own equations as we shall see below.

10.2 The meaning of Goldstein's $\mathbf{a_s}$ and $\mathbf{a_r}$ (and of $\mathbf{v_s}$ and $\mathbf{v_r}$)

Recall (7.4) from above

$$\mathbf{a'_S} = \mathbf{a'} + \dot{\boldsymbol{\omega}} \times \mathbf{r'} + 2\,\boldsymbol{\omega} \times \mathbf{v'} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r'}) \quad . \qquad (7.4)$$

Goldstein states the following equation (he has $\dot{\boldsymbol{\omega}} = 0$ but we include the term anyway)

$$\mathbf{a_s} = \mathbf{a_r} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\,\boldsymbol{\omega} \times \mathbf{v_r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \quad .$$

// Goldstein p 135 (4-105)
// GPS p 175 (4.89)

His subscript r means rotating (same as Marion), while subscript s means space (Marion's fixed system). Comparing our (7.4) to the Goldstein equation above tells us that

G	us	
r	**r'**	
$\mathbf{v_s}$	$\mathbf{v'_S}$	// added to this list since consistent with the $\mathbf{a_s}$ line following
$\mathbf{a_s}$	$\mathbf{a'_S}$	
$\mathbf{v_r}$	**v'**	
$\mathbf{a_r}$	**a'**	

Goldstein uses the name $\mathbf{v_r}$ just the way Marion does. However, for Marion

$$\mathbf{v_f} = \mathbf{v}$$
$$\mathbf{a_f} = \mathbf{a}$$

so we must conclude that $\mathbf{v_f} \neq \mathbf{v_s}$ and $\mathbf{a_f} \neq \mathbf{a_s}$. The only comment Goldstein gives (p 135) is that "$\mathbf{v_s}$ and $\mathbf{v_r}$ are the velocities of the particle relative to the space and rotating set of axes respectively." As we have seen in Section 1.8, there are several *kinds* of "velocity" so this does not quite nail it down. Since

Goldstein is thinking of doing things in terms of a position vector from the "terrestrial frame", it is not unreasonable to think that

Goldstein		Us		
$\mathbf{v}_S = (d\mathbf{r}/dt)_{space}$	→	$(d\mathbf{r}'/dt)_S = \mathbf{v}'_S$	that is	$\mathbf{v}_S = \mathbf{v}'_S$
$\mathbf{v}_r = (d\mathbf{r}/dt)_{rot}$	→	$(d\mathbf{r}'/dt)_{S'} = \mathbf{v}'_{S'}$;	that is	$\mathbf{v}_r = \mathbf{v}'_{S'}$

in agreement with the small table above. So we conclude that these are the correct Goldstein translation rules (us → Goldstein/GPS)

$$\mathbf{r}' \to \mathbf{r} \qquad \mathbf{v}'_S \to \mathbf{v}_S \qquad \mathbf{v}'_{S'} \to \mathbf{v}_r$$
$$\mathbf{a}'_S \to \mathbf{a}_S \qquad \mathbf{a}'_{S'} \to \mathbf{a}_r \ . \tag{10.1}$$

In terms of the discussion in Section 1.8, Goldstein has chosen for his $\mathbf{v}_S$ and $\mathbf{a}_S$ *not* to use the "natural" velocity and acceleration **v** and **a** shown in (1.8.4) and (1.8.6).

Let's test out rules (10.1). Consider our (6.8a) and its Goldstein translation by (10.1) on the second line,

$$\mathbf{v}'_S = \mathbf{v}' + \boldsymbol{\omega} \times \mathbf{r}' \qquad \textit{(6.8a)}$$
$$\mathbf{v}_S = \mathbf{v}_r + \boldsymbol{\omega} \times \mathbf{r} \ . \qquad \text{// Goldstein p 135 (4-104)}$$
$$\text{// GPS p 175 (4.88)}$$

So this is another piece of evidence confirming our interpretation. Notice that (6.8a) is just the G Rule for the vector **r'**, and that is how Goldstein motivates the equation (but with his **r**).

10.3 A hidden approximation is located

In his discussion on page 135 (1950) Goldstein says nothing at all about any approximations, although he does have in mind the specific Earth system. On page 135 he states "the equation of motion" in an equation with no number as

$$\mathbf{F} = m\mathbf{a}_S \ . \qquad \text{// Goldstein page 135, no number}$$
$$\text{// GPS p 175, no number}$$

According to our translation rules, this says that $\mathbf{F} = m\mathbf{a}'_S$. But the correct Newton's Law is

$$\mathbf{F} = m\mathbf{a} \qquad \textit{(8.1)}$$

and according to our (7.5),

$$\mathbf{a} = \mathbf{a}'_S + \ddot{\mathbf{b}}_S \ , \qquad \textit{(7.5)}$$

Newton's Law really says

$$\mathbf{F} = m\mathbf{a'_S} + m\ddot{\mathbf{b}}_S$$
$$= m\mathbf{a}_S + m\ddot{\mathbf{b}}_S \quad . \qquad \text{// Goldstein implied}$$

We conclude that in writing $\mathbf{F} = m\mathbf{a}_S$, Goldstein has made the approximation that $\ddot{\mathbf{b}}_S$ can be neglected. This is the same assumption Marion makes explicitly. So we interpret Goldstein's unnumbered equation as

$$\mathbf{F} = m\mathbf{a}_S + m\ddot{\mathbf{b}}_S \approx m\mathbf{a}_S \quad . \tag{10.2}$$

His remaining two equations can be obtained as follows. We go back to approximate (8.1.10) and translate it using the rules (10.1),

$$m\mathbf{a}' = \mathbf{F'}_{eff} \approx \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad \textit{(8.1.10)}$$
$$m\mathbf{a}_r = \mathbf{F}_{eff} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad . \qquad \text{// Goldstein p 135 (4-106)}$$
$$\text{// GPS p 175 (4.90)}$$

Next, combine (8.1.7) and (8.1.13) to get the first line, then translate for the 2nd line,

$$\mathbf{F'}_{eff} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') - 2m\,\boldsymbol{\omega} \times \mathbf{v}' - m\dot{\boldsymbol{\omega}} \times \mathbf{r}' \qquad \textit{(8.1.7) + (8.1.13)}$$
$$\mathbf{F}_{eff} = \mathbf{F} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\,\boldsymbol{\omega} \times \mathbf{v}_r - m\dot{\boldsymbol{\omega}} \times \mathbf{r} \quad . \qquad \text{// Goldstein p 135 (4-107)}$$
$$\text{// GPS p 175 (4.91)}$$